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A REMARK ON WELL-POSEDNESS FOR HYPERBOLIC EQUATIONS WITH SINGULAR COEFFICIENTS

By

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Abstract. We prove some C^∞ and Gevrey well-posedness results for hyperbolic equations with singular coefficients.

1. Introduction

This work is devoted to the study of the well-posedness of the Cauchy problem for a linear hyperbolic operator whose coefficients depend only on time.

We consider the equation

$$u_{tt} - \sum_{i,j=1}^{n} a_{ij}(t) u_{x_i x_j} = 0 \tag{1.1}$$

in $[0, T] \times \boldsymbol{R}^n$, with initial data

$$u(0, x) = u_0(x), \quad u_t(0, x) = u_1(x) \tag{1.2}$$

in $\boldsymbol{R}^n$. The matrix (a_{ij}) is supposed to be real and symmetric. Setting

$$a(t, \xi) := \sum_{i,j=1}^{n} a_{ij}(t) \xi_i \xi_j / |\xi|^2, \quad (t, \xi) \in [0, T] \times (\boldsymbol{R}^n \backslash \{0\}), \tag{1.3}$$

we assume that $a(\cdot, \xi) \in L^1(0, T)$ for all $\xi \in \boldsymbol{R}^n \backslash \{0\}$.

We suppose that the equation (1.1) is hyperbolic i.e.

$$a(t, \xi) \geq \lambda_0 \geq 0 \tag{1.4}$$

for all $(t, \xi) \in [0, T] \times (\boldsymbol{R}^n \backslash \{0\})$.

In the strictly hyperbolic case (i.e. $\lambda_0 > 0$) it is well known that if the

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coefficients a_{ij} are Lipschitz-continuous then the Cauchy problem (1.1), (1.2) is well-posed in Sobolev spaces. In the same case if the a_{ij}'s are Log-Lipschitz-continuous or Hölder-continuous of index α, (1.1), (1.2) is well-posed in C^∞ or in the Gevrey space $\gamma^{(s)}$ for $s < 1/(1-\alpha)$ respectively (see [1]). In the weakly hyperbolic case (i.e. $\lambda_0 = 0$) if the coefficients are $C^{k,\alpha}$ then the problem (1.1), (1.2) is $\gamma^{(s)}$-well-posed for $s < 1 + (k+\alpha)/2$ (see [4]). Some counter examples show that all these results are sharp (see also [5]).

Recently Colombini, Del Santo and Kinoshita have considered the same problem for operators having coefficients which are C^1 on $[0,T]\backslash\{t_0\}$ with a singularity concentrated at t_0. In this situation, under the main assumptions that

$$(1.5)\qquad \begin{aligned} |t_0 - \cdot|^p a'(\cdot,\xi) = \beta(\cdot,\xi) \in L^\infty(0,T) \quad \text{for all } \xi \in \boldsymbol{R}^n\backslash\{0\} \\ |t_0 - \cdot|^r a(\cdot,\xi) = \alpha(\cdot,\xi) \in L^\infty(0,T) \quad \text{for all } \xi \in \boldsymbol{R}^n\backslash\{0\} \end{aligned}$$

it is possible to show that the Cauchy problem (1.1), (1.2) is $\gamma^{(s)}$-well-posed, the value of s depending on p and r (see [2] and [3]) (here and in the following "$'$" denotes the differentiation with respect to t).

The aim of the present work is to improve the results of [2] and [3] allowing the function β in (1.5) to be in a L^q space and removing the growth assumption on a. We make the following assumptions: let $1 \le q \le +\infty$ and $p \ge 0$ and let $t_0 \in [0,T]$; suppose that

(H1) $a(\cdot,\xi) \in \bigcap_{\varepsilon>0} W^{1,1}(]0, t_0-\varepsilon[\cup]t_0+\varepsilon, T[)$ for all $\xi \in \boldsymbol{R}^n\backslash\{0\}$;
(H2) $|t_0 - \cdot|^p a'(\cdot,\xi) = \beta(\cdot,\xi) \in L^q(0,T)$ for all $\xi \in \boldsymbol{R}^n\backslash\{0\}$.

In the weakly hyperbolic case the results are the following.

Theorem 1. *Assume that* $3 \le (p+1/q)$. *Then the Cauchy problem* (1.1), (1.2) *is* $\gamma^{(\sigma)}$*-well-posed for* $1 \le \sigma < \frac{(p+1/q)-\frac{3}{2}}{(p+1/q)-2}$. *If moreover*

$$(1.6)\qquad |t_0 - \cdot|^r a(\cdot,\xi) = \alpha(\cdot,\xi) \in L^s(0,T) \quad \text{for all } \xi \in \boldsymbol{R}^n\backslash\{0\},$$

with $r \ge 0$, $1 \le s \le +\infty$ *and* $(r+1/s) \le 1$, *then the Cauchy problem* (1.1), (1.2) *is* $\gamma^{(\sigma)}$*-well-posed for* $1 \le \sigma < \frac{(p+1/q)-\frac{3}{2}(r+1/s)}{(p+1/q)-(r+1/s)-1}$.

Theorem 2. *Assume that* $(p+1/q) < 3$. *Then the Cauchy problem* (1.1), (1.2) *is* $\gamma^{(\sigma)}$*-well-posed for all* $1 \le \sigma < \frac{3}{2}$.

The result concerning the strictly hyperbolic case are contained in the following theorems.

Theorem 3. *Assume that* $1 < (p + 1/q) < 3$. *Moreover, assume that* $\lambda_0 > 0$. *Then the Cauchy problem* (1.1), (1.2) *is* $\gamma^{(\sigma)}$*-well-posed for all* $1 \le \sigma < \frac{(p+1/q)}{(p+1/q)-1}$.

Theorem 4. *Assume that* $(p + 1/q) \le 1$. *Moreover, assume that* $\lambda_0 > 0$. *Then the Cauchy problem* (1.1), (1.2) *is* C^∞*-well-posed.*

Remark 1. Adapting to the present situation some counter examples contained in [4], [2], and [3] it is possible to see that the results of Theorems 1–4 are optimal. Let us show this in some detail in the case of Theorem 1. Suppose $p_0 + 1/q_0 = 3$. In this case $\frac{(p_0+1/q_0)-\frac{3}{2}}{(p_0+1/q_0)-2} = \frac{(p_0+1/q_0)-\frac{3}{2}(r_0+1/s_0)}{(p_0+1/q_0)-(r_0+1/s_0)-1} = \frac{3}{2}$; consequently Theorem 2 in [4] shows that this value of the Gevrey index cannot be improved. Consider next the case that $p_0 + 1/q_0 > 3$ and $(r_0 + 1/s_0) \le 1$. Let $\bar{\sigma} > \sigma_0 = \frac{(p_0+1/q_0)-\frac{3}{2}(r_0+1/s_0)}{(p_0+1/q_0)-(r_0+1/s_0)-1}$. We fix $q_1 > q_0$ and $s_1 > s_0$ in such a way that $p_0 + 1/q_1 > 3$, $r_0 + 1/s_1 < 1$ and $\sigma_0 < \sigma_1 := \frac{(p_0+1/q_1)-\frac{3}{2}(r_0+1/s_1)}{(p_0+1/q_1)-(r_0+1/s_1)-1} < \bar{\sigma}$. From Theorem 4 in [3] we have that there exists a function $a : [0,1[\to [1/2,+\infty[$ such that $a \in C^\infty([0,1[)$ and

$$(1-t)^{p_0+1/q_1}a'(t) \in L^\infty, \quad (1-t)^{r_0+1/s_1}a(t) \in L^\infty,$$

and there exist $u_0, u_1 \in \gamma^{(\sigma)}$ for all $\sigma > \sigma_1$ such that the Cauchy problem

$$u_{tt} - a(t)u_{xx} = 0, \quad u(0,x) = u_0(x), \quad u_t(0,x) = u_1(x), \tag{1.7}$$

has no solution in $W^{2,1}([0,1], \mathscr{D}'^{(\sigma)}(\boldsymbol{R}))$ for all $\sigma > \sigma_1$. Consequently

$$(1-t)^{p_0}a'(t) \in L^{q_0}, \quad (1-t)^{r_0}a(t) \in L^{s_0},$$

$u_0, u_1 \in \gamma^{(\bar{\sigma})}$ and the Cauchy problem (1.7) does not have a solution in $W^{2,1}([0,1], \mathscr{D}'^{(\bar{\sigma})}(\boldsymbol{R}))$.

Remark 2. Let us remark that Theorem 1 is a nontrivial improvement of Theorem 2 in [3] also in the case of $q = \infty$. In fact the growth condition on a is removed and the result is sharp (see [3, Th. 4]).

2. Proof of Theorems 1–4

As a preliminary step, let us observe that, since the coefficients a_{ij} are real integrable functions, the Cauchy problem (1.1), (1.2) is well posed in $\mathscr{A}'(\boldsymbol{R}^n)$, the space of real analytic functionals. Moreover, if the initial data vanish in a ball, then the solution vanishes in a cone, whose slope depends on the coefficients a_{ij}.

Therefore it will be sufficient to show that, under the hypotheses of each theorem, if u_0 and u_1 have compact support then the corresponding solution u is not only in $W^{2,1}([0,T],\mathscr{A}'(\boldsymbol{R}^n))$, but it belongs to a Gevrey space in the x variable. Our main tools in doing this will be the Paley-Wiener theorem (in the version of [1, p. 517], to which we refer here and throughout) and some energy estimates.

Denoting by v the Fourier transform of u with respect to x, equation (1.1) reads

$$v''(t,\xi) + a(t,\xi)|\xi|^2 v(t,\xi) = 0. \tag{2.1}$$

Let ε be a positive parameter and for each ε let $a_\varepsilon : [0,T] \times (\boldsymbol{R}^n \backslash \{0\}) \to \boldsymbol{R}$ be a strictly positive real function such that $a_\varepsilon(\cdot,\xi) \in W^{1,1}(0,T)$ for all $\xi \in \boldsymbol{R}^n \backslash \{0\}$. We define the *approximate energy* of v by

$$E_\varepsilon(t,\xi) := a_\varepsilon(t,\xi)|\xi|^2|v(t,\xi)|^2 + |v'(t,\xi)|^2, \quad (t,\xi) \in [0,T] \times (\boldsymbol{R}^n \backslash \{0\}). \tag{2.2}$$

Differentiating E_ε with respect to t and using (2.1) we get

$$\begin{aligned} E_\varepsilon'(t,\xi) &= a_\varepsilon'(t,\xi)|\xi|^2|v(t,\xi)|^2 + 2a_\varepsilon(t,\xi)|\xi|^2 \operatorname{Re}(v'(t,\xi)\bar{v}(t,\xi)) \\ &\quad + 2\operatorname{Re}(v''(t,\xi)\bar{v}'(t,\xi)) \\ &\le \left(\frac{|a_\varepsilon'(t,\xi)|}{a_\varepsilon(t,\xi)} + \frac{|a_\varepsilon(t,\xi) - a(t,\xi)|}{a_\varepsilon(t,\xi)^{1/2}} |\xi| \right) E_\varepsilon(t,\xi). \end{aligned}$$

By Gronwall's lemma we obtain

$$E_\varepsilon(t,\xi) \le E_\varepsilon(0,\xi) \exp\left(\int_0^T \frac{|a_\varepsilon'(t,\xi)|}{a_\varepsilon(t,\xi)}\, dt + |\xi| \int_0^T \frac{|a_\varepsilon(t,\xi) - a(t,\xi)|}{a_\varepsilon(t,\xi)^{1/2}}\, dt \right) \tag{2.3}$$

for all $t \in [0,T]$ and for all $\xi \in \boldsymbol{R}^n$, $|\xi| \ge 1$.

Now we are able to give the

Proof of Theorem 1. First of all, observe that condition (1.6) is always satisfied at least with $r = 0$ and $s = 1$ (recall that $a_{ij} \in L^1(0,T)$).

Since $u_0, u_1 \in \gamma^{(\sigma)} \cap C_0^\infty$, the Paley-Wiener theorem ensures that there exist $M, \delta > 0$ such that

$$|v(0,\xi)|^2 + |v'(0,\xi)|^2 \le M \exp(-\delta|\xi|^{1/\sigma}) \tag{2.4}$$

for all $\xi \in \boldsymbol{R}^n$, $|\xi| \ge 1$. To verify that $u \in W^{2,1}([0,T],\gamma^{(\sigma)})$ it is sufficient to show that there exist $M', \delta' > 0$ such that

(2.5) $$|v(t,\xi)|^2 + |v'(t,\xi)|^2 \le M' \exp(-\delta'|\xi|^{1/\sigma})$$

for all $t \in [0,T]$ and for all $\xi \in \boldsymbol{R}^n$, $|\xi| \ge 1$. We consider first the case $t_0 = T$. For $\varepsilon \in]0,T]$, we set

(2.6) $$a_\varepsilon(t,\xi) := \begin{cases} a(t,\xi) + \varepsilon^{2-(r+1/s)}(T-t)^{-2} & \text{for } 0 \le t \le T-\varepsilon \\ \varepsilon^{-(z+r)}a(t,\xi)(T-t)^{z+r} + \varepsilon^{-(r+1/s)} & \text{for } T-\varepsilon \le t \le T \end{cases}$$

where z is any positive number such that

(2.7) $$z > \max\{1/s, (p+1/q) - r - 1\}.$$

Then

(2.8) $$a_\varepsilon(t,\xi) = \begin{cases} \alpha(t,\xi)(T-t)^{-r} + \varepsilon^{2-(r+1/s)}(T-t)^{-2} & \text{for } 0 \le t \le T-\varepsilon \\ \varepsilon^{-(z+r)}\alpha(t,\xi)(T-t)^{z} + \varepsilon^{-(r+1/s)} & \text{for } T-\varepsilon \le t \le T \end{cases}$$

and

(2.9) $$a'_\varepsilon(t,\xi) = \begin{cases} \beta(t,\xi)(T-t)^{-p} - 2\varepsilon^{2-(r+1/s)}(T-t)^{-3} & \text{for } 0 \le t \le T-\varepsilon \\ \varepsilon^{-(z+r)}(\beta(t,\xi)(T-t)^{z+r-p} - (z+r)\alpha(t,\xi)(T-t)^{z-1}) & \text{for } T-\varepsilon \le t \le T \end{cases}$$

Our choice of z implies that $a_\varepsilon(\cdot,\xi) \in W^{1,1}(0,T)$ for all $\xi \in \boldsymbol{R}^n \backslash \{0\}$. By (2.8) and (2.9) we get

$$\begin{aligned}\int_0^T \frac{|a'_\varepsilon(t,\xi)|}{a_\varepsilon(t,\xi)}\,dt \le{}& \int_0^{T-\varepsilon} \frac{|\beta(t,\xi)|(T-t)^{-p}}{\varepsilon^{2-(r+1/s)}(T-t)^{-2}}\,dt \\ &+ \int_0^{T-\varepsilon} \frac{2\varepsilon^{2-(r+1/s)}(T-t)^{-3}}{\varepsilon^{2-(r+1/s)}(T-t)^{-2}}\,dt \\ &+ \int_{T-\varepsilon}^{T} \frac{\varepsilon^{-(z+r)}|\beta(t,\xi)|(T-t)^{z+r-p}}{\varepsilon^{-(r+1/s)}}\,dt \\ &+ \int_{T-\varepsilon}^{T} \frac{\varepsilon^{-(z+r)}(z+r)|\alpha(t,\xi)|(T-t)^{z-1}}{\varepsilon^{-(r+1/s)}}\,dt\end{aligned}$$

The choice of z allows us to use Hölder inequality; an easy computation shows that

(2.10) $$\int_0^T \frac{|a'_\varepsilon(t,\xi)|}{a_\varepsilon(t,\xi)}\,dt \le C'(1+|\log\varepsilon|)\varepsilon^{-(p+1/q)+(r+1/s)+1},$$

where C' is a constant depending only on C, r, s, p, q and z. On the other hand,

$$\int_0^T \frac{|a_\varepsilon(t,\xi) - a(t,\xi)|}{a_\varepsilon(t,\xi)^{1/2}}\,dt = \int_0^{T-\varepsilon} \frac{\varepsilon^{2-(r+1/s)}(T-t)^{-2}}{\varepsilon^{1-(1/2)(r+1/s)}(T-t)^{-1}}\,dt$$
$$+ \int_{T-\varepsilon}^{T} \frac{\varepsilon^{-(z+r)}\alpha(t,\xi)(T-t)^{z}}{\varepsilon^{-(1/2)(r+1/s)}}\,dt$$
$$+ \int_{T-\varepsilon}^{T} \frac{\varepsilon^{-(r+1/s)}}{\varepsilon^{-(1/2)(r+1/s)}}\,dt + \int_{T-\varepsilon}^{T} \frac{\alpha(t,\xi)(T-t)^{-r}}{\varepsilon^{-(1/2)(r+1/s)}}\,dt.$$

The first three summands on the right hand side can be estimated again by using Hölder inequality. In order to estimate the fourth summand, we shall distinguish the case $(r+1/s) < 1$ and $(r+1/s) = 1$. In the first case, we use once more Hölder inequality; in the second case, we use the fact that $\alpha(t,\xi)(T-t)^{-r} = a(t,\xi) \in L^1(0,T)$. At the end, we get

$$\int_0^T \frac{|a_\varepsilon(t,\xi) - a(t,\xi)|}{a_\varepsilon(t,\xi)^{1/2}}\,dt \le C''(1+|\log\varepsilon|)\varepsilon^{-(1/2)(r+1/s)+1}, \tag{2.11}$$

where C'' is a constant depending only on C, r, s, p, q and z. By (2.3), (2.10) and (2.11) we obtain

$$E(t,\xi) \le E(0,\xi)\exp(\tilde{C}(1+|\log\varepsilon|)(\varepsilon^{-(p+1/q)+(r+1/s)+1} + |\xi|\varepsilon^{-(1/2)(r+1/s)+1})) \tag{2.12}$$

for all $t \in [0,T]$ and for all $\xi \in \boldsymbol{R}^n$, $\xi \ge 1$, where $\tilde{C}$ is a positive constant depending only on C, r, s, p, q and z.

Now, by (2.2) and (2.6), we have

$$E_\varepsilon(0,\xi) \le (a(0,\xi) + T^{-(r+1/s)})|\xi|^2|v(0,\xi)|^2 + |v'(0,\xi)|^2 \tag{2.13}$$

and

$$E_\varepsilon(t,\xi) \ge T^{-2}\varepsilon^{2-(r+1/s)}|\xi|^2|v(t,\xi)|^2 + |v'(t,\xi)|^2. \tag{2.14}$$

Then choosing $\varepsilon := |\xi|^{-[(p+1/q)-(3/2)(r+1/s)]^{-1}}$ we deduce

$$T^{-2}|\xi|^{2-(2-(r+1/s))/((p+1/q)-(3/2)(r+1/s))}|v(t,\xi)|^2 + |v'(t,\xi)|^2$$
$$\le (\tilde{K}|\xi|^2|v(0,\xi)|^2 + |v'(0,\xi)|^2)$$
$$\times \exp(\tilde{C}(1+|\log|\xi||)|\xi|^{((p+1/q)-(r+1/s)-1)/((p+1/q)-(3/2)(r+1/s))}).$$

Using the Paley-Wiener theorem, the well-posedness follows for all $1 \le \sigma < \frac{(p+1/q)-\frac{3}{2}(r+1/s)}{(p+1/q)-(r+1/s)-1}$.

If $t_0 = 0$, for $\varepsilon \in]0,T]$ we set

$$a_\varepsilon(t,\xi) := \begin{cases} \varepsilon^{-(z+r)}a(t,\xi)t^{z+r} + \varepsilon^{-(r+1/s)} & \text{for } 0 \le t \le \varepsilon \\ a(t,\xi) + \varepsilon^{2-(r+1/s)}t^{-2} & \text{for } \varepsilon \le t \le T \end{cases} \tag{2.15}$$

where z satisfies (2.7). Our choice of z implies that $a_\varepsilon(\cdot\,,\xi) \in W^{1,1}(0,T)$ for all $\xi \in \boldsymbol{R}^n\backslash\{0\}$. So, in particular, $a_\varepsilon(\cdot\,,\xi)$ is continuous on $[0,T]$. Arguing as before, we obtain (2.12). An easy computation shows that $|a(t,\xi)| \le \tilde{K}t^{1-(p+1/q)}$ for all $\xi \in \boldsymbol{R}^n\backslash\{0\}$. It follows that

$$a_\varepsilon(0,\xi) = \lim_{\tau\to 0} a_\varepsilon(\tau,\xi) = \lim_{\tau\to 0}(\varepsilon^{-(z+r)}a(\tau,\xi)\tau^{z+r} + \varepsilon^{-(r+1/s)})$$
$$\le \tilde{K}\limsup_{\tau\to 0}(\varepsilon^{-(z+r)}\tau^{z+r+1-(p+1/q)} + \varepsilon^{-(r+1/s)}).$$

By (2.7) we deduce that $a_\varepsilon(0,\xi) \le \tilde{K}\varepsilon^{-(r+1/s)}$. It follows that

$$E_\varepsilon(0,\xi) \le \tilde{K}\varepsilon^{-(r+1/s)}|\xi|^2|v(0,\xi)|^2 + |v'(0,\xi)|^2. \tag{2.16}$$

Moreover, we have also

$$E_\varepsilon(t,\xi) \ge T^{-2}\varepsilon^{2-(r+1/s)}|\xi|^2|v(t,\xi)|^2 + |v'(t,\xi)|^2. \tag{2.17}$$

Then, choosing again $\varepsilon := |\xi|^{-[(p+1/q)-(3/2)(r+1/s)]^{-1}}$, we deduce

$$|\xi|^{2-(2-(r+1/s))/((p+1/q)-(3/2)(r+1/s))}|v(t,\xi)|^2 + |v'(t,\xi)|^2$$
$$\le (\tilde{K}|\xi|^{2+(r+1/s)/((p+1/q)-(3/2)(r+1/s))}|v(0,\xi)|^2$$
$$+ |v'(0,\xi)|^2)\exp(\tilde{C}(1+|\log|\xi|\,|)|\xi|^{((p+1/q)-(r+1/s)-1)/((p+1/q)-(3/2)(r+1/s))}).$$

Using the Paley-Wiener theorem, the well-posedness follows again for all $1 \le \sigma < \frac{(p+1/q)-\frac{3}{2}(r+1/s)}{(p+1/q)-(r+1/s)-1}$.

Finally, if $t_0 \in\,]0,T[$, it will be sufficient to solve first the Cauchy problem in $[0,t_0]$, then to solve the problem in $[t_0,T]$ with the initial data obtained from the previous one and finally to glue together the two solutions. □

In order to prove Theorem 2, we proceed exactly like in the proof of Theorem 1. In this case the role of condition (1.6) is played by the estimate

$$a(t,\xi) \le C'|t-t_0|^{-(p+1/q)+1} \quad \text{for all } \xi \in \boldsymbol{R}^n\backslash\{0\}, \tag{2.18}$$

which is a direct consequence of condition (H2). The function $a_\varepsilon(\cdot\,,\xi)$ is defined by

$$a_\varepsilon(t,\xi) := \begin{cases} a(t,\xi) + \varepsilon^{3-(p+1/q)}(T-t)^{-2} & \text{for } 0 \le t \le T-\varepsilon \\ a(T-\varepsilon,\xi) + \varepsilon^{1-(p+1/q)} & \text{for } T-\varepsilon \le t \le T \end{cases} \tag{2.19}$$

if $t_0 = T$ and by

$$a_\varepsilon(t,\xi) := \begin{cases} a(\varepsilon,\xi) + \varepsilon^{1-(p+1/q)} & \text{for } 0 \le t \le \varepsilon \\ a(t,\xi) + \varepsilon^{3-(p+1/q)} t^{-2} & \text{for } \varepsilon \le t \le T \end{cases} \tag{2.20}$$

if $t_0 = 0$. Arguing like in the proof of Theorem 1, we get

$$\int_0^T \frac{|a'_\varepsilon(t,\xi)|}{a_\varepsilon(t,\xi)}\, dt \le C''(1 + |\log \varepsilon|)\varepsilon^{(p+1/q)-3}, \tag{2.21}$$

and

$$\int_0^T \frac{|a_\varepsilon(t,\xi) - a(t,\xi)|}{a_\varepsilon(t,\xi)^{1/2}}\, dt \le C''(1 + |\log \varepsilon|)\varepsilon^{-(1/2)(p+1/q)+3/2} \tag{2.22}$$

and the conclusion follows by choosing $\varepsilon := |\xi|^{-(2/3)[3-(p+1/q)]^{-1}}$.

Theorem 3 is the strictly hyperbolic version of Theorem 2. We define again a_ε by (2.19) and (2.20), but in this case the positive lower bound for $a(t,\xi)$ allows us to obtain better estimates for $\int_0^T \frac{|a'_\varepsilon(t,\xi)|}{a_\varepsilon(t,\xi)}\, dt$. Let us consider, for example, the case $t_0 = T$. First observe that, by rescaling the x variable if necessary, we can always assume that $\lambda_0 = 1$. Then we can minorize $a_\varepsilon(t,\xi)$ by the constant 1 on $[0, T - \varepsilon^{(1/2)[3-(p+1/q)]}]$ and by $\varepsilon^{3-(p+1/q)}(T-t)^{-2}$ on $[T - \varepsilon^{(1/2)[3-(p+1/q)]}, T - \varepsilon]$. So we obtain that

$$\int_0^T \frac{|a'_\varepsilon(t,\xi)|}{a_\varepsilon(t,\xi)}\, dt \le C''(1 + |\log \varepsilon|)\varepsilon^{(1/2)((p+1/q)-1)((p+1/q)-3)}. \tag{2.23}$$

The conclusion follows by choosing $\varepsilon := |\xi|^{-2[p+1/q]^{-1}[3-(p+1/q)]^{-1}}$.

Finally, we give the

PROOF OF THEOREM 4. Since $u_0, u_1 \in C_0^\infty$, the Paley-Wiener theorem ensures that for all $\zeta > 0$ there exists $M_\zeta > 0$ such that

$$|v(0,\xi)|^2 + |v'(0,\xi)|^2 \le M_\zeta |\xi|^{-\zeta} \tag{2.24}$$

for all $\xi \in \boldsymbol{R}^n$, $|\xi| \ge 1$. To verify that $u \in W^{2,1}([0,T], C_0^\infty)$ it is sufficient to show that for all $\eta > 0$ there exists $M_\eta > 0$ such that

$$|v(t,\xi)|^2 + |v'(t,\xi)|^2 \le M_\eta |\xi|^{-\eta} \tag{2.25}$$

for all $t \in [0,T]$ and for all $\xi \in \boldsymbol{R}^n$, $|\xi| \ge 1$. We give the details only in the case $t_0 = T$. If $q = 1$, then necessarily $p = 0$. This means that $a(\cdot,\xi) \in W^{1,1}(0,T)$ and it is well known that this is enough to detect C^∞-well-posedness of the Cauchy problem (1.1), (1.2). If $q > 1$, for $\varepsilon \in]0,T]$, we set

$$a_\varepsilon(t,\xi) := \begin{cases} a(t,\xi) & \text{for } 0 \le t \le T-\varepsilon \\ a(T-\varepsilon,\xi) & \text{for } T-\varepsilon \le t \le T \end{cases} \tag{2.26}$$

Now observe that

$$\begin{aligned} |a(t,\xi)| &\le |a(0,\xi)| + \int_0^t |a'(\tau,\xi)|\,d\tau \le |a(0,\xi)| + \int_0^t \beta(\tau,\xi)(T-\tau)^{-p}\,d\tau \\ &\le |a(0,\xi)| + \|\beta(\cdot\,,\xi)\|_{L^q}\left(\int_0^t (T-\tau)^{-pq'}\,d\tau\right)^{1/q'} \le C(1+|\log(T-t)|^{1/q'}) \end{aligned}$$

An easy computation shows that

$$\int_0^T |a'_\varepsilon(t,\xi)|\,dt \le C'|\log\varepsilon|^{1/q'} \tag{2.27}$$

and

$$\int_0^T |a_\varepsilon(t,\xi) - a(t,\xi)|\,dt \le C'\varepsilon|\log\varepsilon|^{1/q'}. \tag{2.28}$$

Then we deduce by (2.3) that

$$\begin{aligned} &|\xi|^2|v(t,\xi)|^2 + |v'(t,\xi)|^2 \\ &\quad\le (a(0,\xi)|\xi|^2|v(0,\xi)|^2 + v'(0,\xi)|^2) \\ &\qquad\times \exp(C'|\log\varepsilon|^{1/q'} + C'|\xi|\varepsilon|\log\varepsilon|^{1/q'}). \end{aligned} \tag{2.29}$$

Here, for simplicity, we have assumed that $\lambda_0 = 1$. Choosing $\varepsilon := |\xi|^{-1}$, we obtain

$$\begin{aligned} &|\xi|^2|v(t,\xi)|^2 + |v'(t,\xi)|^2 \\ &\quad\le (a(0,\xi)|\xi|^2|v(0,\xi)|^2 + |v'(0,\xi)|^2)\exp(C'|\log|\xi|\,|^{1/q'}). \end{aligned} \tag{2.30}$$

Now, for $|\xi| \ge e$, we have $|\log|\xi|\,|^{1/q'} \le |\log|\xi|\,|$, and hence

$$|\xi|^2|v(t,\xi)|^2 + |v'(t,\xi)|^2 \le (a(0,\xi)|\xi|^2|v(0,\xi)|^2 + |v'(0,\xi)|^2)|\xi|^{C'}. \tag{2.31}$$

By the Paley-Wiener theorem, the well-posedness in C_0^∞ follows. □

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