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# DIAGONALIZATION OF ELEMENTS OF FREUDENTHAL *R*-VECTOR SPACE AND SPLIT FREUDENTHAL *R*-VECTOR SPACE

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## Introduction

The following diagonalization problems are well known. Any Hermitian matrix $X \in M(n, K)$, $K = \boldsymbol{R}, \boldsymbol{C}$ and $\boldsymbol{H}$, can be transformed to a diagonal form by some element of the groups $SO(n)$, $SU(n)$ and $Sp(n)$, respectively. Moreover we know that any Hermitian matrix $X \in M(3, K)$, $K = \mathfrak{C}, \mathfrak{C}^C$ and $\mathfrak{C}'$, can be transformed to a diagonal form by some element of the groups $F_4$, $E_6$ and $Sp(4)/Z_2$ $(\subset E_{6(6)})$, respectively (Freudenthal [1], Yokota [9], Shukuzawa and Yokota [6]). Recently, it was shown that any element of the Freudenthal $\boldsymbol{C}$-vector space $\mathfrak{P}^C$ can be transformed to a diagonal form by some element of the group $E_7$ (Miyasaka, Yasukura and Yokota [4]).

In this paper, we shall show that any element of the Jordan algebra $\mathfrak{J}^1$ can be transformed to a diagonal form by some element of the group $F_4$ $(\subset E_6(\mathfrak{J}^1) \cong E_{6(-26)})$ (Theorem 4), and any element of the Freudenthal $\boldsymbol{R}$-vector space $\mathfrak{P}$ (resp. split Freudenthal $\boldsymbol{R}$-vector space $\mathfrak{P}'$) can be transformed to a diagonal form by some element of the group $(U(1) \times E_6)/\boldsymbol{Z}_3$ $(\subset E_{7(-25)})$ (resp. $SU(8)/\boldsymbol{Z}_2$ $(\subset E_{7(7)})$) (Theorem 9 (resp. Theorem 13)).

Consequently, we have completed the diagonalization about all of the representation spaces which appear in Freudenthal-Yokota construction of the exceptional Lie groups of type $F_4$, $E_6$ and $E_7$.

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## 1. Preliminaries

### 1.1. Cayley Algebras $\mathfrak{C}$ and $\mathfrak{C}'$

We denote by $\boldsymbol{R}, \boldsymbol{C} = \{a_0 + a_1e_1 \mid a_k \in \boldsymbol{R}\}$ $(e_1{}^2 = -1)$ and $\boldsymbol{H} = \{a_0 + a_1e_1 + a_2e_2 + a_3e_3 \mid a_k \in \boldsymbol{R}\}$ $(e_k{}^2 = -1, e_1e_2 = e_3, e_2e_3 = e_1, e_3e_1 = e_2, e_ke_l = -e_le_k\ (k \neq l))$,

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the fields of real numbers, complex numbers and quaternions, respectively. In a natural way, $\boldsymbol{R} \subset \boldsymbol{C} \subset \boldsymbol{H}$. Now, let $\mathfrak{C} = \boldsymbol{H} \oplus \boldsymbol{H}e$ (resp. $\mathfrak{C}' = \boldsymbol{H} \oplus \boldsymbol{H}e'$) be the Cayley algebra (resp. split Cayley algebra) over $\boldsymbol{R}$ with the multiplication

$$(a + be)(c + de) = (ac - \bar{d}b) + (b\bar{c} + da)e, \quad a + be, c + de \in \mathfrak{C}$$

$$(\text{resp. } (a + be')(c + de') = (ac + \bar{d}b) + (b\bar{c} + da)e', \quad a + be', c + de' \in \mathfrak{C}'),$$

where $\bar{d}$ is the conjugate element of $d \in \boldsymbol{H}$. Moreover, in $\mathfrak{C}$ (resp. $\mathfrak{C}'$), the conjugation and the inner product are defined as follows: $\overline{a + be} = \bar{a} - be$ (resp. $\overline{a + be'} = \bar{a} - be'$), and $(x, y) = \frac{1}{2}(x\bar{y} + y\bar{x})$, $x, y \in \mathfrak{C}$ (resp. $\mathfrak{C}'$).

### 1.2. Jordan Algebras $\mathfrak{J}(3, \mathfrak{C})$, $\mathfrak{J}(3, \mathfrak{C}')$ and $\mathfrak{J}(1, 2, \mathfrak{C})$

Let

$$\mathfrak{J}(3, K) = \{X \in M(3, K) \mid X^* = X\}$$

$$= \left\{X = X(\xi_k, x_k) = \begin{pmatrix} \xi_1 & x_3 & \bar{x}_2 \\ \bar{x}_3 & \xi_2 & x_1 \\ x_2 & \bar{x}_1 & \xi_3 \end{pmatrix} \,\middle|\, \xi_k \in \boldsymbol{R}, x_k \in K\right\}, \quad K = \mathfrak{C}, \mathfrak{C}',$$

$$\mathfrak{J}(1, 2, \mathfrak{C}) = \{X \in M(3, \mathfrak{C}) \mid I_1 X^* I_1 = X\}$$

$$= \left\{X = X(\xi_k, x_k) = \begin{pmatrix} \xi_1 & x_3 & -\bar{x}_2 \\ -\bar{x}_3 & \xi_2 & x_1 \\ x_2 & \bar{x}_1 & \xi_3 \end{pmatrix} \,\middle|\, \xi_k \in \boldsymbol{R}, x_k \in \mathfrak{C}\right\},$$

where $I_1 = \operatorname{diag}(-1, 1, 1)$, be the Jordan algebras with the Jordan multiplication

$$X \circ Y = \frac{1}{2}(XY + YX).$$

Hereafter, we shall briefly denote $\mathfrak{J}(3, \mathfrak{C})$, $\mathfrak{J}(3, \mathfrak{C}')$ and $\mathfrak{J}(1, 2, \mathfrak{C})$ by $\mathfrak{J}$, $\mathfrak{J}'$ and $\mathfrak{J}^1$, respectively. In $\mathfrak{J}$, $\mathfrak{J}'$ and $\mathfrak{J}^1$, we define the inner product $(X, Y)$, the Freudenthal multiplication $X \times Y$ and the determinant $\det X$ respectively by

$$(X, Y) = \operatorname{tr}(X \circ Y),$$

$$X \times Y = \frac{1}{2}(2X \circ Y - \operatorname{tr}(X)Y - \operatorname{tr}(Y)X + (\operatorname{tr}(X)\operatorname{tr}(Y) - (X, Y))E),$$

$$\det X = \frac{1}{3}(X, X \times X),$$

where $E$ is the unit matrix.

### 1.3. Freudenthal $\boldsymbol{R}$-Vector Spaces $\mathfrak{P}$ and $\mathfrak{P}'$

We define the Freudenthal $\boldsymbol{R}$-vector space $\mathfrak{P}$ (resp. split Freudenthal $\boldsymbol{R}$-vector space $\mathfrak{P}'$) by

$$\mathfrak{P} = \mathfrak{J} \oplus \mathfrak{J} \oplus \boldsymbol{R} \oplus \boldsymbol{R} \quad (\text{resp. } \mathfrak{P}' = \mathfrak{J}' \oplus \mathfrak{J}' \oplus \boldsymbol{R} \oplus \boldsymbol{R})$$

with the inner product

$$(P, Q) = (X, Z) + (Y, W) + \xi\zeta + \eta\omega,$$

for $P = (X, Y, \xi, \eta)$, $Q = (Z, W, \zeta, \omega) \in \mathfrak{P}$ (resp. $\mathfrak{P}'$). For $\phi \in \mathfrak{e}_{6(-26)}$ (resp. $\mathfrak{e}_{6(6)}$), $A, B \in \mathfrak{J}$ (resp. $\mathfrak{J}'$) and $v \in \boldsymbol{R}$, we define an $\boldsymbol{R}$-linear mapping $\Phi(\phi, A, B, v) : \mathfrak{P} \to \mathfrak{P}$ (resp. $\mathfrak{P}' \to \mathfrak{P}'$) by

$$\Phi(\phi, A, B, v)\begin{pmatrix} X \\ Y \\ \xi \\ \eta \end{pmatrix} = \begin{pmatrix} \phi X - \dfrac{1}{3}vX + 2B \times Y + \eta A \\ 2A \times X - {}^t\phi Y + \dfrac{1}{3}vY + \xi B \\ (A, Y) + v\xi \\ (B, X) - v\eta \end{pmatrix},$$

where $\mathfrak{e}_{6(-26)} = \mathfrak{e}_6(\mathfrak{J}) = \{\phi \in \mathrm{Hom}_{\boldsymbol{R}}(\mathfrak{J}) \,|\, (\phi X, X \times X) = 0\}$ (resp. $\mathfrak{e}_{6(6)} = \mathfrak{e}_6(\mathfrak{J}'))$ is the Lie algebra of the group

$$E_{6(-26)} = E_6(\mathfrak{J}) = \{\alpha \in \mathrm{Iso}_{\boldsymbol{R}}(\mathfrak{J}) \,|\, \det \alpha X = \det X\}$$

(resp. $E_{6(6)} = E_6(\mathfrak{J}')$) and ${}^t\phi \in \mathfrak{e}_{6(-26)}$ (resp. $\mathfrak{e}_{6(6)}$) is the transpose of $\phi$ with respect to the inner product $(X, Y) : ({}^t\phi X, Y) = (X, \phi Y)$. Next, for $P = (X, Y, \xi, \eta)$, $Q = (Z, W, \zeta, \omega) \in \mathfrak{P}$ (resp. $\mathfrak{P}'$), we define an $\boldsymbol{R}$-linear mapping $P \times Q : \mathfrak{P} \to \mathfrak{P}$ (resp. $\mathfrak{P}' \to \mathfrak{P}'$) by

$$P \times Q = \Phi(\phi, A, B, v), \quad \begin{cases} \phi = -\dfrac{1}{2}(X \vee W + Z \vee Y), \\[2ex] A = -\dfrac{1}{4}(2Y \times W - \xi Z - \zeta X), \\[2ex] B = \dfrac{1}{4}(2X \times Z - \eta W - \omega Y), \\[2ex] v = \dfrac{1}{8}((X, W) + (Z, Y) - 3(\xi\omega + \zeta\eta)), \end{cases}$$

where $X \vee W \in \mathfrak{e}_{6(-26)}$ (resp. $\mathfrak{e}_{6(6)}$) is defined by $(X \vee W)U = \frac{1}{2}(W, U)X + \frac{1}{6}(X, W)U - 2W \times (X \times U)$ for $U \in \mathfrak{J}$ (resp. $\mathfrak{J}'$).

## 2. Diagonalization of $X \in \mathfrak{J}^1$ by $F_4$

### 2.1. Lie Groups $F_4$ and $E_{6(-26)}$

The group

$$F_4 = F_4(\mathfrak{J}) = \{\alpha \in \mathrm{Iso}_{\boldsymbol{R}}(\mathfrak{J}) \mid \alpha(X \circ Y) = \alpha X \circ \alpha Y\}$$

is the simply connected compact simple Lie group of type $F_{4(-52)}$, and the group $E_{6(-26)} = E_6(\mathfrak{J})$ is the simply connected non-compact simple Lie group of type $E_{6(-26)}$ ([1], [11], [13]).

PROPOSITION 1 ([7]). *Let* $E_6(\mathfrak{J}^1) = \{\alpha \in \mathrm{Iso}_{\boldsymbol{R}}(\mathfrak{J}^1) \mid \det \alpha X = \det X\}$. *Then the group* $E_6(\mathfrak{J}^1)$ *is isomorphic to* $E_{6(-26)}$:

$$E_6(\mathfrak{J}^1) \cong E_{6(-26)}.$$

PROOF. Define a mapping $f : \mathfrak{J}^1 \to \mathfrak{J}$ by $fX = I_1 X$. Then,

$$\varphi : E_6(\mathfrak{J}) \to E_6(\mathfrak{J}^1), \quad \varphi(\alpha)X = (f^{-1}\alpha f)X$$

gives an isomorphism as groups: $E_6(\mathfrak{J}^1) \cong E_{6(-26)}$.

PROPOSITION 2. (1) $(E_6(\mathfrak{J}))_E$ $(:=\{\alpha \in E_6(\mathfrak{J}) \mid \alpha E = E\}) = F_4$,
(2) $(E_6(\mathfrak{J}^1))_{I_1}$ $(:=\{\alpha \in E_6(\mathfrak{J}^1) \mid \alpha I_1 = I_1\}) \cong F_4$.

PROOF. (1) is found in [1], [13].

(2) It follows immediately that the restriction $\varphi' := \varphi|F_4$ of $\varphi$ in Proposition 1 to $F_4 = (E_6(\mathfrak{J}))_E$ gives an isomorphism as groups: $(E_6(\mathfrak{J}^1))_{I_1} \cong F_4$.

### 2.2. Diagonalization of $X \in \mathfrak{J}^1$ by $F_4$

We have known the following

PROPOSITION 3 ([1], [13]). *Any element* $X \in \mathfrak{J}$ *can be transformed to a diagonal form by some* $\alpha \in F_4$:

$$\alpha X = \begin{pmatrix} \xi_1 & 0 & 0 \\ 0 & \xi_2 & 0 \\ 0 & 0 & \xi_3 \end{pmatrix}, \quad \xi_1 \le \xi_2 \le \xi_3.$$

*Moreover,* $\xi_1, \xi_2, \xi_3$ *are uniquely determined by* $X$ *independent of the choice of* $\alpha \in F_4$.

Using this proposition, we obtain

Theorem 4. *Any element* $X \in \mathfrak{J}^1$ *can be transformed to a diagonal form by some* $\alpha = \varphi'(\alpha') \in (E_6(\mathfrak{J}^1))_{I_1}$, $\alpha' \in F_4$:

$$\alpha X = \begin{pmatrix} \xi_1 & 0 & 0 \\ 0 & \xi_2 & 0 \\ 0 & 0 & \xi_3 \end{pmatrix}, \quad -\xi_1 \leq \xi_2 \leq \xi_3.$$

*Moreover,* $\xi_1, \xi_2, \xi_3$ *are uniquely determined by* $X$ *independent of the choice of* $\alpha \in (E_6(\mathfrak{J}^1))_{I_1}$.

Proof. For a given element $X \in \mathfrak{J}^1$, $fX \in \mathfrak{J}$ is transformed to a diagonal form by some $\alpha' \in F_4$:

$$\alpha'(fX) = \begin{pmatrix} \xi_1 & 0 & 0 \\ 0 & \xi_2 & 0 \\ 0 & 0 & \xi_3 \end{pmatrix}, \quad \xi_k \in \boldsymbol{R},$$

from Proposition 3. Therefore, let $\alpha = \varphi'(\alpha') \in (E_6(\mathfrak{J}^1))_{I_1}$, and then we have

$$\alpha X = \varphi'(\alpha')X = (f^{-1}\alpha' f)X = \begin{pmatrix} -\xi_1 & 0 & 0 \\ 0 & \xi_2 & 0 \\ 0 & 0 & \xi_3 \end{pmatrix}, \quad \xi_k \in \boldsymbol{R}.$$

The latter half of the theorem is obvious from Proposition 3.

## 3. Diagonalization of $P \in \mathfrak{P}$ by $(U(1) \times E_6)/\boldsymbol{Z}_3$

### 3.1. Lie Group $E_{7(-25)}$ and Its Lie Algebra $\mathfrak{e}_{7(-25)}$

The group

$$E_{7(-25)} = \{\alpha \in \mathrm{Iso}_{\boldsymbol{R}}(\mathfrak{P}) \,|\, \alpha(P \times Q)\alpha^{-1} = \alpha P \times \alpha Q\}$$

is a connected non-compact simple Lie group of type $E_{7(-25)}$, and its Lie algebra $\mathfrak{e}_{7(-25)}$ is given by

$$\mathfrak{e}_{7(-25)} = \{\Phi(\phi, A, B, \nu) \in \mathrm{Hom}_{\boldsymbol{R}}(\mathfrak{P}) \,|\, \phi \in \mathfrak{e}_{6(-26)}, A, B \in \mathfrak{J}, \nu \in \boldsymbol{R}\}$$

([2], [3], [12]). Now, we have already known the following

Proposition 5 ([2], [3]). *A maximal compact subgroup* $(E_{7(-25)})_K$ *of* $E_{7(-25)}$:

$$(E_{7(-25)})_K = \{\alpha \in E_{7(-25)} \,|\, (\alpha P, \alpha Q) = (P, Q)\}$$

*is isomorphic to the group* $(U(1) \times E_6)/\boldsymbol{Z}_3$, *where* $U(1)$ *is the unitary group of degree* 1 *and* $E_6$ *is the simply connected compact simple Lie group of type* $E_{6(-78)}$.

The Lie algebra $(\mathfrak{e}_{7(-25)})_K$ of the group $(E_{7(-25)})_K$ is given by

$$(\mathfrak{e}_{7(-25)})_K = \{\Phi(\delta, A, -A, 0) \in \mathfrak{e}_{7(-25)} \,|\, \delta \in \mathfrak{f}_4, A \in \mathfrak{J}\},$$

where $\mathfrak{f}_4$ $(=\{\delta \in \mathrm{Hom}_{\boldsymbol{R}}(\mathfrak{J}) \,|\, \delta(X \circ Y) = \delta X \circ Y + X \circ \delta Y\})$ is the Lie algebra of the group $F_4$.

### 3.2. Involution $\sigma$ and Subgroup $((E_{7(-25)})_K)^\sigma$

We define three involutive $\boldsymbol{R}$-linear transformations $\sigma_k$ of $\mathfrak{P}$, $k = 1, 2, 3$, by

$$\sigma_k(X, Y, \xi, \eta) = (\sigma_k X, \sigma_k Y, \xi, \eta),$$

which are the extension of the $\boldsymbol{R}$-linear transformations $\sigma_k$ of $\mathfrak{J}$ such that

$$\sigma_k X = S_k X S_k, \quad k = 1, 2, 3,$$

where $S_1 = \mathrm{diag}(-1, 1, 1)$, $S_2 = \mathrm{diag}(1, -1, 1)$ and $S_3 = \mathrm{diag}(1, 1, -1)$. Then $\sigma_k \in F_4 \subset E_{6(-26)} \subset E_{7(-25)}$ and ${}^t\sigma_k = \sigma_k^{-1} = \sigma_k$. Especially, we denote $\sigma_1$ by $\sigma$.

We shall now consider the subgroup $(E_{7(-25)})^\sigma$ of $E_{7(-25)}$:

$$(E_{7(-25)})^\sigma = \{\alpha \in E_{7(-25)} \,|\, \sigma\alpha = \alpha\sigma\}$$

and consider $\boldsymbol{R}$-vector subspaces $\mathfrak{P}_1$ and $\mathfrak{P}_{-1}$ of $\mathfrak{P}$, which are the eigenspaces of $\sigma$, respectively:

$$\mathfrak{P}_1 = \{P \in \mathfrak{P} \,|\, \sigma P = P\}, \quad \mathfrak{P}_{-1} = \{P \in \mathfrak{P} \,|\, \sigma P = -P\}.$$

Then, $\mathfrak{P} = \mathfrak{P}_1 \oplus \mathfrak{P}_{-1}$, and $\mathfrak{P}_1, \mathfrak{P}_{-1}$ are invariant under the action of $(E_{7(-25)})^\sigma$. Besides, if we define $D_1(P)$ for $P \in \mathfrak{P}$ by

$$D_1(P) = (P_1, P_1), \quad P = P_1 + P_{-1}, \quad P_1 \in \mathfrak{P}_1, P_{-1} \in \mathfrak{P}_{-1},$$

then we have the following

LEMMA 6. *For $P \in \mathfrak{P}$ and $\alpha \in (E_{7(-25)})_K \cap (E_{7(-25)})^\sigma$, we have $D_1(\alpha P) = D_1(P)$.*

Hereafter, we shall briefly denote the intersection $(E_{7(-25)})_K \cap (E_{7(-25)})^\sigma$ by $((E_{7(-25)})_K)^\sigma$. Then, the Lie algebra $((\mathfrak{e}_{7(-25)})_K)^\sigma$ of the group $((E_{7(-25)})_K)^\sigma$ is given by

$$((\mathfrak{e}_{7(-25)})_K)^\sigma = \{\Phi(\delta, A, -A, 0) \in (\mathfrak{e}_{7(-25)})_K \,|\, \sigma\delta = \delta\sigma, \sigma A = A\}.$$

### 3.3. Some Elements of $((E_{7(-25)})_K)^\sigma$

From now on, we regard an element $\alpha \in E_{6(-26)}$ as an element $\tilde{\alpha} \in E_{7(-25)}$ by $\tilde{\alpha}(X, Y, \xi, \eta) = (\alpha X, {}^t\alpha^{-1} Y, \xi, \eta)$ ([2]): $E_{6(-26)} \subset E_{7(-25)}$.

Lemma 7. (1) *For* $T \in O(3) = \{T \in M(3, \boldsymbol{R}) \mid {}^tTT = E\}$, *we define an* $\boldsymbol{R}$-*linear mapping* $\tau(T) : \mathfrak{J} \to \mathfrak{J}$ *by*

$$\tau(T)X = TXT^{-1}, \quad X \in \mathfrak{J}.$$

*Then we have*

$$\tau(T) \in F_4 \subset E_{6(-26)} \subset E_{7(-25)}.$$

*In particular, for* $T = \begin{pmatrix} 1 & 0 \\ 0 & T_1 \end{pmatrix} \in O(3)$, $T_1 \in O(2)$, *we denote* $\tau(T)$ *by* $\tau_1(T_1)$. *Then we have*

$$\tau_1(T_1) \in (F_4)^\sigma \subset ((E_{7(-25)})_K)^\sigma.$$

(2) *For* $D_a = \mathrm{diag}(a, \bar{a}, 1)$, $a\bar{a} = 1$, $a \in \mathfrak{C}$, *define an* $\boldsymbol{R}$-*linear mapping* $\delta(a) : \mathfrak{J} \to \mathfrak{J}$ *by*

$$\delta(a)X = D_a X \overline{D}_a = \begin{pmatrix} \xi_1 & a x_3 a & a\bar{x}_2 \\ \bar{a}\bar{x}_3\bar{a} & \xi_2 & \bar{a}x_1 \\ x_2\bar{a} & \bar{x}_1 a & \xi_3 \end{pmatrix}, \quad X = X(\xi_k, x_k) \in \mathfrak{J}.$$

*Then we have*

$$\delta(a) \in (F_4)^\sigma \subset ((E_{7(-25)})_K)^\sigma.$$

(3) *For* $a \in \mathfrak{C}$, $a \neq 0$, *we define an* $\boldsymbol{R}$-*linear mapping* $\alpha(a) : \mathfrak{P} \to \mathfrak{P}$ *by* $\alpha(a)(X(\xi_k, x_k), Y(\eta_k, y_k), \xi, \eta) = (X(\xi'_k, x'_k), Y(\eta'_k, y'_k), \xi', \eta')$:

$$\begin{cases} \xi'_1 = \dfrac{\xi_1 - \xi}{2} + \dfrac{\xi_1 + \xi}{2} \cos 2|a| + \dfrac{(a, y_1)}{|a|} \sin 2|a| \\ \xi'_2 = \xi_2 \\ \xi'_3 = \xi_3 \\ x'_1 = x_1 + \dfrac{(\eta_1 + \eta)a}{2|a|} \sin 2|a| - \dfrac{2(a, x_1)a}{|a|^2} \sin^2|a| \\ x'_2 = x_2 \cos|a| - \dfrac{\overline{y_3 a}}{|a|} \sin|a| \\ x'_3 = x_3 \cos|a| - \dfrac{\overline{a y_2}}{|a|} \sin|a|, \end{cases}$$

$$\begin{cases} \eta_1' = \dfrac{\eta_1 - \eta}{2} + \dfrac{\eta_1 + \eta}{2} \cos 2|a| - \dfrac{(a, x_1)}{|a|} \sin 2|a| \\ \eta_2' = \eta_2 \\ \eta_3' = \eta_3 \\ y_1' = y_1 - \dfrac{(\xi_1 + \xi)a}{2|a|} \sin 2|a| - \dfrac{2(a, y_1)a}{|a|^2} \sin^2|a| \\ y_2' = y_2 \cos|a| + \dfrac{\overline{x_3 a}}{|a|} \sin|a| \\ y_3' = y_3 \cos|a| + \dfrac{\overline{a x_2}}{|a|} \sin|a|, \end{cases}$$

$$\begin{cases} \xi' = -\dfrac{\xi_1 - \xi}{2} + \dfrac{\xi_1 + \xi}{2} \cos 2|a| + \dfrac{(a, y_1)}{|a|} \sin 2|a| \\ \eta' = -\dfrac{\eta_1 - \eta}{2} + \dfrac{\eta_1 + \eta}{2} \cos 2|a| - \dfrac{(a, x_1)}{|a|} \sin 2|a|, \end{cases}$$

*where* $|a| = \sqrt{(a, a)}$. *Then we have*

$$\alpha(a) \in ((E_{7(-25)})_K)^\sigma.$$

(4) *For* $\theta \in \boldsymbol{R}$, *we define an* $\boldsymbol{R}$-*linear mapping* $\beta(\theta) : \mathfrak{P} \to \mathfrak{P}$ *by* $\beta(\theta)(X(\xi_k, x_k), Y(\eta_k, y_k), \xi, \eta) = (X(\xi_k', x_k'), Y(\eta_k', y_k'), \xi', \eta')$:

$$\begin{pmatrix} \xi_1' \\ \eta' \end{pmatrix} = A \begin{pmatrix} \xi_1 \\ \eta \end{pmatrix}, \quad \begin{pmatrix} \xi' \\ \eta_1' \end{pmatrix} = A \begin{pmatrix} \xi \\ \eta_1 \end{pmatrix}, \quad \begin{pmatrix} \eta_2' \\ \xi_3' \end{pmatrix} = A \begin{pmatrix} \eta_2 \\ \xi_3 \end{pmatrix}, \quad \begin{pmatrix} \eta_3' \\ \xi_2' \end{pmatrix} = A \begin{pmatrix} \eta_3 \\ \xi_2 \end{pmatrix},$$

$$\begin{pmatrix} x_1' \\ y_1' \end{pmatrix} = A \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}, \quad \begin{pmatrix} x_2' \\ y_2' \end{pmatrix} = \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}, \quad \begin{pmatrix} x_3' \\ y_3' \end{pmatrix} = \begin{pmatrix} x_3 \\ y_3 \end{pmatrix},$$

*where* $A = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$. *Then we have*

$$\beta(\theta) \in ((E_{7(-25)})_K)^\sigma.$$

Proof. (1) and (2) are easily seen by direct calculations.

(3) follows from $\alpha(a) = \exp(\Phi(0, F_1(a), -F_1(a), 0))$, since $\Phi(0, F_1(a), -F_1(a), 0) \in ((\mathfrak{e}_{7(-25)})_K)^\sigma$, where $F_1(a) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & a \\ 0 & \bar{a} & 0 \end{pmatrix} \in \mathfrak{J}$.

(4) follows from $\beta(\theta) = \exp(\Phi(0, \mathrm{diag}(-\theta, 0, 0)), \mathrm{diag}(\theta, 0, 0), 0)$, since $\Phi(0, \mathrm{diag}(-\theta, 0, 0), \mathrm{diag}(\theta, 0, 0), 0) \in ((\mathfrak{e}_{7(-25)})_K)^\sigma$.

### 3.4. Diagonalization of $P \in \mathfrak{P}$ by $(U(1) \times E_6)/\boldsymbol{Z}_3$

Lemma 8. *Any element $P \in \mathfrak{P}$ can be transformed to the following form by some $\alpha \in ((E_{7(-25)})_K)^\sigma$:*

$$\alpha P = (\begin{pmatrix} \xi_1 & x_3 & \bar{x}_2 \\ \bar{x}_3 & \xi_2 & 0 \\ x_2 & 0 & \xi_3 \end{pmatrix}, \begin{pmatrix} \eta_1 & y_3 & \bar{y}_2 \\ \bar{y}_3 & \eta_2 & 0 \\ y_2 & 0 & \eta_3 \end{pmatrix}, \xi, \eta), \quad \xi_k, \eta_k, \xi, \eta \in \boldsymbol{R}, x_k, y_k \in \mathfrak{C}.$$

Proof. For a given element

$$P = (\begin{pmatrix} \xi_1 & x_3 & \overline{x_2} \\ \overline{x_3} & \xi_2 & x_1 \\ x_2 & \overline{x_1} & \xi_3 \end{pmatrix}, \begin{pmatrix} \eta_1 & y_3 & \overline{y_2} \\ \overline{y_3} & \eta_2 & y_1 \\ y_2 & \overline{y_1} & \eta_3 \end{pmatrix}, \xi, \eta) \in \mathfrak{P},$$

choose $a_1 \in \mathfrak{C}$ such that $(a_1, x_1) = (a_1, y_1) = 0$, $|a_1| = \pi/4$, and operate $\alpha(a_1)$ of Lemma 7.(3) on $P$. Then we have

$$\alpha(a_1)P = (\begin{pmatrix} \xi_1' & * & * \\ * & \xi_2' & x_1' \\ * & \bar{x}_1' & \xi_3' \end{pmatrix}, \begin{pmatrix} \eta_1' & * & * \\ * & \eta_2' & y_1' \\ * & \bar{y}_1' & \eta_3' \end{pmatrix}, -\xi_1', -\eta_1') = P_1, \quad \begin{matrix} \xi_2' = \xi_2, & \xi_3' = \xi_3, \\ \eta_2' = \eta_2, & \eta_3' = \eta_3. \end{matrix}$$

Suppose that $y_1' \neq 0$. Let $a_2 = y_1'/|y_1'|$ and operate $\delta(a_2)$ of Lemma 7.(2) on $P_1$. Then we have

$$\delta(a_2)P_1 = (\begin{pmatrix} \xi_1'' & * & * \\ * & \xi_2'' & x_1'' \\ * & \bar{x}_1'' & \xi_3'' \end{pmatrix}, \begin{pmatrix} \eta_1'' & * & * \\ * & \eta_2'' & y_1'' \\ * & y_1'' & \eta_3'' \end{pmatrix}, -\xi_1'', -\eta_1'') = P_2, \quad \begin{matrix} \xi_k'' = \xi_k', & \eta_k'' = \eta_k', \\ x_1'' = \mathfrak{C}, & y_1'' = \boldsymbol{R}. \end{matrix}$$

Operating $\tau_1(T_1)$ of Lemma 7.(1) such that $T_1\begin{pmatrix} \eta_2'' & y_1'' \\ y_1'' & \eta_3'' \end{pmatrix}T_1^{-1}$ is a diagonal form, on $P_2$, we have

$$\tau_1(T_1)P_2 = (\begin{pmatrix} \xi_1^{(3)} & * & * \\ * & \xi_2^{(3)} & x_1^{(3)} \\ * & \bar{x}_1^{(3)} & \xi_3^{(3)} \end{pmatrix}, \begin{pmatrix} \eta_1^{(3)} & * & * \\ * & \eta_2^{(3)} & 0 \\ * & 0 & \eta_3^{(3)} \end{pmatrix}, -\xi_1^{(3)}, -\eta_1^{(3)}) = P_3,$$

$$\xi_1^{(3)} = \xi_1'', \eta_1^{(3)} = \eta_1'', x_1^{(3)} \in \mathfrak{C}.$$

Suppose that $x_1^{(3)} \neq 0$. Let $a_3 = \pi x_1^{(3)}/4|x_1^{(3)}|$ and operate $\alpha(a_3)$ of Lemma 7.(3) on $P_3$. Then we have

$$\alpha(a_3)P_3 = (\begin{pmatrix} \xi_1^{(3)} & * & * \\ * & \xi_2^{(3)} & 0 \\ * & 0 & \xi_3^{(3)} \end{pmatrix}, \begin{pmatrix} \eta_1^{(4)} & * & * \\ * & \eta_2^{(3)} & 0 \\ * & 0 & \eta_3^{(3)} \end{pmatrix}, -\xi_1^{(3)}, \eta^{(4)}).$$

We have thus proved the lemma.

THEOREM 9. *Any element $P \in \mathfrak{P}$ can be transformed to the following diagonal form by some $\alpha \in (E_{7(-25)})_K$ $(\cong (U(1) \times E_6)/\boldsymbol{Z}_3)$:*

$$\alpha P = (\begin{pmatrix} \xi_1 & 0 & 0 \\ 0 & \xi_2 & 0 \\ 0 & 0 & \xi_3 \end{pmatrix}, \begin{pmatrix} \eta_1 & 0 & 0 \\ 0 & \eta_2 & 0 \\ 0 & 0 & \eta_3 \end{pmatrix}, \eta_1, \xi_1), \quad \xi_k, \eta_k \in \boldsymbol{R}.$$

PROOF. For a general element $P = (X(\xi_k, x_k), Y(\eta_k, y_k), \xi, \eta) \in \mathfrak{P}$, we denote by $D(P)$ the square sum of the diagonal elements of $P$:

$$D(P) = {\xi_1}^2 + {\xi_2}^2 + {\xi_3}^2 + {\eta_1}^2 + {\eta_2}^2 + {\eta_3}^2 + \xi^2 + \eta^2.$$

Then $D_1(P)$ of Section 3.2 is

$$D_1(P) = D(P) + 2|x_1|^2 + 2|y_1|^2, \quad P \in \mathfrak{P}.$$

Now, for a given element $P \in \mathfrak{P}$, consider a space $\mathfrak{X} = \{\alpha P \mid \alpha \in (E_{7(-25)})_K\}$. Since $(E_{7(-25)})_K$ is compact, $\mathfrak{X}$ is also compact. Then, let $D(\tilde{P})$ be the maximal value of $\{D(P') \mid P' \in \mathfrak{X}\}$. We shall show that $\tilde{P} = (X(\tilde{\xi}_k, \tilde{x}_k), Y(\tilde{\eta}_k, \tilde{y}_k), \tilde{\xi}, \tilde{\eta})$ is diagonal. Suppose that $\tilde{x}_1 \neq 0$ or $\tilde{y}_1 \neq 0$. From Lemma 8, $\tilde{P}$ can be transformed to the form

$$\alpha\tilde{P} = (\begin{pmatrix} \widetilde{\xi'}_1 & * & * \\ * & \widetilde{\xi'}_2 & 0 \\ * & 0 & \widetilde{\xi'}_3 \end{pmatrix}, \begin{pmatrix} \widetilde{\eta'}_1 & * & * \\ * & \widetilde{\eta'}_2 & 0 \\ * & 0 & \widetilde{\eta'}_3 \end{pmatrix}, \widetilde{\xi'}, \widetilde{\eta'}) \tag{i}$$

by some $\alpha \in ((E_{7(-25)})_K)^\sigma$. Then

$$D(\tilde{P}) \underset{\left(\substack{\tilde{x}_1 \neq 0 \\ \text{or } \tilde{y}_1 \neq 0}\right)}{<} D_1(\tilde{P}) \underset{\left(\substack{\alpha \in ((E_{7(-25)})_K)^\sigma \\ \text{and Lemma 6}}\right)}{=} D_1(\alpha\tilde{P}) \underset{\text{(i)}}{=} D(\alpha\tilde{P}).$$

This contradicts the maximality of $D(\tilde{P})$. Hence we obtain $\tilde{x}_1 = \tilde{y}_1 = 0$. Similarly we can prove that the other entries of $X(\tilde{\xi}_k, \tilde{x}_k)$ and $Y(\tilde{\eta}_k, \tilde{y}_k)$ except the diagonals are zero by means of $((E_{7(-25)})_K)^{\sigma_k}$, $k = 2, 3$. Hence $\tilde{P}$ is a diagonal form. Finally, we shall show that $\alpha P$ is expressed in the form of the theorem by some $\alpha \in (E_{7(-25)})_K$. Let

$$\tilde{P} = (\begin{pmatrix} \xi_1 & 0 & 0 \\ 0 & \xi_2 & 0 \\ 0 & 0 & \xi_3 \end{pmatrix}, \begin{pmatrix} \eta_1 & 0 & 0 \\ 0 & \eta_2 & 0 \\ 0 & 0 & \eta_3 \end{pmatrix}, \xi, \eta), \quad \xi_k, \eta_k, \xi, \eta \in \boldsymbol{R}.$$

(1) Case of $\eta_1 \neq \eta$. Choose $\theta \in \boldsymbol{R}$ such that $\tan\theta = (\xi - \xi_1)/(\eta_1 - \eta)$ and operate $\beta(\theta)$ of Lemma 7.(4) on $\tilde{P}$. Then we have

$$\beta(\theta)\tilde{P} = (\begin{pmatrix} \xi_1' & 0 & 0 \\ 0 & \xi_2' & 0 \\ 0 & 0 & \xi_3' \end{pmatrix}, \begin{pmatrix} \eta_1' & 0 & 0 \\ 0 & \eta_2' & 0 \\ 0 & 0 & \eta_3' \end{pmatrix}, \xi_1', \eta') = \tilde{P}_1.$$

Operating $\alpha(\pi/4)$ of Lemma 7.(3) on $\tilde{P}_1$, we have

$$\alpha\left(\frac{\pi}{4}\right)\tilde{P}_1 = (\begin{pmatrix} 0 & 0 & 0 \\ 0 & \xi_2'' & x_1'' \\ 0 & x_1'' & \xi_3'' \end{pmatrix}, \begin{pmatrix} \eta_1'' & 0 & 0 \\ 0 & \eta_2'' & y_1'' \\ 0 & y_1'' & \eta_3'' \end{pmatrix}, 0, -\eta_1'') = \tilde{P}_2, \quad \begin{array}{l} \xi_2'' = \xi_2', \quad \xi_3'' = \xi_3', \\ \eta_2'' = \eta_2', \quad \eta_3'' = \eta_3', \\ x_1'', y_1'' \in \boldsymbol{R}. \end{array}$$

Further, operating $\tau_1(T_1)$ of Lemma 7.(1) such that $T_1\begin{pmatrix} \eta_2'' & y_1'' \\ y_1'' & \eta_3'' \end{pmatrix}T_1^{-1}$ is a diagonal form, on $\tilde{P}_2$, we have

$$\tau_1(T_1)\tilde{P}_2 = (\begin{pmatrix} 0 & 0 & 0 \\ 0 & \xi_2^{(3)} & x_1^{(3)} \\ 0 & x_1^{(3)} & \xi_3^{(3)} \end{pmatrix}, \begin{pmatrix} \eta_1^{(3)} & 0 & 0 \\ 0 & \eta_2^{(3)} & 0 \\ 0 & 0 & \eta_3^{(3)} \end{pmatrix}, 0, -\eta_1^{(3)}) = \tilde{P}_3, \quad \begin{array}{l} \eta_1^{(3)} = \eta_1'', \\ x_1^{(3)} \in \boldsymbol{R}. \end{array}$$

Suppose that $x_1^{(3)} \neq 0$. Operate $\alpha(\pi/4)$ of Lemma 7.(3) on $\tilde{P}_3$. Then we have

$$\alpha\left(\frac{\pi}{4}\right)\tilde{P}_3 = (\begin{pmatrix} 0 & 0 & 0 \\ 0 & \xi_2^{(3)} & 0 \\ 0 & 0 & \xi_3^{(3)} \end{pmatrix}, \begin{pmatrix} \eta_1^{(4)} & 0 & 0 \\ 0 & \eta_2^{(3)} & 0 \\ 0 & 0 & \eta_3^{(3)} \end{pmatrix}, 0, \eta^{(4)}) = \tilde{P}_4.$$

Therefore, the required form is obtained by considering $\beta(-\pi/4)\tilde{P}_4$.

(2) Case of $\eta_1 = \eta$. The form of $\beta(\pi/2)\tilde{P}$ is nothing but that of $\tilde{P}_1$ in Case (1), and hence this can be reduced to Case (1).

We have thus completed the proof of the theorem.

## 4. Diagonalization of $P \in \mathfrak{P}'$ by $SU(8)/\boldsymbol{Z}_2$

### 4.1. Lie Group $E_{7(7)}$ and Maximal Compact Subgroup $SU(8)/\boldsymbol{Z}_2$

The group

$$E_{7(7)} = \{\alpha \in \mathrm{Iso}_{\boldsymbol{R}}(\mathfrak{P}') \mid \alpha(P \times Q)\alpha^{-1} = \alpha P \times \alpha Q\}$$

is the connected non-compact simple Lie group of type $E_{7(7)}$ ([10], [12]). Now, we shall recall the several important mapping and give their inverse ones. To do so, we first prepare the following $\boldsymbol{R}$-vector spaces and groups:

$$\mathfrak{J}(4, \boldsymbol{H}) = \{X \in M(4, \boldsymbol{H}) \mid X^* = X\},$$

$$\mathfrak{J}(4, \boldsymbol{H})_0 = \{X \in \mathfrak{J}(4, \boldsymbol{H}) \mid tr(X) = 0\},$$

$$\mathfrak{J}(4, \boldsymbol{H})^C = \{X_0 + iX_1 \mid X_k \in \mathfrak{J}(4, \boldsymbol{H})\}: \text{ complexification of } \mathfrak{J}(4, \boldsymbol{H}),$$

$$\mathfrak{S}(n, \boldsymbol{C}) = \{S \in M(n, \boldsymbol{C}) \mid {}^tS = -S\},$$
$$O(n) = \{A \in M(n, \boldsymbol{R}) \mid {}^tAA = E\},$$
$$U(n) = \{A \in M(n, \boldsymbol{C}) \mid A^*A = E\},$$
$$SU(n) = \{A \in U(n) \mid \det A = 1\}.$$

Then, we have the following mappings and their inverse ones.

(1) $\gamma_2 : \mathfrak{P}' \to \mathfrak{P}'$,

$$\gamma_2(X, Y, \xi, \eta) = (X, \gamma Y, \xi, \eta),$$

where $\gamma : \mathfrak{J}' \to \mathfrak{J}'$, $\gamma X(\xi_k, a_k + b_k e') = X(\xi_k, a_k - b_k e')$.

(1)′ $\gamma_2^{-1} = \gamma_2$.

(2) $\tilde{g} : \mathfrak{P}' \to \mathfrak{J}(4, \boldsymbol{H})^C$,

$$\tilde{g}(X, Y, \xi, \eta) = gX - \frac{\xi}{2}E + i\left(gY - \frac{\eta}{2}E\right),$$

where $g : \mathfrak{J}' \to \mathfrak{J}(4, \boldsymbol{H})_0$ is defined by

$$g\left(\begin{pmatrix} \xi_1 & a_3 + b_3 e' & \bar{a}_2 - b_2 e' \\ \bar{a}_3 - b_3 e' & \xi_2 & a_1 + b_1 e' \\ a_2 + b_2 e' & \bar{a}_1 - b_1 e' & \xi_3 \end{pmatrix}\right) = \begin{pmatrix} \lambda_1 & b_1 & b_2 & b_3 \\ \bar{b}_1 & \lambda_2 & a_3 & \bar{a}_2 \\ \bar{b}_2 & \bar{a}_3 & \lambda_3 & a_1 \\ \bar{b}_3 & a_2 & \bar{a}_1 & \lambda_4 \end{pmatrix},$$

$$\lambda_1 = \frac{1}{2}(\xi_1 + \xi_2 + \xi_3), \quad \lambda_2 = \frac{1}{2}(\xi_1 - \xi_2 - \xi_3),$$

$$\lambda_3 = \frac{1}{2}(\xi_2 - \xi_3 - \xi_1), \quad \lambda_4 = \frac{1}{2}(\xi_3 - \xi_1 - \xi_2).$$

(2)′ $\tilde{g}^{-1} : \mathfrak{J}(4, \boldsymbol{H})^C \to \mathfrak{P}'$,

$$\tilde{g}^{-1}(M + iN) = \left(g^{-1}\left(M - \frac{\mathrm{tr}(M)}{4}E\right), g^{-1}\left(N - \frac{\mathrm{tr}(N)}{4}E\right), -\frac{\mathrm{tr}(M)}{2}, -\frac{\mathrm{tr}(N)}{2}\right).$$

In particular, $g^{-1}(\mathrm{diag}(\lambda_1, \lambda_2, \lambda_3, \lambda_4)) = \mathrm{diag}(\lambda_1 + \lambda_2, \lambda_1 + \lambda_3, \lambda_1 + \lambda_4)$.

(3) $h : \mathfrak{J}(4, \boldsymbol{H})^C \to \mathfrak{S}(8, \boldsymbol{C})$,

$$h(M + iN) = k(M)\boldsymbol{J} + e_1 k(N)\boldsymbol{J},$$

where $k : M(4, \boldsymbol{H}) \to M(8, \boldsymbol{C})$ is the naturally extended mapping of $k : \boldsymbol{H} = \boldsymbol{C} \oplus \boldsymbol{C}e_2 \to M(2, \boldsymbol{C})$, $k(a + be_2) = \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix}$ and $\boldsymbol{J} = \mathrm{diag}(J, J, J, J) \in M(8, \boldsymbol{C})$, $J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

(3)′ $h^{-1} : \mathfrak{S}(8, \boldsymbol{C}) \to \mathfrak{J}(4, \boldsymbol{H})^C$,

$$h^{-1}(S) = -\frac{1}{2}k^{-1}(SJ + J\overline{S}) + \frac{i}{2}k^{-1}(e_1(SJ - J\overline{S})).$$

(4) $\chi : \mathfrak{P}' \to \mathfrak{S}(8, \boldsymbol{C})$,

$$\begin{aligned}\chi(X, Y, \xi, \eta) &= h\tilde{g}\gamma_2(X, Y, \xi, \eta)\\ &= k\left(gX - \frac{\xi}{2}E\right)J + e_1 k\left(g(\gamma Y) - \frac{\eta}{2}E\right)J.\end{aligned}$$

(4)′ $\chi^{-1} : \mathfrak{S}(8, \boldsymbol{C}) \to \mathfrak{P}'$, $\chi^{-1} = \gamma_2\tilde{g}^{-1}h^{-1}$.

Proposition 10 ([10]. cf. [12], [13]). *A maximal compact subgroup* $(E_{7(7)})_K$ *of* $E_{7(7)}$:

$$(E_{7(7)})_K = \{\alpha \in E_{7(7)} \mid \langle \alpha P, \alpha Q \rangle = \langle P, Q \rangle\},$$

*where the positive definite inner product* $\langle P, Q \rangle$ *is defined by* $\langle P, Q \rangle = (P, \gamma Q)$ $(\gamma(X, Y, \xi, \eta) = (\gamma X, \gamma Y, \xi, \eta))$, *is isomorphic to the group* $SU(8)/\boldsymbol{Z}_2$ *by the isomorphism induced from the homomorphism* $\psi : SU(8) \to (E_{7(7)})_K \subset E_{7(7)}$,

$$\psi(A)P = \chi^{-1}(A\chi(P)\,{}^t\!A), \quad P \in \mathfrak{P}'.$$

### 4.2. Diagonalization of $P \in \mathfrak{P}'$ by $SU(8)/\boldsymbol{Z}_2$

Proposition 11 ([5]). *For a given vector* $\boldsymbol{x} = {}^t(x_1, x_2, x_3, \dots, x_n) \in \boldsymbol{C}^n$, *there exists* $D \in U(n-1) \subset U(n)$ *satisfying*

$$D\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} x_1 \\ -s \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad s = \sqrt{\sum_{k=2}^{n} |x_k|^2} \in \boldsymbol{R},$$

*where, identifying* $D \in U(n-1)$ *with* $\begin{pmatrix} 1 & 0 \\ 0 & D \end{pmatrix} \in U(n)$, *we regard* $U(n-1) \subset U(n)$: $U(1) \subset U(2) \subset \cdots \subset U(n-1) \subset U(n)$.
(Note that the first term $x_1$ of $\boldsymbol{x}$ leaves invariant under the action of $D$.)

Lemma 12. *Any complex skew-symmetric matrix $X \in \mathfrak{S}(n, \boldsymbol{C})$ is transformed to the following form by some $A \in SU(n)$:*

$$AX{}^tA = \begin{pmatrix} R_1 & & & \\ & \ddots & & \\ & & R_{[n/2]} & \\ & & & (0) \end{pmatrix}, \quad \begin{array}{l} R_1 = \begin{pmatrix} 0 & c \\ -c & 0 \end{pmatrix}, \quad c \in \boldsymbol{C}, \\ R_k = \begin{pmatrix} 0 & r_k \\ -r_k & 0 \end{pmatrix}, \quad r_k \geq 0 \ (k \geq 2), \\ (0) = \begin{cases} 0 & \textit{if } n \textit{ is odd}, \\ \textit{empty} & \textit{if } n \textit{ is even}. \end{cases} \end{array}$$

Proof. For the first column vector $\boldsymbol{x} = {}^t(0, x_{21}, \dots, x_{n1})$ of $X$, construct $D_1 \in U(n-1) \subset U(n)$ of Proposition 11. Then $X$ is transformed to the matrix

$$X_1 = D_1 X {}^tD_1 = \begin{pmatrix} 0 & * & \cdots & * \\ -s_2 & * & \cdots & * \\ 0 & * & \cdots & * \\ & & \cdots & \\ 0 & * & \cdots & * \end{pmatrix}, \quad s_2 \in \boldsymbol{R}.$$

Since $X_1$ is also skew-symmetric, the first row vector of $X_1$ is $(0, s_2, 0, \dots, 0)$. Thus $X_1$ is of the form

$$X_1 = \begin{pmatrix} 0 & s_2 & 0 & \cdots & 0 \\ -s_2 & & & & \\ 0 & & & & \\ \vdots & & Y & & \\ 0 & & & & \end{pmatrix}, \quad s_2 \in \boldsymbol{R}, Y \in M(n-1, \boldsymbol{C}), {}^tY = -Y.$$

For the complex skew-symmetric matrix $Y \in M(n-1, \boldsymbol{C})$, applying the similar process as above, we can obtain that $X_1$ is transformed by some unitary matrix $A_1 \in U(n)$ to a real skew-symmetric tridiagonal form:

$$X_1 = A_1 X {}^tA_1 = \begin{pmatrix} 0 & s_2 & & & & & \\ -s_2 & 0 & s_3 & & & & \\ & -s_3 & 0 & s_4 & & & \\ & & -s_4 & 0 & & & \\ & & & & \cdots & & \\ & & & & & 0 & s_n \\ & & & & & -s_n & 0 \end{pmatrix}, \quad s_k \in \boldsymbol{R}.$$

Therefore, since $X_1$ is a real skew-symmetric matrix, $X_1$ can be transformed to the canonical form by some orthogonal matrix $A_2 \in O(n) \subset U(n)$. Namely, there exists a unitary matrix $A_3$ $(=A_2A_1) \in U(n)$ such that

$$A_3X\,{}^tA_3 = \mathrm{diag}(R_1, R_2, \dots, R_{[n/2]}, (0)), \quad R_k = \begin{pmatrix} 0 & r_k \\ -r_k & 0 \end{pmatrix}, \quad r_k \geq 0.$$

Further, let det $A_3 = a$ and construct the unitary matrix $A_4 = \mathrm{diag}(a^{-1}, 1, \dots, 1) \in U(n)$. Then we obtain that $A = A_4A_3$ belongs to the special unitary group $SU(n)$ and $AX\,{}^tA$ is of the form as in the lemma.

Theorem 13. *Any element $P \in \mathfrak{P}'$ can be transformed to the following diagonal form by some $\alpha = \psi(A) \in (E_{7(7)})_K$, $A \in SU(8)$:*

$$\alpha P = (\begin{pmatrix} \xi_1 & 0 & 0 \\ 0 & \xi_2 & 0 \\ 0 & 0 & \xi_3 \end{pmatrix}, \begin{pmatrix} \eta & 0 & 0 \\ 0 & \eta & 0 \\ 0 & 0 & \eta \end{pmatrix}, \xi, -\eta), \quad \xi_k, \xi, \eta \in \boldsymbol{R}.$$

Proof. For a given element $P \in \mathfrak{P}'$, $\chi(P) \in \mathfrak{S}(8, \boldsymbol{C})$ is transformed to the following form by some $A \in SU(8)$:

$$A\chi(P)\,{}^tA = \begin{pmatrix} R_1 & & & \\ & R_2 & & \\ & & R_3 & \\ & & & R_4 \end{pmatrix}, \quad \begin{matrix} R_1 = \begin{pmatrix} 0 & r_1 + r_0e_1 \\ -r_1 - r_0e_1 & 0 \end{pmatrix}, \\ R_k = \begin{pmatrix} 0 & r_k \\ -r_k & 0 \end{pmatrix}, \ r_k \geq 0 \ (2 \leq k \leq 4), \end{matrix} \quad r_k \in \boldsymbol{R},$$

from Lemma 12. Then we have

$$\chi^{-1}(A\chi(P)\,{}^tA) = (\frac{1}{2}\begin{pmatrix} r_1 + r_2 - r_3 - r_4 & 0 & 0 \\ 0 & r_1 - r_2 + r_3 - r_4 & 0 \\ 0 & 0 & r_1 - r_2 - r_3 + r_4 \end{pmatrix},$$
$$\frac{1}{2}\begin{pmatrix} r_0 & 0 & 0 \\ 0 & r_0 & 0 \\ 0 & 0 & r_0 \end{pmatrix}, -\frac{1}{2}(r_1 + r_2 + r_3 + r_4), -\frac{1}{2}r_0).$$

from straightforward calculation by using the inverse mapping $\chi^{-1} = \gamma_2\tilde{g}^{-1}h^{-1}$. Thus we have easily obtained that $\alpha P$ is expressed in the form of the theorem by some $\alpha = \psi(A) \in (E_{7(7)})_K$, $A \in SU(8)$. We have just completed the proof of the theorem.

Remark. For $P \in \mathfrak{P}'$, let the canonical form of a complex skew-symmetric matrix $\chi(P)$ by a special unitary matrix

$$A\chi(P)\,{}^tA = \mathrm{diag}(R_1, R_2, R_3, R_4), \quad A \in SU(8),$$

$$R_1 = \begin{pmatrix} 0 & c \\ -c & 0 \end{pmatrix}, \quad c \in \boldsymbol{C}, \quad R_k = \begin{pmatrix} 0 & r_k \\ -r_k & 0 \end{pmatrix}, \quad 0 \le |c| \le r_2 \le r_3 \le r_4.$$

Then, by virtue of Theorem 3 of [8], $c, r_2, r_3, r_4$ are uniquely determined by $P$ (up to permutation of $c, r_2, r_3, r_4$ and sign of $c$) independent of the choice of $A \in SU(8)$. This implies that the minimum number of the parameters which appear in the diagonal form of Theorem 13 above is five.

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