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CHARACTERIZING MANIFOLDS MODELED ON CERTAIN DENSE SUBSPACES OF NON-SEPARABLE HILBERT SPACES

By

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Abstract. For an infinite set Γ, let $\ell_2^f(\Gamma)$ be the linear span of the canonical orthonormal basis of the Hilbert space $\ell_2(\Gamma)$, that is,

$$\ell_2^f(\Gamma) = \{x \in \ell_2(\Gamma) \mid x(\gamma) = 0 \text{ except for finitely many } \gamma \in \Gamma\}.$$

We denote $\ell_2^f = \ell_2^f(\boldsymbol{N})$. Let $Q = [-1,1]^\omega$ be the Hilbert cube. In this paper, we give characterizations of manifold modeled on the following spaces: $\ell_2(\Gamma) \times \ell_2^f$, $\ell_2^f(\Gamma)$ and $\ell_2^f(\Gamma) \times Q$, where $\ell_2(\Gamma) \times \ell_2$ and $\ell_2(\Gamma) \times Q$ are homeomorphic to $\ell_2(\Gamma)$. Our results are obtained by suitable alteration and modification of the separable case due to Bestvina and Mogilski.

1. Introduction

Given a space E, an *E-manifold* is a topological manifold modeled on E, that is, a paracompact Hausdorff space such that each point has an open neighborhood which is homeomorphic to ($\approx$) an open set in E. In [16] (cf. [17]), Toruńczyk gave a characterization of $\ell_2(\Gamma)$-manifolds, where $\ell_2(\Gamma)$ is the Hilbert space of square-summable real-valued function on an infinite set Γ. Let $\ell_2^f(\Gamma)$ be the linear span of the canonical orthonormal basis of $\ell_2(\Gamma)$, that is,

$$\ell_2^f(\Gamma) = \{x \in \ell_2(\Gamma) \mid x(\gamma) = 0 \text{ except for finitely many } \gamma \in \Gamma\}.$$

In case $\Gamma = \boldsymbol{N}$, we denote $\ell_2^f(\boldsymbol{N}) = \ell_2^f$ as well as $\ell_2(\boldsymbol{N}) = \ell_2$. Let $Q = [-1,1]^\omega$ be

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the Hilbert cube. As well-known, the separable Hilbert space ℓ_2 is homeomorphic to the psuedo-interior $s = (-1,1)^\omega$ of Q,

$$\ell_2^f \approx \sigma = \{x \in s \mid x(i) = 0 \text{ except for finitely many } i \in \boldsymbol{N}\} \quad \text{and}$$

$$\ell_2^f \times Q \approx \ell_2^Q = \{x \in \ell_2 \mid \sup_{i \in \boldsymbol{N}} |ix(i)| < \infty\}$$

$$\approx \Sigma = \{x \in Q \mid \sup_{i \in \boldsymbol{N}} |x(i)| < 1\} \approx B(Q) = Q \backslash s.$$

Notice that ℓ_2^Q is a dense subspace of ℓ_2. By Mogilski [8], ℓ_2^f- and $\ell_2^f \times Q$-manifolds were characterized. Furthermore, these were generalized to manifolds modeled on various dense subspaces of ℓ_2 by Bestvina and Mogilski [1]. In particular, $\ell_2 \times \ell_2^f$-manifolds were characterized in addition to ℓ_2^f- and $\ell_2^f \times Q$-manifolds.

In this paper, these results are extended to the non-separable case, that is, we characterize $\ell_2(\Gamma) \times \ell_2^f$-, $\ell_2^f(\Gamma)$- and $\ell_2^f(\Gamma) \times Q$-manifolds for an arbitrary infinite set Γ. One should note that $\ell_2(\Gamma) \times \ell_2^f$ and $\ell_2^f(\Gamma) \times Q$ are regarded as dense subspace of $\ell_2(\Gamma)$. In fact, since $X \times \ell_2(\Gamma) \approx \ell_2(\Gamma)$ for any completely metrizable AR X with weight $w(X) \leq \operatorname{card} \Gamma$ [13], we have

$$\ell_2(\Gamma) \approx \ell_2(\Gamma) \times \ell_2 \approx \ell_2(\Gamma) \times Q.$$

For each open cover $\mathscr{U}$ of Y, two maps $f, g : X \to Y$ are *$\mathscr{U}$-close* (or f is *$\mathscr{U}$-close* to g) if each $\{f(x), g(x)\}$ is contained in some $U \in \mathscr{U}$. A closed set $A \subset X$ is called a (*strong*) *Z-set* in X provided, for each open cover $\mathscr{U}$ of X, there is a map $f : X \to X$ such that f is $\mathscr{U}$-close to id_X and $f(X) \cap A = \emptyset$ ($\operatorname{cl} f(X) \cap A = \emptyset$). When X is an ANR, a closed set A is a Z-set in X if and only if every map $f : \mathbf{I}^k \to X$ $(k \geq 0)$ can be approximated by maps $g : \mathbf{I}^k \to X \backslash A$ (i.e., for each open cover $\mathscr{U}$ of X, there is a map $g : \mathbf{I}^k \to X \backslash A$ which is $\mathscr{U}$-close to f). The union of countably many (strong) Z-sets in X is called a (*strong*) *Z_σ-set* in X. A *Z-embedding* is an embedding whose image is a Z-set.

A space X is said to be *universal for a class* $\mathscr{C}$ (simply, *$\mathscr{C}$-universal*) if every map $f : C \to X$ of $C \in \mathscr{C}$ is approximated by Z-embeddings. It is said that X is *strongly universal for* $\mathscr{C}$ (simply, *strongly $\mathscr{C}$-universal*) when the following condition is satisfied:

($\mathrm{su}_\mathscr{C}$) for each $C \in \mathscr{C}$ and each closed set $D \subset C$, if $f : C \to X$ is a map such that $f|D$ is a Z-embedding, then, for each open cover $\mathscr{U}$ of X, there is a Z-embedding $h : C \to X$ such that $h|D = f|D$ and h is $\mathscr{U}$-close to f.

The following is our main result:

MAIN THEOREM. *Let X be a connected metrizable space and Γ an infinite set with* $\operatorname{card} \Gamma = \tau$.

(1) *X is homeomorpic to* $\ell_2(\Gamma) \times \ell_2^f$ (*or an* $\ell_2(\Gamma) \times \ell_2^f$*-manifold*) *if and only if X is an AR* (*or an ANR*) *with* $w(X) = \tau$, *X is a* σ*-completely metrizable strong* Z_σ*-space and strongly universal for the class of completely metrizable spaces with weight* $\leq \tau$.

(2) *X is homeomorpic to* $\ell_2^f(\Gamma)$ (*or an* $\ell_2^f(\Gamma)$*-manifold*) *if and only if X is an AR* (*or an ANR*) *with* $w(X) = \tau$, *X is a strongly countable-dimensional* σ*-locally compact strong* Z_σ*-space and strongly universal for the class of strongly countable-dimensional locally compact metrizable spaces with weight* $\leq \tau$.

(3) *X is homeomorpic to* $\ell_2^f(\Gamma) \times Q$ (*or an* $\ell_2^f(\Gamma) \times Q$*-manifold*) *if and only if X is an AR* (*or an ANR*) *with* $w(X) = \tau$, *X is a* σ*-locally compact strong* Z_σ*-space and strongly universal for the class of locally compact metrizable spaces with weight* $\leq \tau$.

The above result can be obtained by suitable alteration and modification of [1]. However, one should remind that some arguments in [1] depend on separability (e.g., Lemma 1.4, Propositions 1.7 and 2.3). Thus, we need to take different approaches to obtain non-separable versions of some results in [1].

2. Preliminaries

Throughout of the paper, let τ *be an infinite cardinal and* Γ *an infinite set with* $\operatorname{card}\Gamma = \tau$.

Let $\mathfrak{M}$ be the class of all metrizable spaces. For a class $\mathscr{C} \subset \mathfrak{M}$, we denote by $\mathscr{C}(\tau)$ the subclass of $\mathscr{C}$ consisting of all spaces $X \in \mathscr{C}$ with weight $w(X) \leq \tau$. It is said that

- $\mathscr{C}$ is *topological* if $X \in \mathscr{C}$, $X \approx Y \Rightarrow Y \in \mathscr{C}$,
- $\mathscr{C}$ is *closed* (resp. *open*) *hereditary* if $X \in \mathscr{C}$, $A \subset X$ is closed (resp. open) in $X \Rightarrow A \in \mathscr{C}$,
- $\mathscr{C}$ is *additive* if $X = X_1 \cup X_2$ and $X_1, X_2 \in \mathscr{C}$ are closed in $X \Rightarrow X \in \mathscr{C}$.

By $\mathscr{C}_\sigma$, we denote the class consisting of all metrizable spaces which can be expressed as countable unions of closed subspaces contained in $\mathscr{C}$.

It is convenient to use the notation of [13]:

$$E_1(\Gamma) = \ell_2(\Gamma), \quad E_2(\Gamma) = \ell_2(\Gamma) \times \ell_2^f,$$

$$E_3(\Gamma) = \ell_2^f(\Gamma), \quad E_4(\Gamma) = \ell_2^f(\Gamma) \times Q,$$

$\mathfrak{M}_1 =$ the class of completely metrizable spaces,

$\mathfrak{M}_2 =$ the class of metrizable spaces which are countable unions of completely metrizable closed sets,

$\mathfrak{M}_3 =$ the class of metrizable spaces which are countable unions of locally compact, locally finite-dimensional closed sets,

$\mathfrak{M}_4 =$ the class of metrizable spaces which are countable unions of locally compact closed sets.

The classes $\mathfrak{M}_1$, $\mathfrak{M}_2$, $\mathfrak{M}_3$ and $\mathfrak{M}_4$ are topological, closed hereditary and additive. For each $i = 1, 2, 3, 4$, the following hold:

2.1. *$X \in \mathfrak{M}_i(\tau)$ if and only if X can be embedded into $E_i(\Gamma)$ as a closed set* [13, 1.1].

2.2. *$X \times E_i(\Gamma) \approx E_i(\Gamma)$ for every AR $X \in \mathfrak{M}_i(\tau)$* [13, Theorem 3.2].

2.3. *X is an $E_i(\Gamma)$-manifold if and only if $X \in \mathfrak{M}_i(\tau)$ is an ANR and $X \times E_i(\Gamma) \approx X$* [13, Proposition 4.5].

The following classes are also topological, closed hereditary and additive:

$\mathfrak{M}_0 =$ the class of locally compact metrizable spaces and

$\mathfrak{M}_0^f =$ the class of locally compact, locally finite-dimensional metrizable spaces.

Observe that $\mathfrak{M}_2 = (\mathfrak{M}_1)_\sigma$, $\mathfrak{M}_3 = (\mathfrak{M}_0^f)_\sigma$ and $\mathfrak{M}_4 = (\mathfrak{M}_0)_\sigma$.

We list the necessary results of *non-separable* infinite-dimensional manifolds (cf. Preliminaries of [9]).[1] In the following, let E be a locally convex linear metric space such that $E \approx E^\omega$ or $E \approx E_f^\omega$, where

$$E_f^\omega = \{(x_i)_{i \in \boldsymbol{N}} \in E^\omega \mid x_i = 0 \text{ except for finitely many } i \in \boldsymbol{N}\}.$$

2.4 (TRIANGULATION). *For each E-manifold M, there exists a locally finite-dimensional simplicial complex K such that $M \approx |K| \times E$, where $|K|$ has the metric topology* [14, Theorem 3.4].

A *near-homeomorphism* is a map which can be approximated by homeomorphisms.

2.5 (STABILITY). *For every E-manifold M, the projection of $M \times E$ onto M is a near-homeomorphsim, hence $M \times E \approx M$* [12].

[1] These are discussed in [11].

It is said that $A \subset X$ is *E-deficient* if there exists a homeomorphism $h : X \to X \times E$ such that $h(A) \subset X \times \{0\}$.

2.6. *For a closed set K in an E-manifold M, the following are equivalent* ([2, Theorem 1] and [17, A1]):
(1) *K is a Z-set in M,*
(2) *K is a strong Z-set in M,*
(3) *K is E-deficient in M.*

For an open cover $\mathcal{U}$ of Y, two maps $f, g : X \to Y$ are *$\mathcal{U}$-homotopic* (or f is *$\mathcal{U}$-homotopic* to g) if there is a homotopy $h : X \times \mathbf{I} \to Y$ such that $h_0 = f$, $h_1 = g$ and each $h(\{x\} \times \mathbf{I})$ is contained in some $U \in \mathcal{U}$ (h is called a *$\mathcal{U}$-homotopy*).

2.7 (Z-Set Unknotting). *Let K be a Z-set in an E-manifold M and $\mathcal{U}$ an open cover of M. If a Z-embedding $h : K \to M$ is $\mathcal{U}$-homotopic to* id *then h extends to a homeomorphism $\tilde{h} : M \to M$ which is* st $\mathcal{U}$*-close to* id.

2.8 (Negligibilty of Z_σ-Sets). *In case $E \in \mathfrak{M}_1$, if K is a Z_σ-set in an E-manifold M, then the inclusion of $M \backslash K$ into M is a near-homeomorphism* [4], [2].

A map $f : X \to Y$ is a *fine homotopy equivalence* if, for each open cover $\mathcal{U}$ of Y, there is a map $g : Y \to X$ (called a $\mathcal{U}$-homotopy inverse) such that gf is $\mathcal{U}$-homotopic to id_Y and gf is $f^{-1}(\mathcal{U})$-homotopic to id_X.

2.9. *Every fine homotopy equivalence between E-manifolds is a near-homeomorphism* [6, Theorem 3.4].

3. Alteration of Bestvina-Mogilski's Paper [1]

In this section, we make alteration of §§1–5 of [1]. In order to treat non-separable spaces, we generalize the Strong Discrete Approximation Property. For each $n \in \mathbf{N}$, we say that X has the *τ-discrete n-cells property* if, for each open cover $\mathcal{U}$ of X, every map $f : \mathbf{I}^n \times \Gamma \to X$ is $\mathcal{U}$-close to a map $g : \mathbf{I}^n \times \Gamma \to X$ such that $\{g_\gamma(\mathbf{I}^n)\}_{\gamma \in \Gamma}$ is discrete in X, where $g_\gamma : \mathbf{I}^n \to X$ is defined by $g_\gamma(x) = g(x, \gamma)$. When X has the τ-discrete n-cells property for every $n \in \mathbf{N}$, it is said that it has the *τ-discrete cells property*. The Strong Discrete Approximation Property is no other than the $\aleph_0$-discrete cells property. One should note that if $X \in \mathfrak{M}$ has the τ-discrete 0-cells property then $w(X) \geq \tau$.

Recall that a map $f : X \to Y$ is *closed over $A \subset Y$* if, for each $a \in A$ and each neighborhood U of $f^{-1}(a)$ in X, there exists a neighborhood V of a in Y such that $f^{-1}(V) \subset U$, where it is possible that $f^{-1}(a) = U = \emptyset$, which implies that $f(X) \cap A$ is closed in A.

3.1. Results in § 1 of [1]. First, observe that separability is not used in the proofs of Lemmas 1.1, 1.3 and Corollary 1.2 of [1], hence they are valid for non-separable spaces. In the proof of Lemma 1.4 of [1], it is essential that each P_i is compact because $X \setminus f_{i-1}(P_{i-1})$ need to be open in X. It is a problem to prove Lemma 1.4 of [1] without separability, that is,

Problem 1. In a non-separable ANR X, if A is a Z-set and also a strong Z_σ-set in X, is A a strong Z-set in X?

As same as Lemma 1.4 of [1], separability is required in the proof of Proposition 1.7 of [1]. Then, the following is a problem.

Problem 2. Let $X \in \mathfrak{M}(\tau)$ be an ANR which has the τ-discrete cells property ($\tau > \aleph_0$). Is every Z-set in X a strong Z-set in X?

Instead of Lemma 1.4 and Proposition 1.7 of [1], we can prove the following without separability.

Proposition 3.1. *Let $X \in \mathfrak{M}(\tau)$ be an ANR which has the τ-discrete cells property. If A is a Z-set and also a strong Z_σ-set in X, then A is a strong Z-set in X.*

Proof. We can write $A = \bigcup_{i \in N \cup \{0\}} A_i$, where $A_0 \subset A_1 \subset A_2 \subset \cdots$ are strong Z-sets in X. For each open cover $\mathcal{U}$ of X, let $\mathcal{U}_{-1}$ be an open star-refinement of $\mathcal{U}$. Since X is an ANR, we have a locally finite-dimensional simplicial complex K with $\operatorname{card} K^{(0)} \le w(X)$, $f : X \to |K|$ and $g : |K| \to X$ such that gf is $\mathcal{U}_{-1}$-close to id_X, where $|K|$ admits the weak (Whitehead) topology.

We inductively construct open covers $\mathcal{U}_i$ of X, maps $h_i : |K| \to X$, open sets V_i, V_i' in X and discrete collections $\mathcal{W}_i = \{W_\sigma \mid \sigma \in K^{(i)} \setminus K^{(i-1)}\}$, $\mathcal{W}_i' = \{W_\sigma' \mid \sigma \in K^{(i)} \setminus K^{(i-1)}\}$ of open sets in X, $i \in N \cup \{0\}$, such that

(1) $\operatorname{mesh} \mathcal{U}_i < 2^{-i}$, $\operatorname{st} \mathcal{U}_i \prec \mathcal{U}_{i-1}$, $\mathcal{U}_i \prec \{V_{i-1}, X \setminus \operatorname{cl} V_{i-1}'\}$, $\operatorname{st}(W_\sigma', \mathcal{U}_i) \subset W_\sigma$ for each $\sigma \in K^{(i-1)}$,

(2) h_i is $\mathcal{U}_i$-close to h_{i-1}, $h_i \mid |K^{(i-1)}| = h_{i-1} \mid |K^{(i-1)}|$,

(3) $A_i \subset V_i' \subset \operatorname{cl} V_i' \subset V_i \subset X \setminus h_i(|K|)$,

(4) $\operatorname{cl} W_\sigma' \subset W_\sigma \subset X \setminus A$ and $h_i(\sigma) \subset \bigcup_{\sigma' \le \sigma} W_{\sigma'}'$ for each $\sigma \in K^{(i)}$,

where $h_{-1} = g$. Since $\{W_\sigma' \mid \sigma \in K^{(i)}\}$ is locally finite in X, the condition (4) implies the following condition:

(5) $\operatorname{cl} h_i(|K^{(i)}|) \subset \operatorname{cl} \bigcup_{\sigma \in K^{(i)}} W_\sigma' = \bigcup_{\sigma \in K^{(i)}} \operatorname{cl} W_\sigma' \subset X \setminus A$.

Assume that $\mathscr{U}_j$, h_j, V_j, V_j', $\mathscr{W}_j$ and $\mathscr{W}_j'$ have been obtained for $j < i$. Since $\mathrm{cl}\, V_{i-1}' \subset V_{i-1}$, $\mathrm{cl}\, W_\sigma' \subset W_\sigma$ for each $\sigma \in K^{(i-1)}$ and $\mathscr{W}_{i-1}$ is discrete in X, we can choose an open cover $\mathscr{U}_i$ of X so as to satisfy the condition (1). Let $\mathscr{U}_i'$ be an open star-refinement of $\mathscr{U}_i$. Since $\mathrm{cl}\, h_{i-1}(|K^{(i-1)}|) \cap A_i = \varnothing$ and A_i is a strong Z-set in X, we have a map $h_i' : |K| \to X$ and open neighborhoods V_i, V_i' of A_i in X such that

(6) h_i' is $\mathscr{U}_i'$-close to h_{i-1},

(7) $h_i' \,|\, |K^{(i-1)}| = h_{i-1} \,|\, |K^{(i-1)}|$ and

(8) $\mathrm{cl}\, V_i' \subset V_i \subset \mathrm{cl}\, V_i \subset X \setminus \mathrm{cl}\, h_i'(|K|)$.

Let $\mathscr{U}_i^*$ be an open refinement of $\mathscr{U}_i'$ such that

(9) $\mathscr{U}_i^* \prec \{V_i, X \setminus (\mathrm{cl}\, h_i'(|K|) \cup \mathrm{cl}\, V_i'), X \setminus \mathrm{cl}\, V_i\}$.

Since X is an ANR, $\mathscr{U}_i^*$ has an open refinement $\mathscr{U}_i''$ such that two $\mathscr{U}_i''$-close maps from an arbitrary space to X are $\mathscr{U}_i^*$-homotopic.

For each i-simplex $\sigma \in K$, $U_\sigma = \bigcup_{\sigma' < \sigma} h_{i-1}^{-1}(W_{\sigma'}')$ is an open neighborhood of $\partial\sigma$ in $|K|$. Choose an i-cell c_σ in each i-simplex $\sigma \in K$ so that $\sigma \setminus U_\sigma \subset c_\sigma$ and $\{c_\sigma \,|\, \sigma \in K^{(i)} \setminus K^{(i-1)}\}$ is discrete in $|K|$. Since X has the τ-discrete i-cells property and A is a Z-set in X, we have a map $h_i'' : \bigcup_{\sigma \in K^{(i)} \setminus K^{(i-1)}} c_\sigma \to X$ such that

(10) $h_i''(\bigcup_{\sigma \in K^{(i)} \setminus K^{(i-1)}} c_\sigma) \cap A = \varnothing$,

(11) h_i'' is $\mathscr{U}_i''$-close to h_i' and

(12) $\{h_i''(c_\sigma) \,|\, \sigma \in K^{(i)} \setminus K^{(i-1)}\}$ is discrete in X.

By using the Homotopy Extension Theorem, we can obtain a map $h_i : |K| \to X$ such that

(13) $h_i \,|\, \bigcup_{\sigma \in K^{(i)} \setminus K^{(i-1)}} c_\sigma = h_i''$,

(14) $h_i \,|\, |K^{(i-1)}| = h_i' \,|\, |K^{(i-1)}|$ and

(15) h_i is $\mathscr{U}_i^*$-homotopic to h_i',

whence $h_i \,|\, |K^{(i-1)}| = h_{i-1} \,|\, |K^{(i-1)}|$ and h_i is $\mathscr{U}_i$-close to h_{i-1}, that is, h_i satisfies the condition (2). Since h_i is $\mathscr{U}_i^*$-close to h_i', it follows from (9) that $h_i(|K|) \subset X \setminus \mathrm{cl}\, V_i$, that is, $\mathrm{cl}\, V \subset X \setminus h_i(|K|)$. Thus, the condition (3) is satisfied.

By (12) and (13), for each i-simplex $\sigma \in K$, $h_i(c_\sigma)$ has open neighborhoods W_σ, W_σ' in X such that $\mathrm{cl}\, W_\sigma' \subset W_\sigma \subset X \setminus A$ and $\mathscr{W}_i = \{W_\sigma \,|\, \sigma \in K^{(i)} \setminus K^{(i-1)}\}$ is discrete in X, hence $\mathscr{W}_i' = \{W_\sigma' \,|\, \sigma \in K^{(i)} \setminus K^{(i-1)}\}$ is also discrete in X. For each i-simplex $\sigma \in K^{(i)}$ and $x \in \sigma \setminus c_\sigma \sigma \cap U_\sigma$, choose $\sigma' < \sigma$ so that $h_{i-1}(x) \in W_{\sigma'}'$. Since h_i is $\mathscr{U}_i$-close to h_{i-1}, it follows from (1) that $h_i(x) \in \mathrm{st}(W_{\sigma'}', \mathscr{U}_i) \subset W_{\sigma'}$. Therefore, $h_i(\sigma) \subset \bigcup_{\sigma' \leq \sigma} W_{\sigma'}$. Then, the condition (4) is also satisfied.

By induction, we can obtain $\mathscr{U}_i$, h_i, V_i, $\mathscr{W}_i$ for all $i \in \boldsymbol{N}$. By the condition (2), we can define $h : |K| \to X$ by $h \,|\, |K^{(i)}| = h_i \,|\, |K^{(i)}|$. Then, h is the uniform limit of h_i by (1), hence h is continuous. It follows from (1) and (2) that h is $\mathrm{st}\, \mathscr{U}_{i+1}$-close to h_i, hence h is $\mathscr{U}_i$-close to h_i. In particular, h is $\mathscr{U}_{-1}$-close to $h_{-1} = g$, hence

hf is $\mathscr{U}$-close to id_X. Since $\mathscr{U}_i \prec \{V_i, X\backslash \mathrm{cl}\, V_i'\}$, it follows from (3) that $hf(X) \subset h(|K|) \subset \mathrm{st}(h_i(|K|, \mathscr{U}_i) \subset \mathrm{st}(X\backslash V_i, \mathscr{U}_i) \subset X\backslash \mathrm{cl}\, V_i'$, hence

$$\mathrm{cl}\, hf(X) \cap \bigcup_{i \in \boldsymbol{N} \cup \{0\}} V_i' = \emptyset,$$

which means that $\mathrm{cl}\, hf(X) \cap A = \emptyset$ because $A \subset \bigcup_{i \in \boldsymbol{N} \cup \{0\}} V_i'$. □

By using Lemma 1.4 of [1], Corollary 1.5 of [1] was obtained. But we use Michael's Theorem for local properties [7] to prove the same result without separability, that is,

PROPOSITION 3.2. *A closed set A in an ANR X is a strong Z-set in X if and only if each $a \in A$ has an open neighborhood U in X such that $A \cap U$ is a strong Z-set in U.*

PROOF. The "only if" part is trivial. To see the "if" part, let $\mathscr{P}_A$ be the property of open sets U in X such that $A \cap U$ is a strong Z-set in U. It is enough to prove that $\mathscr{P}_A$ is G-hereditary, that is, (1) if an open set U in X has $\mathscr{P}_A$ then every open set in U has $\mathscr{P}_A$; (2) if two open sets U_1 and U_2 in X have $\mathscr{P}_A$ then $U_1 \cup U_2$ has $\mathscr{P}_A$; (3) for a dicrete collection $\{U_\lambda\}_{\lambda \in \Lambda}$ open sets in X, if each U_λ has $\mathscr{P}_A$, then $\bigcup_{\lambda \in \Lambda} U_\lambda$ has $\mathscr{P}_A$. Since Lemma 1.3 of [1] is valid without separability, we have (1). And (3) is trivial.

To see (2), assume that U_1 and U_2 are open sets in X such that $A \cap U_i$ is a strong Z-set in U_i. We write $A \cap (U_1 \cup U_2) = A_1 \cup A_2$ such that $A_i \subset U_i$ and A_i is closed in $U_1 \cup U_2$, whence A_i is a strong Z-set in U_i. For each open cover $\mathscr{U}$ of $U_1 \cup U_2$, let $\mathscr{V}_1$ be an open star-refinement of $\mathscr{U}$. Then, we have a map $f_1 : U_1 \to U_1$ and an open neighborhood V_1 of A_1 in U_1 such that $V_1 \cap f_1(U_1) = \emptyset$, f_1 is $\mathscr{V}_1$-close to id and f_1 can be extended to $\tilde{f}_1 : U_1 \cup U_2 \to U_1 \cup U_2$ by $\tilde{f}_1 \mid U_2 \backslash U_1 = \mathrm{id}$, whence $V_1 \cap \tilde{f}_1(U_1 \cup U_2) = \emptyset$. Choose an open set W_1 in $U_1 \cup U_2$ so that $(U_1 \cup U_2) \cap \mathrm{cl}\, W_1 \subset V_1$. let $\mathscr{V}_2$ be an open cover of $U_1 \cup U_2$ such that

$$\mathscr{V}_2 \prec \mathscr{V}_1 \text{ and } \mathscr{V}_2 \prec \{V_1, (U_1 \cup U_2) \backslash \mathrm{cl}\, W_1\}.$$

Then, we have a map $f_2 : U_2 \to U_2$ and an open neighborhood V_2 of A_2 in U_2 such that $V_2 \cap f_2(U_2) = \emptyset$, f_2 is $\mathscr{V}_2$-close to id and f_2 can be extended to $\tilde{f}_2 : U_1 \cup U_2 \to U_1 \cup U_2$ by $\tilde{f}_2 \mid U_1 \backslash U_2 = \mathrm{id}$, whence $V_2 \cap \tilde{f}_2(U_1 \cup U_2) = \emptyset$. Observe that $W_1 \cap \tilde{f}_2 \tilde{f}_1(U_1 \cup U_2) = \emptyset$. Hence,

$$(W_1 \cup V_2) \cap \tilde{f}_2 \tilde{f}_1(U_1 \cup U_2) = \emptyset.$$

Thus, $A \cap (U_1 \cup U_2)$ is a strong Z-set in $U_1 \cup U_2$. □

Note that Corollary 1.6 of [1] is proved by Curtis [3, Lemma 7.2] without separability.

In the proof of Corolary 1.8 of [1], the following is shown without separability:

Lemma 3.3. *Let X be an ANR which has the Strong Discrete Approximation Property. Then, every compact set in X is a Z-set.*

This extends as follows:

Proposition 3.4. *Let $X \in \mathfrak{M}(\tau)$ be an ANR which has the τ-discrete cells property. Then, every closed set A in X with $w(A) < \tau$ is a Z-set in X.*

Proof. For each $n \in \boldsymbol{N}$ and each map $f : \mathbf{I}^n \to X$, let $\tilde{f} : \mathbf{I}^n \times \Gamma \to X$ be the map defined by $\tilde{f}(x, \gamma) = f(x)$. For each open cover $\mathscr{U}$ of X, $\tilde{f}$ is $\mathscr{U}$-close to a map $g : \mathbf{I}^n \times \Gamma \to X$ such that $\{g_\gamma(\mathbf{I}^n)\}_{\gamma \in \Gamma}$ is discrete in X by the τ-discrete cells property. Since $w(A) < \tau$, it is easy to see that $A \cap g_\gamma(\mathbf{I}^n) = \emptyset$ for some $\gamma \in \Gamma$, whence g_γ is $\mathscr{U}$-close to f. Then, A is a Z-set in X. □

Problem 3. In Proposition 3.4 above, is A a strong Z-set in X?

We call X a *Z_σ-space* (or a *strong Z_σ-space*) if X itself is a Z_σ-set (or a strong Z_σ-set) in X. By Baire's Theorem, any completely metrizable spaces is not a (strong) Z_σ-space. It is a problem whether Lemma 1.9 of [1] can be generalized to non-separable spaces, that is,

Problem 4. Let $X \in \mathfrak{M}(\tau)$ be an ANR which is a strong Z_σ-space ($\tau > \aleph_0$). Does X have the τ-discrete cells property?

Lemmas 1.10 and 1.11 of [1] are valid for non-separable spaces (cf. their proofs).

3.2. Results in §2 of [1]. Observe that Propositions 2.1 and 2.2 of [1] are proved whitout separability. Thus, they are valid for non-separable spaces.

In the proof of Proposition 2.3 of [1], Lemmas 1.4, 1.9 and Proposition 1.7 of [1] are applied, where separability is necessary. Moreover, separability is also used in the proof of 2.3 of [1] itself (the last paragraph). By adding the condition on $\mathscr{C}$ that $\mathbf{I}^n \times \Gamma \in \mathscr{C}$ for each $n \in \boldsymbol{N}$, we can extend the result to ANR's X with

$w(X) = \tau$. The proof is basically same as [1]. Since the proof in [1] contains some misprints and some of details are not easy to follow, we give a complete proof, where we make some small changes in the arguments to make the proof clear.

PROPOSITION 3.5. *Let $\mathscr{C}$ be a closed hereditary additive topological class of spaces such that $\mathbf{I}^n \times \Gamma \in \mathscr{C}$ for each $n \in \boldsymbol{N}$, and let X be an ANR with $w(X) = \tau$. If X is a strongly $\mathscr{C}$-universal strong Z_σ-space, then X is strongly $\mathscr{C}_\sigma$-universal.*

PROOF. Since $\mathbf{I}^n \times \Gamma \in \mathscr{C}$ for each $n \in \boldsymbol{N}$, if X is strongly $\mathscr{C}$-universal then X has the τ-discrete cells property. By Proposition 3.1, every Z-set in X is a strong Z-set. Then, by Proposition 2.2 of [1], it suffices to show that each open set $U \neq \emptyset$ in X is $\mathscr{C}_\sigma$-universal. Note that U is an ANR with $w(U) = \tau$. Since U is an F_σ-set in X, U is a strong Z_σ-space. It follows from Proposition 2.1 of [1] that U is strongly $\mathscr{C}$-universal. Thus, we may assume that $U = X$, whence it suffices to show that X is $\mathscr{C}_\sigma$-universal.

Let $f : C \to X$ be a map of $C \in \mathscr{C}_\sigma$. In case C is an open set in some member of $\mathscr{C}$, it is proved by the same way as [1] that f can be approximated by Z-embeddings. We now consider the general case $C \in \mathscr{C}_\sigma$, that is, $C = \bigcup_{i \in \boldsymbol{N}} C_i$, where $C_1 \subset C_2 \subset \cdots$ are closed in C and $C_i \in \mathscr{C}$.[2] We write $X = \bigcup_{i \in \boldsymbol{N}} X_i$, where $X_1 \subset X_2 \subset \cdots$ are strong Z-sets in X. Given an admissible metric d for X, let $C(\mathbf{I}^k, X)$ be the space of all (continuous) maps from $\mathbf{I}^k$ to X with the sup-metric induced by d. For each $k \in \boldsymbol{N}$, since $C(\mathbf{I}^k, X)$ has the same weight as X, there is a map $g_k : \mathbf{I}^k \times \Gamma \to X$ such that $\{g_{k,\gamma} \mid \gamma \in \Gamma\}$ is dense in $C(\mathbf{I}^k, X)$, where $g_{k,\gamma} : \mathbf{I}^k \to X$ is defined by $g_{k,\gamma}(x) = g_k(x, \gamma)$. Given an open cover $\mathscr{U}$ of X, let $\mathscr{U}_0$ be an open star-refinement of $\mathscr{U}$. By induction, we shall construct maps $f_i : C \to X$, $g_k^i : \mathbf{I}^k \times \Gamma \to X$ $(k \le i)$, and open covers $\mathscr{U}_i$ of $X \setminus (f_i(C_i) \cup X_i)$, $i \in \boldsymbol{N}$, such that

(1) $f_i | C_i$ is a Z-embedding,
(2) $f_i | C_{i-1} = f_{i-1} | C_{i-1}$,
(3) $f_i(C \setminus C_i) \cap f_i(C_i) = \emptyset$,
(4) f_i is closed over $f_i(C_i) \cup X_i$,
(5) $f_i \mid C \setminus C_{i-1}$ is $\mathscr{U}_{i-1}$-close to $f_{i-1} \mid C \setminus C_{i-1}$,
(6) $\operatorname{cl} f_i(C \setminus C_{i-1}) \cap (X_i \setminus (f_{i-1}(C_{i-1}) \cup X_{i-1})) = \emptyset$,
(7) $\operatorname{st} \mathscr{U}_i \prec \mathscr{U}_{i-1}$,
(8) $\operatorname{diam} U < \min\{2^{-i}, \frac{1}{2} d(U, f_i(C_i) \cup X_i)\}$ for each $U \in \mathscr{U}_i$,
(9) $g_k^i(\mathbf{I}^k \times \Gamma)$ is a Z-set in X,

[2] In the case C is an open set in some member of $\mathscr{C}$, we can assume that $C_i \subset \operatorname{int} C_{i+1}$. However, this assumption cannot be used in the general case.

(10) $f_i(C) \cap \bigcup_{k \le j \le i} g_k^j(\mathbf{I}^k \times \Gamma) = \emptyset$,

(11) $\{g_{k,\gamma}^i \,|\, \gamma \in \Gamma\}$ is 2^{-i}-dense in $C(\mathbf{I}^k, X)$, that is, each $g \in C(\mathbf{I}^k, X)$ is 2^{-i}-close to some $g_{k,\gamma}^i$,

where $f_0 = f$ and $C_0 = X_0 = \emptyset$.

Assume that f_{i-1}, g_k^{i-1} $(k \le i-1)$ and $\mathscr{U}_{i-1}$ have been obtained. Since $f_{i-1}(C_{i-1})$ is a Z-set in X by (1) and $\mathbf{I}^k \times \Gamma \in \mathscr{C}$, we can apply the strong $\mathscr{C}$-universality of X to find Z-embeddings $g_k^i : \mathbf{I}^k \times \Gamma \to X$ $(k \le i)$ such that

$$g_k^i(\mathbf{I}^k \times \Gamma) \cap f_{i-1}(C_{i-1}) = \emptyset,$$

and each g_k^i is $2^{-(i+1)}$-close to g_k, hence it satisfies (9) and (11).

Now, we denote

$$W = X \backslash (f_{i-1}(C_{i-1}) \cup X_{i-1}).$$

Then, $\mathscr{U}_{i-1}$ is an open cover of W. Let $\mathscr{V}$ be an open star-refinement of $\mathscr{U}_{i-1}$. Since W is open in X, W is a strong Z_σ-space and has τ-discrete cells property. By Proposition 3.1, each Z-set in W is a strong Z-set. Note that $X_i \cap W$ is a strong Z-set in W by Proposition 3.2 and W is strongly $\mathscr{C}$-universal by Proposition 2.1 of [1]. We apply the special case to the open set $C_i \backslash C_{i-1}$ in $C_i \in \mathscr{C}$, and use the Homotopy Extension Theorem to construct a map $h : C \backslash C_{i-1} \to W$ such that

(12) $h \,|\, C_i \backslash C_{i-1}$ is a Z-embedding,

(13) h is $\mathscr{V}$-close to $f_{i-1} \,|\, C \backslash C_{i-1}$,

(14) $\operatorname{cl} h(C \backslash C_{i-1}) \cap W \cap (X_i \cup \bigcup_{k \le j \le i} g_k^j(\mathbf{I}^k \times \Gamma)) = \emptyset$.

Since $h(C_i \backslash C_{i-1}) \cup (X_i \cap W)$ is a strong Z-set in W, we apply Lemma 1.1 of [1] to obtain a map $\tilde{h} : C \backslash C_{i-1} \to W$ such that

(15) $\tilde{h}$ is $\mathscr{V}$-close to h, hence it is $\mathscr{U}_{i-1}$-close to $f_{i-1}|C \backslash C_{i-1}$ by (13),

(16) $\operatorname{cl} \tilde{h}(C \backslash C_{i-1}) \cap W \cap (X_i \cup \bigcup_{k \le j \le i} g_k^j(\mathbf{I}^k \times \Gamma)) = \emptyset$,

(17) $\tilde{h} \,|\, C_i \backslash C_{i-1} = h \,|\, C_i \backslash C_{i-1}$,

(18) $\tilde{h}(C \backslash C_i) \cap \tilde{h}(C_i \backslash C_{i-1}) = \emptyset$,

(19) $\tilde{h}$ is closed over $\tilde{h}(C_i \backslash C_{i-1}) \cup (X_i \cap W)$.

For each $z \in C_{i-1}$ and $\varepsilon > 0$, since f_{i-1} is continuous, we have a neighborhood V of z in C such that $y \in V$ implies $d(f_{i-1}(y), f_{i-1}(z)) < \varepsilon/2$. For each $y \in V \backslash C_{i-1}$, choose $U \in \mathscr{U}_{i-1}$ so that $\tilde{h}(y), f_{i-1}(y) \in U$, whence we have $d(\tilde{h}(y), f_{i-1}(y)) < \frac{1}{2} d(f_{i-1}(y), f_{i-1}(z))$ by (8) for $i-1$. Then, we have

$$d(\tilde{h}(y), f_{i-1}(z)) \le d(\tilde{h}(y), f_{i-1}(y)) + d(f_{i-1}(y), f_{i-1}(z))$$
$$< \tfrac{3}{2} d(f_{i-1}(y), f_{i-1}(z)) < \varepsilon.$$

Therefore, as an extension of $\tilde{h}$, we can obtain the map $f_i : C \to X$ satisfying (2), which clearly satsfies (3), (5), (6) and (10) (cf. (18), (15), (16)).

Since $f_i|C_{i-1} = f_{i-1}|C_{i-1}$ and $f_i \mid C_i \backslash C_{i-1} = h \mid C_i \backslash C_{i-1}$ are injective and

$$f_i(C_i \backslash C_{i-1}) \cap f_{i-1}(C_{i-1}) = \tilde{h}(C_i \backslash C_{i-1}) \cap f_{i-1}(C_{i-1}) = \emptyset,$$

it follows that $f_i|C_i$ is injective. If f_i satisfies (4), that is, f_i is closed over $f_i(C_i) \cup X_i$, then $f_i|C_i$ is an embedding.

Suppose that f_i is not closed over $f_i(C_i) \cup X_i$. Then, there exist $a \in f_i(C_i) \cup X_i$, a neighborhood U of $f_i^{-1}(a)$ in C (we allow $U = f_i^{-1}(a) = \emptyset$) and a sequence $(z_n)_{n \in \boldsymbol{N}}$ in $C \backslash U$ with $\lim f_i(z_n) = a$. Since $f_i|C_{i-1} = f_{i-1}|C_{i-1}$ is a closed embedding into X by (1) for $i-1$, we have $z_n \in C \backslash C_{i-1}$ for sufficiently large $n \in \boldsymbol{N}$. Since $f_i \mid C \backslash C_{i-1}$ is closed over $f_i(C_i \backslash C_{i-1}) \cup (X_i \cap W)$ by (19), it follows that $a \notin f_i(C_i \backslash C_{i-1}) \cup (X_i \cap W)$. Recall $a \in f_i(C_i) \cup X_i$. Then, we have

$$a \in f_i(C_{i-1}) \cup (X_i \backslash W) = f_{i-1}(C_{i-1}) \cup X_{i-1}.$$

For suffiently large $n \in \boldsymbol{N}$, we can choose $U_n \in \mathscr{U}_{i-1}$ so that $f_i(z_n), f_{i-1}(z_n) \in U_n$ by (5), whence

$$d(f_{i-1}(z_n), a) \le d(f_i(z_n), f_{i-1}(z_n)) + d(f_i(z_n), a) < \tfrac{3}{2} d(f_i(z_n), a).$$

Then, $\lim f_{i-1}(z_n) = a$, which implies that $f_{i-1}^{-1}(a) \neq \emptyset$ by (4) for $i-1$. Since $f_{i-1}^{-1}(a) \subset C_{i-1}$ by (3) for $i-1$, it follows from (2) that $f_{i-1}^{-1}(a) \subset f_i^{-1}(a) \subset U$. Again by (4) for $i-1$, we have a neighborhood V of a in X such that $f_{i-1}^{-1}(V) \subset U$. For sufficiently large $n \in \boldsymbol{N}$, $f_{i-1}(z_n) \in V$, hence $z_n \in f_{i-1}^{-1}(V) \subset U$. This is a contradiction. Therefore, f_i satisfies (4).

To see (1), it remains to show that $f_i(C_i)$ is a Z-set in X. Observe that

$$X \backslash (f_i(C_i) \cup X_{i-1}) = W \backslash h(C_i \backslash C_{i-1}),$$

which is open in W. Then, $f_i(C_i) \cup X_{i-1}$ is closed in X, hence $f_i(C_i) \cup X_i$ is also closed in X. Since $f_{i-1}(C_{i-1}) \cup X_i$ is a Z-set in X and $f_i(C_i \backslash C_{i-1}) = h(C_i \backslash C_{i-1})$ is a Z-set in $W = X \backslash (f_{i-1}(C_{i-1}) \cup X_i)$, it follows that $f_i(C_i) \cup X_i$ is a Z-set in X. By (3) and (4), we can see that $f_i(C_i)$ is closed in $f_i(C_i) \cup X_i$. Therefore, $f_i(C_i)$ is a Z-set in X.

Finally, by choosing an open cover $\mathscr{U}_i$ of $X \backslash (f_i(C_i) \cup X_i)$ so as to satisfy (7) and (8), we can obtain f_i, g_k^i $(k \le i)$ and $\mathscr{U}_i$ which satisfy all conditions (1)–(11).

By (2), we can define $f_* : C \to X$ defined by $f_*|C_i = f_i|C_i$. It follows from (5) and (8) that f_* is 2^{-i+1}-close to f_i. Thus, f_* is the uniform limit of $(f_i)_{i \in \boldsymbol{N}}$, so f_* is continuous. By (1) and (3), f_* is injective. Then, to see that f_* is a Z-embedding, it remains to show that f_* is a closed map and $f_*(C)$ is a Z-set in X.

Now, assume that f_* is not closed. Then, we have a sequence $(z_n)_{n \in \boldsymbol{N}}$ in

C such that $(z_n)_{n \in \boldsymbol{N}}$ has no convergent subsequences but $(f_*(z_n))_{n \in \boldsymbol{N}}$ converges to some $a \in X$. Let $a \in X_m \backslash X_{m-1}$. Then, $z_n \in C \backslash C_m$ for sufficiently large $n \in \boldsymbol{N}$. Otherwise, C_m contains a subsequence of $(z_n)_{n \in \boldsymbol{N}}$, which is convergent because $f_* | C_m = f_m | C_m$ is a closed embedding. From (2), (5) and (7), it follows that $f_* \mid C \backslash C_m$ is st $\mathscr{U}_m$-close to $f_m \mid C \backslash C_m$. By (8), we have $x_n, y_n \in X$ for sufficiently large $n \in \boldsymbol{N}$ such that

$$d(f_*(z_n), x_n) < \tfrac{1}{2} d(f_*(z_n), a),$$

$$d(x_n, y_n) < \tfrac{1}{2} d(x_n, a) \quad \text{and}$$

$$d(y_n, f_m(z_n)) < \tfrac{1}{2} d(y_n, a).$$

Then, $(f_m(z_n))_{n \in \boldsymbol{N}}$ also converges to a, hence

$$a \in \operatorname{cl} f_m(C \backslash C_m) \subset \operatorname{cl} f_m(C \backslash C_{m-1}),$$

which implies that $a \in f_{m-1}(C_{m-1})$ by (6). By (1), (2) and (3), there is unique $c \in C_{m-1}$ such that $f_m(c) = f_{m-1}(c) = a$. Since $(z_n)_{n \in \boldsymbol{N}}$ does not converge to c and f_m is closed over $f_m(C_m)$ by (4), we have a neighborhood V of a in X such that infinitely many z_n are not contained in $f_m^{-1}(V)$, that is, infinitely many $f_m(z_n)$ are not contained in V. This is a contradiction. Therefore, f_* is a closed map.

To see that $f_*(C)$ is a Z-set in X, let $g : \mathbf{I}^k \to X$ be a map and $\varepsilon > 0$. Choose $j \in \boldsymbol{N}$ so that $2^{-j} < \varepsilon$. Then, g is ε-close to some $g^j_{k,\gamma}$ by (11), whence $f_i(C_i) \cap g^j_{k,\gamma}(\mathbf{I}^k) = \varnothing$ for every $i \geq j$ by (10). Since $f_*(C) = \bigcup_{i \geq j} f_i(C_i)$, it follows that $f_*(C) \cap g^j_{k,\gamma}(\mathbf{I}^k) = \varnothing$. Hence, $f_*(C)$ is a Z-set in X. □

By the above version of Proposition 2.3 of [1], Corollary 2.4 of [1] is valid for spaces X with $w(X) = \tau$ if $\mathbf{I}^n \times \Gamma \in \mathscr{C}$ for each $n \in \boldsymbol{N}$.

In this paper, the weak product of a space X with a basepoint $* \in X$ is denoted by X^ω_f intead of $W(X, *)$. In the proof of Proposition 2.5 of [1],[3] when $w(X) = \tau > \aleph_0$, we have $\tilde{X}^\omega \approx \ell_2(\Gamma)$ by Theorem 5.1 of [16]. Then, X^ω and X^ω_f can be regarded as homotopy dense subsets of $\ell_2(\Gamma)$. Hence, every Z-set in X^ω (or X^ω_f) is a strong Z-set. In any other part, separability is not necessary.[4] Then, Proposition 2.5 [1] valid for a non-separable AR X.

Proposition 2.6 of [1] is also valid for non-separable spaces because the proof does not require separability.

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[3] In Proposition 2.5 of [1], X should be an AR (see the proof).

[4] p. 302 of [1], lines 4 and 5: $\frac{1}{\delta(f(c))} - k$ should be $\frac{2^{-k}}{\delta(f(c))} - 1$.

—, line 10: $\delta(f(c)) \leq 2\delta(f'(c))$ should be $\frac{2}{3}\delta(f'(c)) \leq \delta(f(c)) \leq 2\delta(f'(c))$.

In the proof of Proposition 2.7 of [1], we cannot assume that $\mathscr{U}$ is countable when X is non-separable. However, by Stone's Theorem (cf. [5, 4.4.1]) and Proposition 2.1 of [1], we can assume that $\mathscr{U}$ is locally finite σ-discrete, whence it is not difficult to modify the proof to be valid for non-separable spaces. We can also apply Michael's Theorem for local properies [7] to prove this proposition without separability.

3.3. Results in § 3 of [1]. A subset $X \subset M$ is said to be *homotopy dense* if there exists a deformation $h : M \times \mathbf{I} \to M$ such that $h_0 = \mathrm{id}$ and $h_t(M) \subset X$ for $t > 0$. By [15], X is homotopy dense in an ANR M if and only if $M \backslash X$ is locally homotopy negligible in M. A strongly $\mathscr{C}$-universal homotopy dense Z_σ-set $X \subset M$ is called a *$\mathscr{C}$-absorbing set* in M. By just replacing "s-manifold" by "$\ell_2(\Gamma)$-manifold" in § 3 of [1], we can obtain the non-separable version of Theorems 3.1, 3.2 and 3.3 of [1]. In fact, all facts used in the proofs hold in the non-separable case (cf. 2.6–2.9).

3.4. Results in § 4 of [1]. Observe that Lemma 4.1 of [1] is valid for $\ell_2(\Gamma)$-manifolds (cf. 2.6, 2.7 and [16, Proposition 2.1]). In Theorem 4.2 of [1], if Y is non-separable but $w(Y) \le \tau$, we have an $\ell_2(\Gamma)$-manifold $M = \tilde{Y} \times \ell_2(\Gamma)$, where $\tilde{Y} \in \mathfrak{M}_1(\tau)$ is an ANR which contains Y as a homotopy dense set (cf. [15, Proposition 4.1], [10]). Note that the projection $\mathrm{pr}_1 : \tilde{Y} \times \ell_2(\Gamma) \to \tilde{Y}$ is a fine homotopy equivalence. Thus, we have

THEOREM 3.6. *For each ANR $Y \in \mathfrak{M}(\tau)$, there exists an $\ell_2(\Gamma)$-manifold M such that, for every $\mathscr{C}$-absorbing set $X \subset M$, there is a fine homotopy equivalence $f : X \to Y$.* □

Then, we have the non-separable version of Corollary 4.3 of [1], where "s-manifold" is just replaced by "$\ell_2(\Gamma)$-manifold".

3.5. Results in § 5 of [1]. In Lemma 5.2 of [1], if M is an $\ell_2(\Gamma)$-manifold, then $\tilde{\Omega}$ and $\tilde{X}$ in the proof are $\ell_2(\Gamma)$-manifolds by Toruńczyk characterization of $\ell_2(\Gamma)$-manifolds, and $\tilde{i} : \tilde{\Omega} \to \tilde{X}$ is a near-homeomorphism by [2, Corollary]. Thus, by just replacing "s-manifold" by "$\ell_2(\Gamma)$-manifold", we have the non-separable version of Lemma 5.2 of [1].

In the proof of Theorem 5.1 of [1], Theorem 2.3 of [1] is used. As saw in the above, the condition that $\mathbf{I}^n \times \Gamma \in \mathscr{C}$ for each $n \in \mathbf{N}$ is required when $w(X) = \tau$. Then, the non-seprable version of Theorem 5.1 of [1] is as follows:

Theorem 3.7. *Let $\mathscr{C}$ be a closed hereditary additive topological class of spaces such that $\mathbf{I}^n \times \Gamma \in \mathscr{C}$ for each $n \in \boldsymbol{N}$. Suppose that Ω is a $\mathscr{C}$-absorbing set in an $\ell_2(\Gamma)$-manifold M and X is a strong $\mathscr{C}$-universal ANR with $w(X) = \tau$ which is written as $X = \bigcup_{i \in \boldsymbol{N}} X_i$, where each X_i is a strong Z-set in X and $X_i \in \mathscr{C}$. Then, every fine homotopy equivalence $f : \Omega \to X$ is a near-homeomorphism.* □

Thus, we have the following non-separable version of Theorem 5.3 of [1]

Theorem 3.8. *Let $\mathscr{C}$ be a closed hereditary additive topological class of spaces such that $\mathbf{I}^n \times \Gamma \in \mathscr{C}$ for each $n \in \boldsymbol{N}$. Suppose that there exists a $\mathscr{C}$-absorbing set Ω in $\ell_2(\Gamma)$. Then, X is homeomorphic to Ω if and only if $X \in \mathscr{C}_\sigma$, X is a strongly $\mathscr{C}$-universal AR which is a strong Z_σ-space.* □

The non-separable version of Corollary 5.4 of [1] is true when "*s*" is replaced by "$\ell_2(\Gamma)$" and the condition $\Gamma \in \mathscr{C}$ is added. Corollary 5.5 is valid for non-separable spaces.

Theorem 3.9. *Let $\mathscr{C}$ be a closed hereditary additive topological class of spaces such that $\Gamma \in \mathscr{C}$ and $C \times \mathbf{I} \in \mathscr{C}$ for each $C \in \mathscr{C}$.*[5] *Suppose that there exists a $\mathscr{C}$-absorbing set Ω in $\ell_2(\Gamma)$. Then, the following hold*:

(1) *Every $\ell_2(\Gamma)$-manifold contains a $\mathscr{C}$-absorbing set.*
(2) (Triangulation) *X is a Ω-manifold if and only if there exists a locally finite-dimensional simplicial complex K with* card $K^{(0)} \leq \tau$ *such that $X \approx |K| \times \Omega$, where $|K|$ admits the metric topology.*
(3) (Open Embedding) *Every connected Ω-manifold can be embedded in Ω as an open set.*[6]
(4) *Every $\mathscr{C}$-absorbing set in an $\ell_2(\Gamma)$-manifold is a Ω-manifold, and every Ω-manifold can be embedded in an $\ell_2(\Gamma)$-manifold as a $\mathscr{C}$-absorbing set.*

The assertions (1), (2) and (3) are the non-separable versions of Corollaries 5.6 and 5.7 of [1]. For the assertion (4), the first half and the second half are respectively the non-separable versions of the facts implicitly showed in the proofs of Corolaries 5.7 and 5.6(ii) of [1].

As the above results are based on the existence of an $\mathscr{C}$-absorbing set in $\ell_2(\Gamma)$, the following problem is fundamental:

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[5] By induction, we have $C \times \mathbf{I}^n \in \mathscr{C}$ for each $C \in \mathscr{C}$ and $n \in \boldsymbol{N}$. In particular, $\mathbf{I}^n \times \Gamma \in \mathscr{C}$ for each $n \in \boldsymbol{N}$.

[6] To avoid the case that X has components more that τ, we have to assume that X is connected or $w(X) = \tau$.

PROBLEM 5. For what class $\mathscr{C}$, does there exist a $\mathscr{C}$-absorbing set in $\ell_2(\Gamma)$? Or, for given a model space $E \in \mathfrak{M}(\tau)$, let $\mathscr{C}_E$ be the class of spaces which can be embedded in E as closed sets. Can E be embedded in $\ell_2(\Gamma)$ as a $\mathscr{C}_E$-absorbing set?

4. The Proof of Main Theorem

The following is the answer to Problem 5 for $\mathfrak{M}_2(\tau)$, $\mathfrak{M}_3(\tau)$ and $\mathfrak{M}_4(\tau)$.

PROPOSITION 4.1. *For each* $i = 2, 3, 4$, *the space* $E_i(\Gamma)$ *can be embedded in* $\ell_2(\Gamma)$ *as an* $\mathfrak{M}_i(\tau)$*-absorbing set.*

PROOF. Note that $E_i(\Gamma)_f^\omega \approx E_i(\Gamma)$ (cf. [13, p. 61, Footnote (3)]). It follows from [1, Proposition 2.5] that $E_i(\Gamma)$ is strongly $\mathfrak{M}_i(\tau)$-universal. Since $E_i(\Gamma)$ is a Z_σ-space, it follows from [17, A1] that $E_i(\Gamma)$ is a strong Z_σ-space. It remains to show that each $E_i(\Gamma)$ can be embedded in $\ell_2(\Gamma)$ as a homotopy dense set.

First, $E_3(\Gamma) = \ell_2^f(\Gamma)$ itself is homotopy dense in $\ell_2(\Gamma)$. Then, it follows that $E_4(\Gamma) = \ell_2^f(\Gamma) \times Q$ is homotopy dense in $\ell_2(\Gamma) \times Q \approx \ell_2(\Gamma)$. Since ℓ_2^f is homotopy dense in ℓ_2, it follows that $E_2(\Gamma) = \ell_2(\Gamma) \times \ell_2^f$ is homotopy dense in $\ell_2(\Gamma) \times \ell_2^f \approx \ell_2(\Gamma)$. Thus, each $E_i(\Gamma)$ can be embedded in $\ell_2(\Gamma)$ as a homotopy dense set. □

By combining the following proposition and Proposition 3.5, we can obtain Main Theorem.

PROPOSITION 4.2. *Let X be a connected metrizable space. For each* $i = 2, 3, 4$, *X is an $E_i(\Gamma)$-manifold* (*or* $X \approx E_i(\Gamma)$) *if and only if* $X \in \mathfrak{M}_i(\tau)$ *is an ANR* (*or an AR*) *which is a strongly* $\mathfrak{M}_i(\tau)$*-universal strong* Z_σ*-space.*

PROOF. First, we show the "only if" part. By 2.3, $X \in \mathfrak{M}_i(\tau)$ is an ANR (or an AR) and $X \approx X \times E_i(\Gamma)$. Since every Z-set in X is a strongly Z-set by [17, A1] and $E_i(\Gamma)$ is strongly $\mathfrak{M}_i(\tau)$-universal, it follows from [1, Proposition 2.6] that X is strongly $\mathfrak{M}_i(\tau)$-universal. Moreover, X is a strong Z_σ-space because so is $E_i(\Gamma)$.

Next, we prove the "if" part. By Theorem 3.6, we have an $\ell_2(\Gamma)$-manifold M such that, for every $\mathfrak{M}_i(\tau)$-absorbing set W in M, there is a fine homotopy equivalence $\varphi : W \to X$. By Theorem 3.9(1), M contains an $\mathfrak{M}_i(\tau)$-absorbing set W. Then, we have a fine homotopy equivalence $\varphi : W \to X$, which is a near-homeomorphism by Theorem 3.7. Hence, $X \approx W$ is an $E_i(\Gamma)$-manifold by Theorem 3.9(4). If X is an AR, $M \approx \ell_2(\Gamma)$ in the above, whence we have $X \approx W \approx E_i(\Gamma)$ by [1, Theorem 3.1] and Proposition 4.1. □

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