

KILLING VECTOR FIELDS ON TANGENT BUNDLES WITH CHEEGER-GROMOLL METRIC

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Abstract. This paper deals with properties of the Cheeger-Gromoll metric \bar{g} introduced in 1988 by Musso and Tricerri on the tangent bundle TM associated to a given Riemannian metric (M, g) .

One can find here essentially the two following results:

1. A classification of Killing vector fields on (TM, \bar{g})
2. A generalization of a result of M. Sekizawa concerning the non rigidity of the Cheeger-Gromoll metric.

1. Introduction

Let (M, g) be a Riemannian manifold and TM its tangent bundle with the natural projection $\pi : TM \rightarrow M$.

Although the Sasaki metric Tg on TM [9] is a “naturally defined” Riemannian metric, it is “extremely rigid” [8]. O. Kowalski [6] has shown that it is never locally symmetric unless the base metric is locally Euclidean. E. Musso and F. Tricerri [8] have generalized this fact: they have shown that it has constant scalar curvature if and only if the base metric is flat. To surmount this hindrance E. Musso and F. Tricerri have suggested the Cheeger-Gromoll metric \bar{g} on TM [3] as an alternative. Indeed, M. Sekizawa [10] has shown that its scalar curvature is never constant if the original metric on the base manifold has constant curvature. The authors have also studied some other properties of this metric [1], many of which will be used here.

Throughout this work, TM will be endowed with the Cheeger-Gromoll metric \bar{g} . We give, firstly, a theorem on general forms of all Killing vector fields on

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(TM, \bar{g}) (Theorem 3.6). If some assumptions are imposed, then the forms of Killing vector fields become simpler ones (Theorems 4.1, 4.2 and 4.3); some ideas of these results are inspired from similar ones for the case of the Sasaki metric [12]. In §5, we give a generalization of a result of M. Sekizawa [10] concerning the non rigidity of the Cheeger-Gromoll metric on the tangent bundle (Theorem 5.7).

In this paper, manifolds are assumed to be connected and smooth and the so-called ‘‘Einstein’s summation’’ will be used everywhere.

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2. Preliminaries

A local coordinate neighborhood $\{(U; x^i, i = 1, \dots, n)\}$ in M induces on TM a local coordinate neighborhood $\{\pi^{-1}(U); x^i, v^i, i = 1, \dots, n\}$. Let $X = (X^i)$ be a vector field on M . Then the complete lift X^c , the vertical lift X^v and the horizontal lift X^h of X are given, with respect to the induced coordinates, by: $X^c = (X^i, v^j(\partial X^i/\partial x^j))$, $X^v = (0, X^i)$ and $X^h = (X^i, -\Gamma_{jk}^i v^j X^k)$ respectively, where (Γ_{jk}^i) denote the Christoffel’s symbols of g . (cf. [9], [15])

On the other hand, to any $(1, 1)$ -tensor field P on M , one can associate two vector fields ιP and $*P$ defined on TM by:

$$\iota P = (0, P_j^i v^j) \quad \text{and} \quad *P = (P_j^i v^j, -\Gamma_{lk}^i P_j^l v^j v^k).$$

Then ιP is a vertical vector field and $*P$ is a horizontal one, and we have $X^c = X^h + \iota(\nabla X)$, where $\nabla X = (\nabla_j X^i)$ and ∇ denotes the covariant differentiation with respect to the Levi-Civita connection of g .

In this paper, we need new lifts of vector fields on M . For any vector field X on M , let X_A and X_B be vector fields on TM defined at any point $(x, u) \in TM$ by

$$X_A(x, u) = (1 - r^2)X^v(x, u) + g_x(u, X(x))U(x, u),$$

$$X_B(x, u) = (1 - r^2)X^v(x, u) + g_x(u, X(x))U(x, u) + *(C(X))(x, u),$$

where $r^2 = g_x(u, u)$, U is the canonical vertical vector field defined, locally, by $U = (0, v^i)$, i.e., $U(x, u) = (0, u^i)$, and $C(X)$ is $(1, 1)$ -tensor field on M defined by $g(C(X)Y, Z) = -g(Y, \nabla_Z X)$ for all vector fields Y and Z on M . The components of $C(X)$ with respect to the local coordinates are $(-g^{ik} g_{jl} \nabla_k X^l)$. Clearly the lifts X_A and X_B are smooth vector fields on TM .

Remark that X_B is a vertical field; it is the vertical part of X_A .

The Cheeger-Gromoll metric \bar{g} on TM has components [1]:

$$\begin{cases} \bar{g}_{ij} = g_{ij} + v^\lambda v^\tau \Gamma_{i\lambda}^k \Gamma_{j\tau}^l G_{kl} \\ \bar{g}_{in+j} = v^\tau \Gamma_{i\tau}^k G_{kj} \\ \bar{g}_{n+in+j} = G_{ij} \end{cases} \quad i, j = 1, \dots, n,$$

where G_{ij} is the function on $\pi^{-1}(U)$ defined by:

$$G_{ij} = 1/(1 + r^2)(g_{ij} + g_{ik}g_{jl}v^k v^l).$$

3. Classification of Killing Vector Fields on (TM, \bar{g})

The general forms of Killing vector fields on (TM, \bar{g}) are given by

THEOREM 3.1. *Let (TM, \bar{g}) be the tangent bundle with the Cheeger-Gromoll metric of a Riemannian manifold (M, g) . Let*

- (i) *X be a Killing vector field on (M, g) ;*
- (ii) *P be a $(1, 1)$ -tensor field on (M, g) which is*
 (P_1) Parallel with respect to g and
 (P_2) skew-symmetric with respect to g , when considered as a vector field valued linear mapping on the space of vector fields on M ;
- (iii) *Y be a vector field on (M, g) which satisfies*
 (Y_1) the second covariant derivation $\nabla^2 Y$ of Y is skew-symmetric, when considered as a vector field valued bilinear mapping on the space of vector fields on M ;

$$\begin{aligned} (Y_2) \quad & -R(X, C(Y)(W))V - R(X, C(Y)(V))W \\ & = 2[g(\nabla_X Y, V)W + g(\nabla_X Y, W)V - 2g(V, W)\nabla_X Y] \end{aligned}$$

for all vector fields X, V and W on M .

Then the vector field Z on TM defined by

$$(TK) \quad Z = X^c + \iota P + Y_B \text{ is a Killing vector field on } (TM, \bar{g}).$$

Conversely, every Killing vector field on (TM, \bar{g}) is of the form (TK) .

PROOF. The proof will be divided into a series of propositions and lemmas.

We have shown in [1] (Theorem 5.11), that the vertical lift to TM of a non-zero vector field on M is never a Killing vector on (TM, \bar{g}) contrary to the Sasaki metric case where the vertical lift to TM of a parallel vector field on M is a Killing vector field on (TM, Tg) [9]. This fact for the Sasaki metric was a fundamental tool on which S. Tanno had based to give general forms of all Killing vector fields on (TM, Tg) [12]. Now, we will give an analogous of Tanno's the-

orem (Theorem A. [12]) on the classification of Killing vector fields, in the case of Cheeger-Gromoll metric.

Recall that a vector field $Z = (Z^I)$ is a Killing vector or an infinitesimal isometry with respect to the metric $\bar{g} = (\bar{g}_{IJ})$ if and only if $L_Z\bar{g} = 0$, where L_Z denotes the Lie derivative with respect to Z , i.e.,

$$Z^K \partial_K \bar{g}_{IJ} + \partial_I Z^K \bar{g}_{KJ} + \partial_J Z^K \bar{g}_{KI} = 0; \quad I, J = 1, \dots, 2n.$$

Putting $(I, J) = (i, j)$, $(i, n + j)$ and $(n + i, n + j)$, the last equation is transformed to the system:

$$(K) \left\{ \begin{array}{l} (K_1) \quad Z^\gamma \partial / \partial x^\gamma (g_{ij} + v^\lambda v^\tau \Gamma_{i\lambda}^k \Gamma_{j\tau}^l G_{lk}) + Z^{n+\gamma} \partial / \partial v^\gamma (g_{ij} + v^\lambda v^\tau \Gamma_{i\lambda}^k \Gamma_{j\tau}^l G_{kl}) \\ \quad + \partial Z^\gamma / \partial x^i (g_{\gamma j} + v^\lambda v^\mu \Gamma_{\gamma\lambda}^k \Gamma_{j\mu}^l G_{kl}) + \partial Z^{n+\gamma} / \partial x^i v^\mu \Gamma_{j\mu}^k G_{k\gamma} \\ \quad + \partial Z^\gamma / \partial x^j (g_{\gamma i} + v^\lambda v^\mu \Gamma_{\gamma\lambda}^k \Gamma_{i\mu}^l G_{kl}) + \partial Z^{n+\gamma} / \partial x^j v^\lambda \Gamma_{i\lambda}^k G_{k\gamma} = 0, \\ (K_2) \quad Z^\gamma \partial / \partial x^\gamma (v^\lambda \Gamma_{i\lambda}^k G_{kj}) + Z^{n+\gamma} \partial / \partial v^\gamma (v^\lambda \Gamma_{i\lambda}^k G_{kj}) + \partial Z^\gamma / \partial x^i v^\lambda \Gamma_{\gamma\lambda}^k G_{kj} \\ \quad + \partial Z^{n+\gamma} / \partial x^i G_{\gamma j} + \partial Z^\gamma / \partial v^j (g_{\gamma i} + v^\lambda v^\tau \Gamma_{i\lambda}^k \Gamma_{\gamma\tau}^l G_{kl}) \\ \quad + \partial Z^{n+\gamma} / \partial v^j v^\lambda \Gamma_{i\lambda}^k G_{k\gamma} = 0, \\ (K_3) \quad Z^\gamma \partial G_{ij} / \partial x^\gamma + Z^{n+\gamma} \partial G_{ij} / \partial v^\gamma + \partial Z^\gamma / \partial v^i v^\lambda \Gamma_{\gamma\lambda}^k G_{kj} + \partial Z^{n+\gamma} / \partial v^i G_{\gamma j} \\ \quad + \partial Z^\gamma / \partial v^j v^\lambda \Gamma_{\gamma\lambda}^k G_{ki} + \partial Z^{n+\gamma} / \partial v^j G_{\gamma i} = 0. \end{array} \right.$$

First of all, we shall study the particular cases X^c , ιP and X_B .

PROPOSITION 3.2 [1]. *In order that a complete lift X^c to TM of a vector field X on M be a Killing vector field of (TM, \bar{g}) , it is necessary and sufficient that X itself is a Killing vector field of (M, g) .*

The following lemma is immediate from (K):

LEMMA 3.3.

1- *If $P = (P_j^i)$ is a $(1, 1)$ -tensor field on M , then:*

$$L_{\iota P} \bar{g} = \begin{pmatrix} 0 & G_{\gamma i} \nabla_j P_r^\gamma v^r \\ G_{\gamma j} \nabla_i P_r^\gamma v^r & \frac{1}{1+r^2} (A_{ij} + B_{ij} - C_{ij}) \end{pmatrix},$$

where

$$A_{ij} = (g_{\gamma j} P_i^\gamma + g_{\gamma i} P_j^\gamma) + (g_{\gamma i} P_r^\gamma + g_{\gamma r} P_i^\gamma) v^r v_j,$$

$$B_{ij} = (g_{\gamma j} P_r^\gamma + g_{\gamma r} P_j^\gamma) v^r v_i,$$

$$C_{ij} = (g_{\gamma s} P_r^\gamma + g_{\gamma r} P_s^\gamma) v^r v^s G_{ij}.$$

2- If Y is a vector field on M , then

$$L_{Y_B} \bar{g} = \begin{pmatrix} -(\nabla_i \nabla_j Y^k + \nabla_j \nabla_i Y^k) v^k & D_{ij} \\ D_{ji} & 0 \end{pmatrix},$$

where $D_{\xi\zeta} = -g^{lk} G_{k\xi} v^r v^s [R_{ls\zeta\gamma} \nabla^\gamma Y_r - \nabla_\zeta Y^\gamma (g_{lr} g_{\gamma s} + g_{ls} g_{\gamma r} - 2g_{l\gamma} g_{rs})]$, $\xi, \zeta = 1, \dots, n$.

As corollaries of lemma 3.3, we obtain

PROPOSITION 3.4. *Let P be a $(1, 1)$ -tensor field on (M, g) satisfying the conditions (P_1) and (P_2) in Theorem 3.1. Then ιP is a Killing vector field on (TM, \bar{g}) .*

PROPOSITION 3.5. *Let Y be a vector field on (M, g) satisfying the conditions (Y_1) and (Y_2) in Theorem 3.1. Then Y_B is a Killing vector field on (TM, \bar{g}) .*

Now, we consider the 0-section ($v^i = 0$) in the coordinate neighborhood $(\pi^{-1}(U); x^i, v^i, i = 1, \dots, n)$ in TM induced from $(U; x^i, i = 1, \dots, n)$ in M , and its neighborhood W . For a vector field Z on TM , and $(x, v) = (x^i, v^i)$ in W , we can write, by Taylor's theorem,

$$Z^i(x, v) = Z^i(x, 0) + \partial Z / \partial v^r(x, 0) v^r + \frac{1}{2} (\partial / \partial v^r (\partial Z^i / \partial v^s))(x, 0) v^r v^s + \dots [*\]_\lambda^i,$$

$$\begin{aligned} Z^{n+i}(x, v) &= Z^{n+i}(x, 0) + \partial Z^{n+i} / \partial v^r(x, 0) v^r \\ &\quad + \frac{1}{2} (\partial / \partial v^r (\partial Z^{n+i} / \partial v^s))(x, 0) v^r v^s + \dots [*\]_\lambda^{n+i}, \end{aligned}$$

where $[*\]_\lambda^I$ ($I = 1, \dots, 2n$) is of the form: $[*\]_\lambda^I = \frac{1}{\lambda!} (\partial^\lambda Z^I / \partial v^{i_1} \partial v^{i_2} \dots \partial v^{i_\lambda}) \cdot (x^a, \theta(x, v) v^b) v^{i_1} v^{i_2} \dots v^{i_\lambda}$; $1 \leq i_1, \dots, i_\lambda \leq n$.

The following lemma is valid.

LEMMA 3.6 [12]. *In the above situation, the following*

$$X = (X^i(x)) = (Z^i(x, 0)), \quad Y = (Y^i(x)) = (Z^{n+i}(x, 0)),$$

$$K = (K_r^i(x)) = (\partial Z^i / \partial v^r(x, 0)),$$

$$E = (E_{rs}^i(x)) = (\partial / \partial v^r (\partial Z^i / \partial v^s)(x, 0)),$$

$$P = (P_r^i(x)) = ((\partial Z^{n+i} / \partial v^r)(x, 0) - \partial Z^i / \partial x^r(x, 0))$$

are tensor fields on M .

With the notations of lemma 3.6, we can write:

$$(T_1) \quad Z^i(x, v) = X^i + K_r^i v^r + \frac{1}{2} E_{rs}^i v^r v^s + \cdots + [*]_{\lambda}^i,$$

$$(T_2) \quad Z^{n+i}(x, v) = Y^i + \tilde{P}_r^i v^r + \frac{1}{2} Q_{rs}^i v^r v^s + \cdots + [*]_{\lambda}^{n+i},$$

where \tilde{P}_r^i and Q_{rs}^i are given by

$$\tilde{P}_r^i(x) = \partial Z^{n+i} / \partial v^r(x, 0) \quad \text{and} \quad Q_{rs}^i = \partial^2 Z^{n+i} / \partial v^r \partial v^s(x, 0),$$

and we have:

$$(T_3) \quad \partial Z^i / \partial v^j(x, v) = K_j^i + E_{rj}^i v^r + \cdots + \langle * \rangle_{\lambda-1}^i,$$

$$(T_4) \quad \partial Z^{n+i} / \partial v^j(x, v) = \tilde{P}_j^i + Q_{rj}^i v^r + \cdots + \langle * \rangle_{\lambda-1}^{n+i}, \dots \text{ etc.}$$

When we apply Taylor's theorem to the left hand sides of (K) to some order λ , the results are the same as one obtains by substituting (T_1) , (T_2) , (T_3) , $(T_4) \dots$ into (K) up to order $\lambda - 1$. Furthermore the vanishing of the right hand sides of the equations of (K) implies the vanishing of each coefficient (up to order $\lambda - 1$).

Substituting (T_1) and (T_2) into (K_1) and taking the part which does not contain v^r , we have

$$X^\gamma \partial g_{ij} / \partial x^\gamma + \partial X^\gamma / \partial x^i g_{\gamma j} + \partial X^\gamma / \partial x^j g_{\gamma i} = 0.$$

Hence, $X = (X^i)$ is a Killing vector field on (M, g) . Since, by proposition 3.2, X^c is a Killing vector field on (TM, \bar{g}) , $Z - X^c$ is also a Killing vector field. Therefore, in the following, denoting $Z - X^c$ by the same letter Z , one may assume that $X^i = 0$ in (T_1) . Then $(\tilde{P}_r^i) = (P_r^i)$ is a tensor field on M by lemma 3.6.

Putting (T_1) and (T_2) into (K_2) (from now on, we omit this statement) and taking the part which does not contain v^r , we get: $Y^\gamma \Gamma_{i\gamma}^k g_{kj} + \partial Y^\gamma / \partial x^i g_{\gamma j} + K_j^\gamma g_{\gamma i} = 0$, that is

$$(D_1) \quad K_j^\gamma g_{i\gamma} = -g_{ij} \nabla_i Y^\gamma = -\nabla_i Y_j, \text{ i.e.,}$$

$$(D_1') \quad K_j^i = -\nabla^i Y_j.$$

Taking the coefficient of v^s in (K_1) , we get

$$K_s^\gamma \partial g_{ij} / \partial x^\gamma + Y^\gamma (\Gamma_{is}^k \Gamma_{j\gamma}^l g_{lk} + \Gamma_{i\gamma}^k \Gamma_{js}^l g_{lk}) + \partial K_s^\gamma / \partial x^i g_{\gamma j} \\ + \partial Y^\gamma / \partial x^i \Gamma_{js}^k g_{k\gamma} + \partial K_s^\gamma / \partial x_j g_{\gamma i} + \partial Y^\gamma / \partial x^j \Gamma_{is}^k g_{k\gamma} = 0.$$

Using the equality $\partial g_{ij} / \partial x^\gamma = \Gamma_{\gamma i}^t g_{tj} + \Gamma_{\gamma j}^t g_{ti}$ and (D_1) , we see that the above equation can be simplified to

$$(D_2) \quad \nabla_i \nabla_j Y_s + \nabla_j \nabla_i Y_s = 0.$$

Taking the part which does not contain v^r , in (K_3) , we get

$$(D_3) \quad P_i^\gamma g_{\gamma j} + P_j^\gamma g_{\gamma i} = 0.$$

Taking the coefficient of v^s in (K_2) , we get

$$P_s^\gamma \Gamma_{i\gamma}^k g_{kj} + \partial P_s^\gamma / \partial x^i g_{\gamma j} + E_{sj}^\gamma g_{\gamma i} + P_j^\gamma \Gamma_{is}^k g_{k\gamma} = 0,$$

or equivalently

$$g_{i\gamma}E_{js}^\gamma + \nabla_i P_s^\gamma g_{j\gamma} = 0, \quad \text{i.e. } g_{i\gamma}E_{js}^\gamma + \nabla_i P_{sj} = 0,$$

where $P_{sj} = P_s^\gamma g_{j\gamma}$.

Since E_{js}^γ is symmetric in j and s , and $(\nabla_i P_s^\gamma)g_{j\gamma} = \nabla_i P_{sj}$ is skew-symmetric in j and s by (D₃), we have

$$(D_4) \quad E_{js}^\gamma = 0 \quad \text{and} \quad \nabla_i P_s^\gamma = 0.$$

Taking the coefficient of v^s in (K₃), we get

$$(D_5) \quad K_s^\gamma \partial g_{ij} / \partial x^\gamma + Y^\gamma (g_{is}g_{j\gamma} + g_{i\gamma}g_{js} - 2g_{\gamma s}g_{ij}) + K_i^\gamma \Gamma_{\gamma s}^k g_{kj} \\ + Q_{st}^\gamma g_{\gamma j} + K_j^\gamma \Gamma_{\gamma s}^k g_{ki} + Q_{sj}^\gamma g_{\gamma i} = 0.$$

We put $Q_{rs}^i = -(\Gamma_{kr}^i K_s^k + \Gamma_{ks}^i K_r^k) - 2Y^i g_{rs} + (Y^k g_{kr} \delta_s^i + Y^k g_{ks} \delta_r^i) + T_{rs}^i$.

By simple calculation using (D₅), we can verify that

$$T_{is}^r g_{rj} + T_{js}^r g_{ri} = 0.$$

If we put $T_{isj} = T_{is}^r g_{rj}$, then T_{isj} is symmetric in i and s , and skew-symmetric in i and j . Hence $T_{isj} = 0$. That is

$$(D_6) \quad Q_{sr}^i = -(\Gamma_{kr}^i K_s^k + \Gamma_{ks}^i K_r^k) - 2Y^i g_{rs} + (Y^k g_{kr} \delta_s^i + Y^k g_{ks} \delta_r^i).$$

Taking the coefficient of $v^s v^t$ in (K₂), we get by virtue of (D₆)

$$(D_7) \quad (K_s^\gamma R_{it\gamma}^k + K_t^\gamma R_{is\gamma}^k) g_{kj} + 2\nabla_i Y^\gamma (g_{\gamma s} g_{jt} + g_{\gamma t} g_{js} - 2g_{\gamma j} g_{st}) \\ + \partial / \partial v^s (\partial / \partial v^t (\partial Z^\gamma / \partial v^j))(x, 0) g_{\gamma i} = 0.$$

Taking the skew-symmetric part in s and j of (D₇), we get

$$(D_8) \quad 2R_{jsi\gamma} K_t^\gamma + R_{jti\gamma} K_s^\gamma - R_{sti\gamma} K_j^\gamma + 2\nabla_i Y^\gamma (3g_{\gamma s} g_{jt} - 3g_{\gamma j} g_{st}) = 0.$$

Taking the symmetric part in s and t of (D₈), we get

$$(D_9) \quad R_{jsi\gamma} K_t^\gamma + R_{jti\gamma} K_s^\gamma = -2\nabla_i Y^\gamma (g_{\gamma s} g_{jt} + g_{\gamma t} g_{js} - 2g_{\gamma j} g_{st}).$$

Now, by (D₃) and (D₄), we see that ιP is a Killing vector on (TM, \bar{g}) by Proposition 3.4. By (D₁), (D₂), (D₉) and Proposition 3.5, Y_B is a Killing vector field on (TM, \bar{g}) .

We put $[Z] = Z - \iota P - Y_B$ which is a Killing vector field on (TM, \bar{g}) . Using the preceding arguments, we see that $([Z]^A)(x, 0) = (\partial [Z]^A / \partial x^B)(x, 0) = 0$, for $A = i, n+i$ and $B = j, n+j$ at a point $p = (x, 0)$.

Since a Killing vector field $[Z]$ (or more generally an infinitesimal affine transformation) is determined by the values of its components and their first partial derivatives at a point (cf. [5, p. 232]), we have $[Z] = 0$ on TM .

Finally, propositions 3.2, 3.4 and 3.5 and the facts that have already been shown complete the proof of Theorem 3.1.

4. Classification of Killing Vector Fields on (TM, \bar{g}) in Particular Cases

Let (M, g) be a compact oriented Riemannian manifold with volume element dv . We can define the global scalar product $\langle \alpha, \beta \rangle_g$ of two p -forms α and β as follows: $\langle \alpha, \beta \rangle_g = \int_M g(\alpha, \beta) dv$, where $g(\alpha, \beta)$ denotes the scalar product of α and β , with respect to g , locally defined by $g(\alpha, \beta) = \alpha_{i_1 \dots i_p} \beta^{i_1 \dots i_p}$.

The norm associated to the global product $\langle \cdot, \cdot \rangle_g$ is denoted by $\| \cdot \|_{\langle \cdot, \cdot \rangle_g}$.

THEOREM 4.1. *In theorem 3.1, if (M, g) is compact, then Y satisfying (Y_1) and (Y_2) is parallel. Hence (TK) is $Z = X^c + \iota P + Y_A$.*

PROOF. By considering the double covering manifold if necessary, we can assume that M is orientable. Let Y be a vector field on M satisfying (Y_1) . We can write, using (Y_1) , $\nabla_i \nabla^i Y_r = \nabla_i (g^{ij} \nabla_j Y_r) = g^{ij} \nabla_i \nabla_j Y_r = 0$. In this case, we have

$$(\alpha) \nabla_i (Y_r \nabla^i Y^r) = \nabla_i Y_r \nabla^i Y^r + Y_r \nabla_i \nabla^i Y^r = \nabla_i Y_r \nabla^i Y^r.$$

If we denote by Y^* the 1-form on M obtained by contraction of Y with respect to g , then the quantity $Y_r \nabla^i Y^r$ is the local expression of the vector field V obtained by contraction of the $(2, 1)$ -tensor field $Y^* \otimes \nabla_* Y$; $\nabla_* Y$ being defined locally by, $(\nabla_* Y)^{ij} = \nabla^i Y^j$.

On the other hand, the quantity $\nabla_i Y_r \nabla^i Y^r$ is nothing but $g(\nabla Y^*, \nabla Y^*)$. The equality (α) is then transformed to $\nabla_i V^i = (\nabla Y^*)_{ir} (\nabla Y^*)^{ir}$, that is

$$(\beta) \operatorname{div} V = g(\nabla Y^*, \nabla Y^*).$$

Integrating both sides of the last equation (β) , we get

$$(\gamma) \int_M \operatorname{div} V dv = \|\nabla Y^*\|_{\langle \cdot, \cdot \rangle_g}^2.$$

Applying Green's theorem ($\int_M \operatorname{div} V dv = 0$ [5], I.p. 281), it follows from (γ) that $\|\nabla Y^*\|_{\langle \cdot, \cdot \rangle_g}^2 = 0$.

Thus $\nabla Y^* = 0$ i.e. $\nabla Y = 0$. Therefore $Y_B = Y_A$, which gives the result.

THEOREM 4.2. *In theorem 3.1, if (M, g) satisfies one of the following three conditions:*

- (i) M is compact and orientable with vanishing first Betti number $b_1(M)$;
- (ii) M is compact and orientable with strictly positive Ricci Curvature;
- (iii) (M, g) is an Einstein space with non-zero scalar curvature different from $2n(1 - n)$, then Y satisfying (Y_1) and (Y_2) vanishes, in which case (TK) becomes $Z = X^c + \iota P$.

PROOF. (i) M being compact, Y satisfying (Y_1) and (Y_2) is parallel, by theorem 4.1. Applying Hodge's theorem ([13]), we have $Y = 0$ by virtue of $b_1(M) = 0$.

- (ii) According to [2, theorem 1.76], $b_1(M) = 0$. So, we can apply (i).
- (iii) contracting (Y_2) with respect to i and s , we obtain

$$R_{sy}\nabla^y Y_t + R_{ty}\nabla^y Y_s = 2(\nabla_t Y_s + \nabla_s Y_t - 2\nabla^r Y_r g_{st}),$$

where (R_{ij}) denotes the Ricci curvature tensor.

By assumption, (M, g) satisfies: $R_{sy} = S/ng_{sy}$, for non-zero scalar curvature S . Hence, we have

$$(S/n - 2)(\nabla_s Y_t + \nabla_t Y_s) = -4\nabla^r Y_r g_{st}.$$

Multiplying by g^{st} both sides of the last equation and summing over s and t , we get $(2n^2 - 2n + S)\nabla^s Y_s = 0$.

Since $S \neq 2n(1 - n)$, we have $\nabla^s Y_s = 0$, which means that $\nabla_r Y^r = 0$. Now, using (Y_1) and the Ricci identity,

$$\nabla_i \nabla_j Y^r - \nabla_j \nabla_i Y^r = R_{tji}^r Y^t,$$

we get

$$\nabla_i \nabla_j Y^r = \frac{1}{2} R_{tji}^r Y^t.$$

Contracting with respect to r and j , and applying $\nabla_r Y^r = 0$, we get $R_{ti} Y^t = 0$, that is $S/ng_{ti} Y^t = S/n Y_i = 0$. Therefore $Y = 0$, since $S \neq 0$.

THEOREM 4.3. *Let (TM, \bar{g}) be the tangent bundle with the Cheeger-Gromoll metric \bar{g} of a Riemannian manifold (M, g) . If M is compact and orientable, and if the first and the second Betti numbers vanish, then the Lie algebra of Killing vector fields on (M, g) and the Lie algebra of Killing vector fields on (TM, \bar{g}) are isomorphic, via the correspondence $X \rightarrow X^c$.*

PROOF. Applying theorem 4.2(i), it follows from $b_1(M) = 0$ that $Y = 0$. Furthermore, if the $(1, 1)$ -tensor field P is parallel then, by $b_2(M) = 0$ and Hodge's theorem, we have $P = 0$. Therefore, every Killing vector field on (TM, \bar{g}) is of the form X^c for some Killing vector field X on (M, g) . On the other hand, for any vector fields X and X' on M , it is known that $[X, X']^c = [X^c, X'^c]$.

REMARK 4.4. Let (M, g) be a compact orientable Riemannian manifold whose first and second Betti numbers vanish. The isomorphism stated in theorem 4.3 is also true in the case of the Sasaki metric [12, Theorem D]. Generally, one can consider the following interesting questions: Consider on TM a Riemannian metric G so that:

- (i) The projection $\pi : (TM, G) \rightarrow (M, g)$ is a Riemannian submersion;
 - (ii) vertical and horizontal distributions are orthogonal,
1. Is the complete lift of any Killing vector field on (M, g) a Killing vector field on (TM, G) ?
 2. If yes, are the Lie algebras of Killing vector fields of (M, g) and (TM, G) isomorphic with the correspondence $X \rightarrow X^c$?

5. The Non-Rigidity of the Cheeger-Gromoll Metric

Concerning the rigidity of the Cheeger-Gromoll metric, M. Sekizawa had shown in [10, p. 417], the following result

THEOREM 5.1 [10]. *The scalar curvature of the tangent bundle (TM, \bar{g}) with the Cheeger-Gromoll metric \bar{g} of a Riemannian manifold of constant sectional curvature (M, g) is never constant. In particular, (TM, \bar{g}) is never a space of constant sectional curvature if (M, g) is of constant sectional curvature.*

Our aim in this section is to generalize the particular situation of the previous theorem. We begin with a classical result concerning the characterization of spaces of constant sectional curvature in terms of dimension of the Lie algebra of Killing vector fields. Such a result is well known in literature (See for example [5, I. p. 238]).

THEOREM 5.2 [5, I]. *The Lie algebra $i(M)$ of infinitesimal isometries of a connected Riemannian manifold M is of dimension at most $n(n+1)/2$, where $n = \dim M$. If $\dim i(M) = n(n+1)/2$, then M is a space of constant curvature.*

Now, we consider three subsets of $i(TM)$

$$i_c(TM) = \{X^c \mid X \in i(M)\},$$

$$i_i(TM) = \{iP \mid P \text{ is a } (1,1)\text{-tensor field on } M \text{ satisfying } (P_1) \text{ and } (P_2)\} \text{ and}$$

$$i_B(TM) = \{Y_B \mid Y \text{ is a vector field on } M \text{ satisfying } (Y_1) \text{ and } (Y_2)\}.$$

PROPOSITION 5.3.

1. $i_c(TM)$, $i_i(TM)$ and $i_B(TM)$ are \mathbf{R} -vector subspaces of $i(TM)$. Furthermore, $\dim i_c(TM) \leq (1/2)n(n+1)$, $\dim i_i(TM) \leq (1/2)n(n-1)$, and $\dim i_B(TM) \leq n^2 + n$.

2. $i(TM) = i_c(TM) \oplus i_i(TM) \oplus i_B(TM)$.

3. $\dim i(TM)$ is maximal if and only if the dimensions of $i_c(TM)$, $i_i(TM)$ and $i_B(TM)$ are maximal equal, respectively, to $(1/2)n(n+1)$, $(1/2)n(n-1)$ and n^2+n .

PROOF. 1. Using Theorem 5.2, it is clear that $\dim i_c(TM) \leq (1/2)n(n+1)$, since $\dim i(M) \leq (1/2)n(n+1)$. On the other hand, since elements of $i_i(TM)$ raise from skew-symmetric parallel $(1,1)$ -tensor fields, we see that $\dim i_i(TM) \leq (1/2)n(n-1)$. Finally, condition (Y_1) on elements of $i_B(TM)$ let us assert that $\dim i_B(TM) \leq n^2+n$. 2. Follows immediately from Theorem 3.1. 3. Follows from 1 and 2 [Remark that $(1/2)n(n+1) + (1/2)n(n-1) + n^2+n = n(2n+1)$, which is the maximal dimension of $i(TM)$].

Now, if (M, g) is of constant sectional curvature which is assumed to be different from -2 and 0 , then we can easily verify that (M, g) is an Einstein space with scalar curvature different from $2n(1-n)$ and 0 ; hence $i_B(TM)$ is reduced to the 0-section of TM by virtue of Theorem 4.2.

Applying 3 of Proposition 5.3, we have $\dim i(TM) < n(2n+1)$. This implies, by Theorem 5.2, that (TM, \bar{g}) is of non constant sectional curvature.

REMARK 5.4. The additional condition on the constant sectional curvature of (M, g) is not a geometrical constraint. It is nothing but a technical condition which comes from the analysis of equation (Y_1) in Theorem 4.2. In fact, by Theorem 5.1, we can say that (TM, \bar{g}) is never a space of constant sectional curvature if (M, g) is a space of constant sectional curvature.

On the other hand, if we suppose that (M, g) is a space of non constant sectional curvature, then by Theorem 5.2, $\dim i(M) < (1/2)n(n+1)$ and so $\dim i_c(TM) < (1/2)n(n+1)$. Therefore, $\dim i(TM) < n(2n+1)$, by 3 of Proposition 5.3.

Applying again Theorem 5.2 to the case of TM , we see that (TM, \bar{g}) is a space of non constant sectional curvature. We have then proved the following

THEOREM 5.5. *The sectional curvature of the tangent bundle (TM, \bar{g}) with the Cheeger-Gromoll metric \bar{g} of a Riemannian manifold (M, g) is never constant.*

REMARK 5.6. Using the same arguments as in the proof of [12, Theorem E], one can state a similar result for the Cheeger-Gromoll metric. i.e., (M, g) is of constant curvature $k > 1$ if and only if the restriction of the geodesic flow vector field to T_rM ($r = 1/\sqrt{k} - 1$), is a Killing vector field with respect to the induced metric of (TM, \bar{g}) .

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