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A SPLITTING THEOREM FOR CAT(0) SPACES WITH THE GEODESIC EXTENSION PROPERTY

By

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Abstract. In this paper, we show the following splitting theorem: For a proper CAT(0) space X with the geodesic extension property, if a group $\Gamma = G_1 \times G_2$ acts geometrically (i.e., properly discontinuously and cocompactly by isometries) on X, then X splits as a product $X_1 \times X_2$ and there exist geometric actions of G_1 and some subgroup of finite index in G_2 on X_1 and X_2, respectively.

1. Introduction and Preliminaries

The purpose of this paper is to study CAT(0) spaces. We say that a metric space (X, d) is a *geodesic space* if for each $x, y \in X$, there exists an isometry $\xi : [0, d(x, y)] \to X$ such that $\xi(0) = x$ and $\xi(d(x, y)) = y$ (such ξ is called a *geodesic*). Let (X, d) be a geodesic space and let T be a geodesic triangle in X. A *comparison triangle* for T is a geodesic triangle T' in the Euclidean plane $\mathbf{R}^2$ with same edge lengths as T. Choose two points x and y in T. Let x' and y' denote the corresponding points in T'. Then the inequality

$$d(x, y) \le d_{\mathbf{R}^2}(x', y')$$

is called the *CAT(0)-inequality*, where $d_{\mathbf{R}^2}$ is the natural metric on $\mathbf{R}^2$. A geodesic space (X, d) is called a *CAT(0) space* if the CAT(0)-inequality holds for all geodesic triangles T and for all choices of two points x and y in T. A CAT(0) space X is said to have the *geodesic extension property* if every geodesic can be extended to a geodesic line $\mathbf{R} \to X$.

A metric space X is said to be *proper*, if every closed metric ball in X is compact. A subset M of a metric space X is *quasi-dense* if there exists a number $N > 0$ such that each point of X is N-close to some point of M.

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The minimal set $\mathrm{Min}(\gamma)$ of an isometry γ is defined as follows: Let X be a metric space and let γ be an isometry of X. Then the *translation length* of γ is defined as $|\gamma| = \inf\{d(x, \gamma x) \mid x \in X\}$, and the *minimal set* of γ is defined as $\mathrm{Min}(\gamma) = \{x \in X \mid d(x, \gamma x) = |\gamma|\}$. If Γ is a group acting by isometries on X, then $\mathrm{Min}(\Gamma) := \bigcap_{\gamma \in \Gamma} \mathrm{Min}(\gamma)$.

P. L. Bowers and K. Ruane proved the following theorem in [1].

Theorem 1.1 ([1, Proposition 1.1], [2, Theorem II.7.1]). *Let G be a group and let A be a free abelian group of rank n. Suppose that $\Gamma = G \times A$ acts geometrically on a proper $CAT(0)$ space X. Then $\mathrm{Min}(A) = \bigcap_{\alpha \in A} \mathrm{Min}(\alpha)$ is a closed, Γ-invariant, convex and quasi-dense subset of X that splits as a product $Y \times \mathbf{R}^n$, and there exist geometric actions of G and A on Y and $\mathbf{R}^n$, respectively. Moreover if X has the geodesic extension property, then $X = \mathrm{Min}(A) = Y \times \mathbf{R}^n$.*

The last sentence of the above theorem is obtained from the following lemma.

Lemma 1.2 ([2, Lemma II.6.16]). *Let X be a complete $CAT(0)$ space with the geodesic extension property and let α be an isometry of X. If there exists a group Γ which acts cocompactly by isometries on X such that α commutes with Γ, then $\mathrm{Min}(\alpha) = X$.*

Also the following splitting theorem is known.

Theorem 1.3 ([2, Proposition II.6.23, Lemma II.6.24]). *Suppose that a group $\Gamma = G_1 \times G_2$ acts geometrically on a proper $CAT(0)$ space X with the geodesic extension property. If G_2 has the finite center, then X splits as a product $X_1 \times X_2$, the subspaces of the form $X_1 \times \{x_2\}$ are the closed convex hulls of the G_1-orbits, G_1 acts geometrically on X_1, and there exists a geometric action of G_2 on X_2.*

In this paper, using Theorems 1.1 and 1.3, we prove the following splitting theorems as an extension.

Theorem A. *Suppose that a group G acts geometrically (i.e., properly discontinuously and cocompactly by isometries) on a proper $CAT(0)$ space X. Then there exist subgroups $G', A \subset G$ such that*

(1) *$G' \times A$ is a subgroup of finite index in G,*

(2) *G' has the finite center, and*

(3) *A is isomorphic to $\mathbf{Z}^n$ for some n,*

and there exist convex subspaces $X', Z \subset X$ such that

(1) *$X' \times Z$ is a quasi-dense subspace of X,*
(2) *there exists a geometric action of G' on X', and*
(3) *Z is isometric to $\mathbf{R}^n$.*
Moreover if X has the geodesic extension property, then $X = X' \times Z$.

THEOREM B. *Suppose that a group $\Gamma = G_1 \times G_2$ acts geometrically on a proper $CAT(0)$ space X with the geodesic extension property. Then X splits as a product $X_1 \times X_2$ and there exist geometric actions of G_1 and some subgroup of finite index in G_2 on X_1 and X_2, respectively.*

2. Proof of Main Theorems

To prove Main Theorems, the following lemma plays a key role.

LEMMA 2.1. *Suppose that a group $\Gamma = G \times H$ acts geometrically on a proper $CAT(0)$ space X. Then there exist subgroups $G', A \subset G$ such that*
(1) *$G' \times A$ is a subgroup of finite index in G,*
(2) *G' has the finite center, and*
(3) *A is isomorphic to $\mathbf{Z}^n$ for some n.*

We first recall some properties of CAT(0) spaces.

DEFINITION 2.2. An isometry γ of a metric space X is called *semi-simple* if $\mathrm{Min}(\gamma)$ is nonempty.

The following results are known.

LEMMA 2.3 ([2, Proposition II.6.10 (2)]). *Suppose that a group Γ acts geometrically on a proper metric space X. Then every element of Γ is a semi-simple isometry of X.*

LEMMA 2.4 ([2, p. 439, Theorem 1.1 (1), (4)]). *If a group Γ acts geometrically on a proper $CAT(0)$ space, then*
(1) *Γ is finitely presented, and*
(2) *every abelian subgroup of Γ is finitely generated.*

LEMMA 2.5 ([2, p. 439, Theorem 1.1 (iv)]). *Let G be a finitely generated group that acts properly discontinuously by semi-simple isometries on a proper $CAT(0)$ space X. If $A \cong \mathbf{Z}^n$ is central in G, then there exists a subgroup of finite index in G that contains A as a direct factor.*

Using lemmas above, we prove Lemma 2.1.

Proof of Lemma 2.1. Since the center $C(G)$ of G is an abelian subgroup of Γ, $C(G)$ is finitely generated by Lemma 2.4 (2). If the center $C(G)$ is finite, then $G' := G$ and $A := 0$ satisfy the three conditions of this lemma. Suppose that $C(G)$ is infinite.

Let A_1 be the free abelian subgroup of $C(G)$ such that $C(G) = A_1 \times B_1$, where B_1 is the torsion subgroup of $C(G)$. Since $G \times H$ is finitely presented by Lemma 2.4 (1), G is finitely generated. By Lemma 2.3, G acts properly discontinuously by semi-simple isometries on X. Hence, by Lemma 2.5, there exists a subgroup $G_1 \subset G$ such that $G_1 \times A_1$ is a subgroup of finite index in G. If G_1 has the finite center, then $G' := G_1$ and $A := A_1$ satisfy the three conditions of this lemma. Suppose that the center $C(G_1)$ of G_1 is infinite.

Let A_2 be the free abelian subgroup of $C(G_1)$ such that $C(G_1) = A_2 \times B_2$, where B_2 is the torsion subgroup of $C(G_1)$. Since $[G \times H : G_1 \times A_1 \times H] < \infty$, $G_1 \times A_1 \times H$ acts geometrically on X. Hence $G_1 \times A_1 \times H$ is finitely generated by Lemma 2.4 (1), i.e., G_1 is finitely generated. By Lemma 2.3, G_1 acts properly discontinuously by semi-simple isometries on X. By Lemma 2.5, there exists a subgroup $G_2 \subset G_1$ such that $G_2 \times A_2$ is a subgroup of finite index in G_1. Then $G_2 \times A_2 \times A_1 \times H$ acts geometrically on X.

By the same argument, we have a sequence

$$G \supset G_1 \times A_1 \supset G_2 \times A_2 \times A_1 \supset \cdots \supset G_m \times (A_m \times \cdots \times A_1),$$

where each index is finite and $A_i \cong \mathbf{Z}^{n_i}$ for some $n_i \geq 1$. By Lemma 2.4 (2), this is a finite sequence, i.e., G_m has the finite center for some m. Then $G' := G_m$ and $A := A_1 \times \cdots \times A_m$ satisfy the three conditions of this lemma. □

We obtain Theorem A from Lemma 2.1 and Theorem 1.1.

Proof of Theorem A. Since $G \times 0$ acts geometrically on X, by Lemma 2.1, there exist subgroups $G', A \subset G$ such that

(1) $G' \times A$ is a subgroup of finite index in G,

(2) G' has the finite center, and

(3) $A \cong \mathbf{Z}^n$ for some n.

Since $[G : G' \times A] < \infty$, $G' \times A$ acts geometrically on X. By Theorem 1.1, $\mathrm{Min}(A)$ is closed, $(G' \times A)$-invariant, convex and quasi-dense subset of X, it splits as a product $X' \times \mathbf{R}^n$, and there exists a geometric action of G' on X'. Moreover, if X has the geodesic extension property, then $X = \mathrm{Min}(A) = X' \times \mathbf{R}^n$. □

Using Lemma 2.1 and Theorems 1.1 and 1.3, we prove Theorem B.

Proof of Theorem B. By Lemma 2.1, there exist subgroups $G_2', A_2 \subset G_2$ such that

(1) $G_2' \times A_2$ is a subgroup of finite index in G_2,

(2) G_2' has the finite center, and

(3) $A_2 \cong \mathbf{Z}^n$ for some n.

Then $G_1 \times G_2' \times A_2$ acts geometrically on X because $[G_1 \times G_2 : G_1 \times G_2' \times A_2] < \infty$. Since $A_2 \cong \mathbf{Z}^n$, by Theorem 1.1, $X = \mathrm{Min}(A_2)$ splits as a product $Y \times Z$, where $Z \cong \mathbf{R}^n$, and there exist geometric actions of $G_1 \times G_2'$ and A_2 on Y and Z, respectively. Since G_2' has the finite center, by Theorem 1.3, Y splits as a product $X_1 \times Y'$ and there exist geometric actions of G_1 and G_2' on X_1 and Y', respectively. Therefore

$$X = Y \times Z = (X_1 \times Y') \times Z = X_1 \times (Y' \times Z),$$

and $G_2' \times A_2$ acts geometrically on $Y' \times Z$ by product. Here $G_2' \times A_2$ is a subgroup of finite index in G_2. $\square$

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