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RAPID DECAY OF THE TOTAL ENERGY FOR DISSIPATIVE WAVE EQUATIONS

By

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1. Introduction

In this paper, we are concerned with the dissipative wave equation of the form:

$$\begin{cases} w_{tt} - \Delta w + b(t,x)w_t + c(t,x)w = 0 & (t,x) \in (0,\infty) \times \mathbf{R}^N, \\ w(0,x) = w_0(x),\ w_t(0,x) = w_1(x) & x \in \mathbf{R}^N, \end{cases} \tag{1.1}$$

where $w = w(t,x)$, $\Delta = \sum_{j=1}^{N}(\partial^2/\partial x^2)$, $b(t,x)$ and $c(t,x)$ are some non-negative continuous functions.

In Saeki and Ikehata [12] the authors obtained the following decay estimate for (1.1) with $c(t,x) \equiv 0$: *Suppose that* $N \geq 3$, $b(t,x) = b(x) \geq b_0$ *in* $\mathbf{R}^N$ *for some positive constant* b_0. *If* $(w_0, w_1) \in H^1 \cap L^{2,1} \times L^{2,1}$, *then the estimate*

$$(1+t)^2 \|w(t)\|_E^2 \leq C_1 \|(w_0,w_1)\|_{H^1 \cap L^{2,1} \times L^{2,1}}^2$$

holds with a positive constant C_1, *where*

$$L^{2,1} = \left\{ f \mid \|f\|_{L^{2,1}}^2 = \int_{\mathbf{R}^N} (1+|x|^2)|f(x)|^2\,dx < \infty \right\},$$

$\|w(t)\|_E^2$ *is the total energy at time* t:

$$\|w(t)\|_E^2 = \frac{1}{2}\int_{\mathbf{R}^N} \{|\nabla w(t,x)|^2 + w_t(t,x)^2\}\,dx$$

and

$$\|(w_0,w_1)\|_{H^1 \cap L^{2,1} \times L^{2,1}}^2 \equiv \|w_0\|_{H^1}^2 + \|w_0\|_{L^{2,1}}^2 + \|w_1\|_{L^{2,1}}^2.$$

Our aim is to derive more rapid decay estimate for the total energy without the assumption for the data as above. We consider our problem under the following

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two kinds of assumptions on the dimension number, initial data and coefficients, that is,

$$\begin{cases} N \geq 1,\ (w_0, w_1) \in H^1 \times L^2,\ b_0 \leq b(t,x) \leq b_1, \\ b_t(t,x) \leq 0 \ \text{ or } \ |b_t(t,x)| \leq \gamma_b(t), \\ c_t(t,x) \leq 0 \ \text{ or } \ |c_t(t,x)| \leq \gamma_c(t)c(t,x) \end{cases} \tag{1.2}$$

or

$$\begin{cases} N \geq 1,\ (w_0, w_1) \in H^1 \times L^2, \\ b_0(1+t)^{-\sigma} \leq b(t) \leq b_1(1+t)^{-\sigma}, \\ b_2(1+t)^{-\sigma-1} \leq -b_t(t) \leq b_3(1+t)^{-\sigma-1}, \\ b_0 \leq b_1 \leq \sigma^{-1}b_2 \leq \sigma^{-1}b_3, \\ c_t(t,x) \leq 0 \ \text{ or } \ |c_t(t,x)| \leq \gamma_c(t)c(t,x) \end{cases} \tag{1.3}$$

for some positive constants b_0, b_1, b_2, b_3 and $\sigma \leq 1$, and for some functions $\gamma_b(t)$, $\gamma_c(t) \in L^1((0,\infty))$. Then our main results are described as follows:

THEOREM 1.1. *Let w be a solution to the Cauchy problem* (1.1). *Assume* (1.2). *Then the following inequality holds*:

$$(1+t)\|w(t)\|_E^2 + \|w(t)\|_{L^2}^2 + \int_0^t \left\{ (1+\tau)\|w_t(\tau)\|_{L^2}^2 + \|\nabla w(\tau)\|_{L^2}^2 + \int_{\mathbf{R}^N} c(\tau,x)w(\tau,x)^2\,dx \right\} d\tau \leq C_2, \tag{1.4}$$

where C_2 depends only on $\|w_0\|_{H^1}$, $\|w_1\|_{L^2}$ and b_0, and

$$\|w(t)\|_E^2 = \frac{1}{2}\int_{\mathbf{R}^N} (|\nabla w(t,x)|^2 + w_t(t,x)^2 + c(t,x)w(t,x)^2)\,dx$$

is the total energy at t. Moreover, we have

$$\lim_{t\to+\infty}(1+t)\|w(t)\|_E^2 = 0. \tag{1.5}$$

Particularly, if $b(t,x) = b(t)$ behaves like $b_0(1+t)^{-\sigma}$ with $b_0 > 2$ and $\sigma \in (1/2, 1]$, then more rapid energy decay estimate holds:

THEOREM 1.2. *Assume* (1.3) *with $b_0 > 2$ and $\sigma \in (1/2, 1]$. Then the following inequality holds*:

$$(1+t)^{\sigma+1}\|w(t)\|_E^2 + \|w(t)\|_{L^2}^2 + \int_0^t \left\{ (1+\tau)\|w_t(\tau)\|_{L^2}^2 + (1+\tau)^\sigma\|\nabla w(\tau)\|_{L^2}^2 + \chi(\sigma)(1+\tau)^{-1}\|w(\tau)\|_{L^2}^2 + (1+\tau)^\sigma\int_{\mathbf{R}^N} c(\tau,x)w(\tau,x)^2\,dx \right\} d\tau \leq C_3, \tag{1.6}$$

where $\chi(\sigma) = 1$ *(if* $\sigma = 1$*),* $= 0$ *(if* $\sigma \neq 1$*), and* C_3 *depends on* $\|w_0\|_{H^1}$, $\|w_1\|_{L^2}$ *and* $b(t,x)$. *Moreover, we have*

$$\lim_{t\to+\infty} (1+t)^{\sigma+1}\|w(t)\|_E^2 = 0. \tag{1.7}$$

In particular, if $\sigma = 1$, *then we find*

$$\lim_{t\to+\infty} \|w(t)\|_{L^2}^2 = 0. \tag{1.8}$$

The proofs of these theorems are done by the weighted energy method, which were used in [1], [3], [5], [7], [8], [10], [13] and [14], for instance. Among them, in [13] it was proved that the total energy $\|w(t)\|_E^2$ decays like $O(t^{-\mu})$, where μ is a positive real number satisfying $\mu \le (1+t)b(t,x)$ and $(\mu-1)(\mu-2) - (\mu-1)(1+t)b(t,x) - (1+t)^2 b_t(t,x) \ge 0$. This shows that one can take μ at most $\mu = 2$. Moreover, in [14], it is obtained that $C_1(1+t)^{-\mu_1} \le \|w(t)\|_E^2 \le C_2(1+t)^{-\mu_2}$ for some positive constants C_1 and C_2, where $\mu_2 = \min\{b_0, 2\}$, $\mu_1 = 2b_0$. Recently the latter of the present authors [10] generalized the result of [12] to $o(t^{-2})$.

We mention other results on (1.1) with $c(t,x) \equiv 0$. Energy non-decay and scattering problem is considered in [4], [5], [6], [7], [9] and [11]. In [2] it has been proved that if the dissipation is of spatial anisotropy, then (1.1) does not have uniform decay property.

The content of the present paper is organized as follows. In Section 2, we give the proof of Theorem 1.1. Theorem 1.2 is shown in Section 3. In Section 4, we state related results without proofs.

2. Proof of Theorem 1.1

Let $\varphi = \varphi(t)$ and $\psi = \psi(t)$ be non-negative smooth functions, to be chosen later. Multiplying $\varphi w_t + \psi w$ by the equation of (1.1) and integrating it over $\mathbf{R}^N$, we have

$$\frac{d}{dt}\left(\int_{\mathbf{R}^N} X(t,x)\,dx\right) + \int_{\mathbf{R}^N} Z(t,x)\,dx = 0, \tag{2.9}$$

where

$$X(t,x) = \frac{\varphi}{2}(w_t^2 + |\nabla w|^2 + cw^2) + \psi w_t w + \frac{b\psi - \psi_t}{2} w^2 \tag{2.10}$$

and

$$Z(t,x) = \frac{2b\varphi - \varphi_t - 2\psi}{2} w_t^2 + \frac{2\psi - \varphi_t}{2} |\nabla w|^2 + \frac{1}{2}\{\psi_{tt} - (b\psi)_t + 2\psi c - (\varphi c)_t\} w^2. \tag{2.11}$$

Let us take

$$\varphi(t) = \frac{5}{2} b_0 + b_0^2 t \quad \text{and} \quad \psi(t) = b_0^2. \tag{2.12}$$

(i) Estimate of $Z(t,x)$. Using the condition on $b(t,x)$ in (1.2), we easily obtain

$$2b\varphi - \varphi_t - 2\psi \ge C_4(b_0)(1+t) \quad \text{and} \quad 2\psi - \varphi_t = b_0^2,$$

where $C_4(b_0) = 2b_0^2 \min\{1, b_0\}$. Noting the condition on b_t in (1.2), we have

$$\psi_{tt} - (b\psi)_t \ge \begin{cases} 0 & \text{if } b_t \le 0, \\ -b_0^2 \gamma_b(t) & \text{if } |b_t| \le \gamma_b. \end{cases}$$

Moreover since $2\psi c - (\varphi c)_t = b_0^2 c - \varphi c_t$, using the condition on $c(t,x)$ in (1.2), we find

$$2\psi c - (\varphi c)_t \ge \begin{cases} b_0^2 c & \text{if } c_t \le 0, \\ b_0^2 c - C_5(b_0)(1+t)\gamma_c(t) c & \text{if } |c_t| \le \gamma_c c, \end{cases}$$

where we put $C_5(b_0) = (b_0/2)\min\{5, 2b_0\}$. It then follows from the arguments above that

$$\int_{\mathbf{R}^N} Z(t,x)\,dx \ge C_4(b_0)(1+t)\frac{\|w_t(t)\|_{L^2}^2}{2} + b_0^2 \frac{\|\nabla w(t)\|_{L^2}^2}{2} + \frac{b_0^2}{2}\int_{\mathbf{R}^N} c(t,x) w(t,x)^2\,dx - \frac{1}{2} W_1(t), \tag{2.13}$$

where

$$W_1(t) = \begin{cases} 0 & \text{if } b_t \le 0 \text{ and } c_t \le 0, \\ b_0^2 \gamma_b(t) \|w(t)\|_{L^2}^2 & \text{if } |b_t| \le \gamma_b \text{ and } c_t \le 0, \\ C_5(b_0)(1+t)\gamma_c(t) \int_{\mathbf{R}^N} c(t,x) w(t,x)^2\,dx & \text{if } b_t \le 0 \text{ and } |c_t| \le \gamma_c c, \\ b_0^2 \gamma_b(t) \|w(t)\|_{L^2}^2 & \\ \quad + C_5(b_0)(1+t)\gamma_c(t) \int_{\mathbf{R}^N} c(t,x) w(t,x)^2\,dx & \text{if } |b_t| \le \gamma_b \text{ and } |c_t| \le \gamma_c c. \end{cases}$$

(ii) Estimate of $X(t,x)$. Put

$$A = \frac{\varphi}{2}, \quad B = \psi, \quad C = \frac{b\psi - \psi_t}{2},$$

and for the convenience of proving

$$X_1 = X_2 = \frac{1}{2}(Aw_t^2 + Bw_tw + Cw^2).$$

Then we have

$$X(t,x) = \frac{\varphi}{2}(|\nabla w|^2 + cw^2) + X_1 + X_2. \tag{2.14}$$

Noting $A, B, C > 0$ by (2.12), we find

$$2X_1 = C\left(w + \frac{B}{2C}w_t\right)^2 + \left(A - \frac{B^2}{4C}\right)w_t^2 \geq \left(A - \frac{B^2}{4C}\right)w_t^2 \geq C_6(b_0)(1+t)w_t^2$$

with $C_6(b_0) = (b_0/4)\min\{3, 2b_0\}$. Similar argument is applicable for X_2 to conclude

$$2X_2 \geq \left(C - \frac{B^2}{4A}\right)w^2 \quad \text{with } C - \frac{B^2}{4A} \geq \frac{3b_0^3}{10} > 0.$$

From the arguments above and the estimate $A \geq (1/2)(A - B^2/4C)$, it then follows that

$$\begin{aligned} X(t,x) &\geq \frac{1}{2}\left(A - \frac{B^2}{4C}\right)(w_t^2 + |\nabla w|^2 + cw^2) + \frac{1}{2}\left(C - \frac{B^2}{4A}\right)w^2 \\ &\geq C_6(b_0)(1+t)\frac{1}{2}(w_t^2 + |\nabla w|^2 + cw^2) + \frac{3b_0^3}{20}w^2. \end{aligned} \tag{2.15}$$

On the other hand, we can easily obtain

$$\int_{\mathbf{R}^N} X(0,x) \leq C_7(\|w(0)\|_E^2 + \|w_0\|_{L^2}^2) \equiv C_8. \tag{2.16}$$

(iii) Derivation of (1.4). Integrating (2.9) on $[0,t]$ and using (2.13), (2.15) and (2.16), we find

$$\begin{aligned} &C_6(b_0)(1+t)\|w(t)\|_E^2 + \frac{3b_0^3}{20}\|w(t)\|_{L^2}^2 \\ &\quad + \int_0^t \left\{ C_4(b_0)(1+\tau)\frac{\|w_t(\tau)\|_{L^2}^2}{2} + b_0^2\frac{\|\nabla w(\tau)\|_{L^2}^2}{2} + \frac{b_0^2}{2}\int_{\mathbf{R}^N} c(\tau,x)w(\tau,x)^2\,dx \right\} d\tau \\ &\leq C_8 + \frac{1}{2}\int_0^t W_1(\tau)\,d\tau. \end{aligned}$$

In the following, we give a proof of only the case of $|b_t| \le \gamma_b$ and $|c_t| \le \gamma_c c$ since we can estimate the other cases more easily. Put

$$C_9(b_0) = \min\left\{C_6(b_0), \frac{3b_0^3}{20}, \frac{C_4(b_0)}{2}, \frac{b_0^2}{2}\right\},$$

$$C_{10}(b_0) = \max\left\{C_8, \frac{b_0^2}{2}, \frac{C_5(b_0)}{2}\right\} \quad \text{and} \quad C_{11}(b_0) = \frac{C_{10}(b_0)}{C_9(b_0)}.$$

Then the inequality above has the following form:

$$\begin{aligned}(1+t)\|w(t)\|_E^2 &+ \|w(t)\|_{L^2}^2 \\ &+ \int_0^t \left\{(1+\tau)\|w_t(\tau)\|_{L^2}^2 + \|\nabla w(\tau)\|_{L^2}^2 + \int_{\mathbf{R}^N} c(\tau,x)w(\tau,x)^2\,dx\right\}d\tau \\ &\le F(t),\end{aligned} \tag{2.17}$$

where

$$F(t) = C_{11}(b_0)\left\{1 + \int_0^t \left(\gamma_b(\tau)\|w(\tau)\|_{L^2}^2 + (1+\tau)\gamma_c(\tau)\int_{\mathbf{R}^N} c(\tau,x)w(\tau,x)^2\,dx\right)d\tau\right\}.$$

From (2.17), we especially find

$$(1+t)\int_{\mathbf{R}^N} c(t,x)w(t,x)^2\,dx \le 2F(t) \tag{2.18}$$

and

$$\|w(t)\|_{L^2}^2 \le F(t). \tag{2.19}$$

Differentiating $F(t)$ defined above, and using (2.18) and (2.19), we have

$$F'(t) \le C_{11}(b_0)(\gamma_b(t) + 2\gamma_c(t))F(t).$$

From this we set

$$F(t) \le C_{12} \equiv C_{11}(b_0)e^{C_{13}} < \infty$$

with

$$C_{13} \equiv C_{11}(b_0)\int_0^\infty (\gamma_b(\tau) + 2\gamma_c(\tau))\,d\tau.$$

This and (2.17) give (1.4).

(iv) Derivation of (1.5). It follows from (1.4) that $\|w(t)\|_E^2 \in L^1([0,\infty))$, therefore we find

$$\liminf_{t\to\infty}(1+t)\|w(t)\|_E^2 = 0.$$

So, we have only to show that $\{(1+t)\|w(t)\|_E^2\}_{t\geq 0}$ is a Cauchy sequence with respect to t. For this aim, we consider the following identity:

$$\frac{d}{dt}\{(1+t)\|w(t)\|_E^2\} = \|w(t)\|_E^2 + (1+t)\frac{d}{dt}\|w(t)\|_E^2.$$

Using the energy identity

$$\frac{d}{dt}\|w(t)\|_E^2 + \int_{\mathbf{R}^N}\left(b(t,x)w_t(t,x)^2 - \frac{c_t(t,x)}{2}w(t,x)^2\right)dx = 0,$$

which is derived from (2.9) with $\varphi = 1$ and $\psi = 0$, we find

$$\begin{aligned}\frac{d}{dt}&\{(1+t)\|w(t)\|_E^2\}\\ &= \|w(t)\|_E^2 - (1+t)\int_{\mathbf{R}^N} b(t,x)w_t(t,x)^2\,dx + \frac{(1+t)}{2}\int_{\mathbf{R}^N} c_t(t,x)w(t,x)^2\,dx.\end{aligned}$$

Integrating the equation above on $t\in[t_1,t_2]$ $(0<t_1<t_2<\infty)$, we have

$$\begin{aligned}|(1+t_2)&\|w(t_2)\|_E^2 - (1+t_1)\|w(t_1)\|_E^2|\\ &\leq \int_{t_1}^{t_2}\left\{\|w(t)\|_E^2 + (1+t)\left(\int_{\mathbf{R}^N} b(t,x)w_t(t,x)^2 + \frac{|c_t(t,x)|}{2}w(t,x)^2\,dx\right)\right\}dt.\end{aligned} \tag{2.20}$$

Using (1.2), we find that the right hand side of (2.20) is estimated from above by

$$\int_{t_1}^{t_2}\left\{\|w(t)\|_E^2 + b_1(1+t)\|w_t(t)\|_{L^2}^2 + \frac{\gamma_c(t)(1+t)}{2}\int_{\mathbf{R}^N} c(t,x)w(t,x)^2\,dx\right\}dt. \tag{2.21}$$

Since

$$(1+t)\int_{\mathbf{R}^N} c(t,x)w(t,x)^2\,dx \leq 2C_2$$

by (1.4), it follows that

$$\frac{1}{2}\int_{t_1}^{t_2}\gamma_c(t)(1+t)\int_{\mathbf{R}^N} c(t,x)w(t,x)^2\,dxdt \leq C_2\int_{t_1}^{t_2}\gamma_c(t)\,dt. \tag{2.22}$$

Therefore we obtain

$$|(1+t_2)\|w(t_2)\|_E^2 - (1+t_1)\|w(t_1)\|_E^2|$$
$$\leq \int_{t_1}^{t_2} (\|w(t)\|_E^2 + b_1(1+t)\|w_t(t)\|_{L^2}^2 + C_2\gamma_c(t))\, dt,$$

to conclude (1.5).

3. Proof of Theorem 1.2

From the condition on b_0 and σ, we may assume $2b_0 > \sigma + 3$. So, we take $\varepsilon \in (0,1)$ as

$$0 < \varepsilon < 1 - \frac{2}{2b_0 - \sigma - 1}. \tag{3.23}$$

Put

$$\varphi(t) = b_0^2(1-\varepsilon)(1+t)^{\sigma+1}, \quad \psi(t) = b_0^2(1+t)^{\sigma}, \tag{3.24}$$

and consider (2.9)–(2.11).

(i) Estimate of $Z(t,x)$. Easy computations give

$$2b\varphi - \varphi_t - 2\psi \geq C_{14}(b_0,\sigma)(1+t) \tag{3.25}$$

and

$$2\psi - \varphi_t \geq C_{15}(b_0,\sigma)(1+t)^{\sigma}, \tag{3.26}$$

where

$$C_{14}(b_0,\sigma) = 2b_0^2(1-\varepsilon)\left(b_0 - \frac{\sigma+1}{2} - \frac{1}{1-\varepsilon}\right)$$

and

$$C_{15}(b_0,\sigma) = b_0^2\{2 - (\sigma+1)(1-\varepsilon)\}.$$

Note that

$$\psi_{tt} = \begin{cases} 0 & \text{if } \sigma = 1, \\ -\gamma_\psi(t) & \text{if } 0 < \sigma < 1, \end{cases} \quad \text{and} \quad -(b\psi)_t \geq \begin{cases} b_0^2 b_2 (1+t)^{-1} & \text{if } \sigma = 1, \\ 0 & \text{if } 0 < \sigma < 1, \end{cases}$$

where

$$\gamma_\psi(t) = b_0^2(1-\sigma)(1+t)^{\sigma-2} \in L^1(0,\infty).$$

So, we find

$$\psi_{tt} - (b\psi)_t \geq \begin{cases} b_0^2 b_1 (1+t)^{-1} & \text{if } \sigma = 1, \\ -\gamma_\psi(t) & \text{if } 0 < \sigma < 1. \end{cases} \tag{3.27}$$

By the use of the condition on $c(t,x)$ and (3.26), we obtain

$$2\psi c - (\varphi c)_t \geq C_{15}(b_0,\sigma)(1+t)^\sigma c - \begin{cases} 0 & \text{if } c_t \leq 0, \\ C_{16}(b_0)(1+t)^{\sigma+1}\gamma_c c & \text{if } |c_t| \leq \gamma_c c, \end{cases} \tag{3.28}$$

where we have used the estimate

$$-\varphi c_t \geq -C_{16}(b_0)(1+t)^{\sigma+1}\gamma_c c$$

with $C_{16}(b_0) = b_0^2(1-\varepsilon)$. It then follows from (3.25)–(3.28) that

$$\begin{aligned} \int_{\mathbf{R}^N} Z(t,x)\, dx &\geq C_{14}(b_0,\sigma)(1+t)\frac{\|w_t(t)\|_{L^2}^2}{2} \\ &\quad + C_{15}(b_0,\sigma)(1+t)^\sigma \left\{ \frac{\|\nabla w(t)\|_{L^2}^2}{2} + \frac{1}{2}\int_{\mathbf{R}^N} c(t,x) w(t,x)^2\, dx \right\} - \frac{1}{2} W_2(t), \end{aligned} \tag{3.29}$$

where

$$W_2(t) = \begin{cases} -b_0^2 b_1 (1+t)^{-1} \|w(t)\|_{L^2}^2 & \text{if } \sigma = 1 \text{ and } c_t \leq 0, \\ -b_0^2 b_1 (1+t)^{-1} \|w(t)\|_{L^2}^2 & \\ \quad + C_{16}(b_0)(1+t)^{\sigma+1}\gamma_c \int_{\mathbf{R}^N} c(t,x) w(t,x)^2\, dx & \text{if } \sigma = 1 \text{ and } |c_t| \leq \gamma_c c, \\ \gamma_\psi \|w(t)\|_{L^2}^2 & \text{if } 0 < \sigma < 1 \text{ and } c_t \leq 0, \\ \gamma_\psi \|w(t)\|_{L^2}^2 & \\ \quad + C_{16}(b_0)(1+t)^{\sigma+1}\gamma_c \int_{\mathbf{R}^N} c(t,x) w(t,x)^2\, dx & \text{if } 0 < \sigma < 1 \\ & \text{and } |c_t| \leq \gamma_c c. \end{cases}$$

(ii) Estimate of $X(t,x)$. We use the same notation as in Section 2 (ii). By (3.24), we find $A, B > 0$ and $C \geq b_0^2(b_0 - \sigma) > 0$. Therefore, it follows

$$A - \frac{B^2}{4C} \geq C_{17}(b_0,\sigma)(1+t)^{\sigma+1} \tag{3.30}$$

with $C_{17}(b_0,\sigma) = (1-\varepsilon)b_0^2(1 - 1/2(b_0-\sigma)) > 0$. Similar arguments give

$$C - \frac{B^2}{4A} \geq C_{18}(b_0,\sigma) \tag{3.31}$$

with $C_{18}(b_0,\sigma) = (b_0^2/4)(b_0 - \sigma - 1/(1-\varepsilon)) > 0$. It then follows from the arguments as in Section 2 (iii), (3.30) and (3.31) that

$$\int_{\mathbf{R}^N} X(t,x)\,dx \ge C_{17}(b_0,\sigma)(1+t)^{\sigma+1}\|w(t)\|_E^2 + C_{18}(b_0,\sigma)\|w(t)\|_{L^2}^2. \quad (3.32)$$

On the other hand, the following is easily obtained:

$$\int_{\mathbf{R}^N} |X(0,x)|\,dx \le C_{19}(\|w_0\|_{H^1}^2 + \|w_1\|_{L^2}^2) \equiv C_{20} \quad (3.33)$$

for a positive constant C_{19}.

(iii) Derivation of (1.6). Integrating (2.9) on $t \in [0,t]$ and using (3.29), (3.32) and (3.33) we have

$$\begin{aligned} &C_{17}(b_0,\sigma)(1+t)^{\sigma+1}\|w(t)\|_E^2 + C_{18}(b_0,\sigma)\|w(t)\|_{L^2}^2 + C_{21}(b_0,b_1,b_2,\sigma)\int_0^t \frac{\|w(\tau)\|_{L^2}^2}{1+\tau}\,d\tau \\ &\quad + \int_0^t \left\{ \frac{C_{14}(b_0,\sigma)}{2}(1+\tau)\|w_t(\tau)\|_{L^2}^2 \right. \\ &\qquad \left. + \frac{C_{15}(b_0,\sigma)}{2}(1+\tau)^\sigma\left(\|\nabla w(\tau)\|_{L^2}^2 + \int_{\mathbf{R}^N} c(\tau,x)w(\tau,x)^2\,dx\right)\right\} d\tau \\ &\le C_{20} + \frac{1}{2}\int_0^t \tilde{W}_2(\tau)\,d\tau, \end{aligned}$$

where

$$\tilde{W}_2(t) = \begin{cases} 0 & (\sigma = 1, c_t \le 0), \\ C_{16}(b_0)(1+t)^{\sigma+1}\gamma_c(t)\int_{\mathbf{R}^N} c(t,x)w(t,x)^2\,dx & \text{if } \sigma = 1 \text{ and } |c_t| \le \gamma_c c, \\ \gamma_\psi(t)\|w(t)\|_{L^2}^2 & \text{if } \frac{1}{2} < \sigma < 1 \text{ and } c_t \le 0, \\ \gamma_\psi(t)\|w(t)\|_{L^2}^2 & \\ \quad + C_{16}(b_0)(1+t)^{\sigma+1}\gamma_c(t)\int_{\mathbf{R}^N} c(t,x)w(t,x)^2\,dx, & \frac{1}{2} < \sigma < 1 \text{ and } |c_t| \le \gamma_c c \end{cases}$$

and

$$C_{20}(b_0,b_2) = \begin{cases} \frac{b_0^2 b_2}{2} & \text{if } \sigma = 1, \\ 0 & \text{if } 0 < \sigma < 1. \end{cases}$$

In the following, we consider only the case $1/2 < \sigma < 1$ and $|c_t| \le \gamma_c c$ since the other cases are more easily treated than this case. Put

$$C_{21} = \min\left\{\frac{C_{14}(b_0,\sigma)}{2}, \frac{C_{15}(b_0,\sigma)}{2}, C_{17}(b_0,\sigma), C_{18}(b_0,\sigma), C_{21}(b_0,b_2)\right\},$$

$$C_{22} = \max\left\{\frac{C_{16}(b_0)}{2}, C_{20}\right\} \quad \text{and} \quad C_{23} = \frac{C_{22}}{C_{21}}.$$

It then follows from the inequality above that

$$(1+t)^{\sigma+1}\|w(t)\|_E^2 + \|w(t)\|_{L^2}^2 + \begin{cases} \int_0^t (1+\tau)^{-1}\|w(\tau)\|_{L^2}^2\, d\tau & \text{if } \sigma = 1 \\ 0 & \text{if } 0 < \sigma < 1 \end{cases}$$
$$+ \int_0^t \left\{(1+\tau)\|w_t(\tau)\|_{L^2}^2 + (1+\tau)^\sigma\left(\|\nabla w(\tau)\|_{L^2}^2 + \int_{\mathbf{R}^N} c(\tau,x)w(\tau,x)^2\, dx\right)\right\} d\tau$$
$$\leq G(t), \tag{3.34}$$

where

$$G(t) = C_{23}\left\{1 + \int_0^t (1+\tau)^{\sigma+1}\gamma_c(\tau)\int_{\mathbf{R}^N} c(\tau,x)w(\tau,x)^2\, dxd\tau + \int_0^t \gamma_\psi(\tau)\|w(\tau)\|_{L^2}^2\, d\tau\right\}.$$

From this we find

$$(1+t)^{\sigma+1}\int_{\mathbf{R}^N} c(t,x)w(t,x)^2\, dx \leq 2G(t) \tag{3.35}$$

and

$$\|w(t)\|_{L^2}^2 \leq G(t). \tag{3.36}$$

Therefore it holds that

$$G'(t) \leq C_{23}\{2\gamma_c(t) + \gamma_\psi(t)\}G(t)$$

and we conclude

$$G(t) \leq C_{23}e^{C_{24}}$$

with

$$C_{24} = C_{23}\int_0^t \{2\gamma_c(t) + \gamma_\psi(t)\}\, dt.$$

This and (3.34) give (1.6).

(iv) Derivation of (1.7). Since the same arguments as in Section 2 (iv) give the desired result (1.7), we omit the detailed proof.

(v) Derivation of (1.8). If $\sigma = 1$, we find from (3.34) that

$$\int_0^\infty (1+t)\|w_t(t)\|_{L^2}^2\, dt < \infty \tag{3.37}$$

and

$$\int_0^\infty (1+t)^{-1}\|w(t)\|_{L^2}^2\,dt < \infty. \tag{3.38}$$

It then follows from (3.38) that

$$\liminf_{t\to+\infty}\|w(t)\|_{L^2}^2 = 0.$$

So, we have only to show that $\{\|w(t)\|_{L^2}^2\}_{t\geq 0}$ becomes a Cauchy sequences with respect to $t \in [0,\infty)$.

Integrating from t_1 to t_2 $(0 \leq t_1 \leq t_2 < \infty)$

$$\frac{d}{dt}(\|w(t)\|_{L^2}^2) = 2(w(t), w_t(t))_{L^2},$$

we have

$$\begin{aligned}
&|\,\|w(t_2)\|_{L^2}^2 - \|w(t_1)\|_{L^2}^2| \\
&\leq 2\int_{t_1}^{t_2} |(w(t), w_t(t))_{L^2}|\,dt \\
&\leq 2\left(\int_{t_1}^{t_2}(1+t)^{-1}\|w(t)\|_{L^2}^2\,dt\right)^{1/2}\left(\int_{t_1}^{t_2}(1+t)\|w_t(t)\|_{L^2}^2\,dt\right)^{1/2} \to 0
\end{aligned}$$

as $t_1, t_2 \to +\infty$ by (3.37) and (3.38). This proves (1.8).

4. Final Remarks

In the last section, we state several results without the proof for (i) the generalization of Theorem 1.2, (ii) the initial-boundary value problem, and for (iii) the case b depends on t and x.

(i) Generalizations of Theorem 1.2 with $c(t,x) \equiv 0$.

Consider (1.1) with $c(t,x) \equiv 0$ under the assumption (1.3). Without the condition on the size of b_0 as in Theorem 1.2, we obtain the following:

THEOREM 4.1. *Assume* (1.3). *If* $\mu > 0$ *satisfies*

$$\sigma < \mu < \min\left\{b_0, \sigma + \min\left\{\frac{b_0}{2}, \frac{b_2}{b_1}, \frac{b_1 + 1 - \sqrt{(b_1+1)^2 - 4b_2}}{2}\right\}\right\},$$

then the following inequality holds

$$(1+t)^{\mu}\|w(t)\|_E^2 + (1+t)^{\mu-\sigma-1}\|w(t)\|_{L^2}^2 + \int_0^t \{(1+\tau)^{\mu-\sigma}\|w(\tau)\|_E^2 + (1+\tau)^{\mu-2\sigma-1}\|w(\tau)\|_{L^2}^2\}\,d\tau \le C_{25}(\|w(0)\|_E^2 + \|w(0)\|_{L^2}^2)$$

for some positive constant C_{25}. *Therefore, we obtain*

$$\lim_{t\to\infty}(1+t)^{\mu}\|w(t)\|_E^2 = 0.$$

REMARK 4.1. In the above theorem, we have $\mu < 2$.

(ii) Exterior initial-boundary value problem.

Assume $N \ge 1$ and let $\Omega = \mathbf{R}^N \setminus \mathcal{O}$, where $\mathcal{O}$ is some bounded domain with smooth boundary $\partial\mathcal{O}$. We consider the equation:

$$\begin{cases} w_{tt} - \Delta w + b(t,x)w_t + c(t,x)w = 0 & (t,x) \in (0,\infty)\times\Omega, \\ w_n + a(t,x)w = 0 & (t,x) \in (0,\infty)\times\partial\Omega, \\ w(0,x) = w_0(x),\ w_t(0,x) = w_1(x) & x \in \Omega, \end{cases}$$

where n denotes the outer unit normal of $\partial\Omega$. We state the assumption on $a(t,x)$:

$$0 \le a(t,x) \le a_0 < \infty, \quad (t,x) \in (0,\infty)\times\partial\Omega, \tag{4.39}$$

$$a_t(t,x) \le 0, \quad (t,x) \in (0,\infty)\times\partial\Omega \tag{4.40}$$

and

$$|a_t(t,x)| \le \gamma_a(t)a(t,x), \quad (t,x) \in (0,\infty)\times\partial\Omega, \tag{4.41}$$

where a_0 is some constant and $\gamma_a \in L^1((0,\infty))$. In the following, we use the notation

$$\|w(t)\|_{E(\Omega)}^2 = \frac{1}{2}\left\{\int_\Omega (w_t(t,x)^2 + |\nabla w(t,x)|^2 + c(t,x)w(t,x)^2)\,dx + \int_{\partial\Omega} a(t,x)w(t,x)^2\,dS\right\}$$

and

$$\|w(t)\|_{L^2(\Omega)}^2 = \int_\Omega w(t,x)^2\,dx.$$

THEOREM 4.2. (i) *Assume* (1.2) *for* N, b *and* c, *and* (4.39), (4.40) *for* a. *Then we have for some* $C_{26} > 0$,

$$\begin{aligned}&(1+t)\|w(t)\|^2_{E(\Omega)} + \|w(t)\|^2_{L^2(\Omega)} \\ &\qquad + \int_0^t \Big\{(1+\tau)\|w_t(\tau)\|^2_{L^2(\Omega)} + \|\nabla w(\tau)\|^2_{L^2(\Omega)} + \int_{\mathbf{R}^N} c(t,x)w(t,x)^2\,dx \\ &\qquad\qquad + \int_{\partial\Omega} a(\tau,x)w(\tau,x)^2\,dS\Big\}\,d\tau \le C_{26}\end{aligned}$$

and

$$\lim_{t\to\infty}(1+t)\|w(t)\|^2_{E(\Omega)} = 0.$$

(ii) *Assume* (1.3) *for* N *and* c, *the condition as in Theorem* 1.2 *for* b *and* (4.39), (4.41) *for* a. *Then we have for some* $C_{27} > 0$,

$$\begin{aligned}&(1+t)^{\sigma+1}\|w(t)\|^2_{E(\Omega)} + \|w(t)\|^2_{L^2(\Omega)} + \int_0^t \Big[(1+\tau)\|w_t(\tau)\|^2_{L^2(\Omega)} \\ &\quad + (1+\tau)^{\sigma}\Big\{\|\nabla w(\tau)\|^2_{L^2(\Omega)} + \int_\Omega c(\tau,x)w(\tau,x)^2\,dx + \int_{\partial\Omega} a(\tau,x)w(\tau,x)^2\,dS\Big\} \\ &\quad + \chi(\sigma)(1+\tau)^{-1}\|w(\tau)\|^2_{L^2(\Omega)}\Big]\,d\tau \le C_{27},\end{aligned}$$

$$\lim_{t\to\infty}\{(1+t)^{\sigma+1}\|w(t)\|^2_{E(\Omega)} + \chi(\sigma)\|w(t)\|^2_{L^2(\Omega)}\} = 0.$$

REMARK 4.2. We can obtain the similar results for Dirichlet or Neumann problem.

(iii) The case where dissipation b depends on t and x.

For the sake of simplicity, we state the results only for the equation of the form:

$$\begin{cases} w_{tt} - \Delta w + b(t,x)w_t = 0 & (t,x) \in (0,\infty)\times\Omega, \\ w = 0 & (t,x) \in (0,\infty)\times\partial\Omega, \\ w(0,x) = w_0(x),\ w_t(0,x) = w_1(x) & x \in \Omega, \end{cases}$$

where Ω is the same exterior domain as in (i).

THEOREM 4.3. *Assume* $N \ge 1$,

$$b_0(1+|x|+t)^{-\mu} \le b(x,t) \le b_1(1+|x|+t)^{-\mu}$$

and

$$\mu b_2(1+|x|+t)^{-\mu-1} \le -b_t(t,x) \le \mu b_3(1+|x|+t)^{-\mu-1}$$

for any $(t,x) \in [0,\infty) \times \Omega$, *where* $\mu \in [0,1]$ *and* b_0, b_1, b_2 *and* b_3 *are some positive constants.*

(i) *If the data satisfy*

$$(1+|x|)^{\mu+1}\nabla w_0, \quad w_0, w_1 \in L^2(\Omega),$$

then there exists a positive number σ *such that*

$$\lim_{t\to\infty}(1+t)\int_{\mathbf{R}^N}(1+|x|+t)^{\sigma-1}(|\nabla w(t,x)|^2 + w_t(t,x)^2)\,dx = 0,$$

where $\sigma \ge 1$ *if* $\mu < 1$ *and* $\sigma = 1$ *if* $\sigma > 0$.

(ii) *If the data satisfies* $(w_0, w_1) \in H_0^1(\Omega) \times L^2(\Omega)$, *then there exists a positive number* σ *such that*

$$\lim_{t\to\infty}(1+t)^{\sigma}\|w(t)\|_{E(\Omega)}^2 = 0,$$

where

$$\|w(t)\|_{E(\Omega)}^2 = \frac{1}{2}\int_{\Omega}(|\nabla w(t,x)|^2 + w_t(t,x)^2)\,dx$$

is the total energy and

$$\sigma = \begin{cases} \mu + \frac{\mu b_2}{b_1} & \text{if } \mu < 1, \\ \min\left\{b_0, \frac{b_1+3-\sqrt{(b_1-1)^2+4(b_2-b_1)}}{2}\right\} & \text{if } \mu = 1 \text{ and } b_0 \ne 2, \\ \frac{b_1+3-\sqrt{(b_1-1)^2+4(b_2-b_1)}}{2} - \varepsilon & \text{if } \mu = 1 \text{ and } b_0 = 2 \end{cases}$$

for any $\varepsilon > 0$.

Corollary 4.1. *If* $(w_0, w_1) \in H^1(\Omega) \times L^2(\Omega)$ *and* $b_0 = b_1$, *we have*

$$\lim_{t\to\infty}(1+t)^{\mu+1}\|w(t)\|_{E(\Omega)}^2 = 0 \quad \text{if } \mu < 1,$$

$$\lim_{t\to\infty}(1+t)^{\min\{b_0,2\}}\|w(t)\|_{E(\Omega)}^2 = 0 \quad \text{if } \mu = 1 \text{ and } b_0 \ne 2,$$

and for any fixed $\varepsilon > 0$, $\mu = 1$ *and* $b_0 = 2$

$$\lim_{t\to\infty}(1+t)^{2-\varepsilon}\|w(t)\|_{E(\Omega)}^2 = 0.$$

Remark 4.3. $\mu = 1$ and $b_0 = 2$ of Corollary 4.1 is the critical case, and such a pair of critical numbers appears in the estimate (3.32). If $b(t, x) = 2(t+1)^{-1}$, that is, $\mu = 1$ and $b_0 = 2$, then it is possible that the energy decays exponential order with a pair of initial data and boundary condition. Indeed, if $N = 1$, $\Omega = [0, \infty)$, $w_0(x) = e^{-x}$, $w_1(x) = -2e^{-x}$ and $w(t, 0) = (1+t)^{-1}e^{-t}$, then the solution is explicitly represented by $w(t, x) = (1+t)^{-1}e^{-x-t}$. This implies that the energy decays exponential order.

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