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SPINORS, CALIBRATIONS AND GRASSMANNIANS

By

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Abstract. In this paper we use Clifford algebra and spinor calculus to study the calibrations on Riemannian manifolds and the Grassmann manifolds. Show that for every Grassmannian, there is a map $\pi : G(k, \boldsymbol{R}^m) \to M$ such that every $\xi \in M$ is a calibration on $\boldsymbol{R}^m$ and $\pi^{-1}(\xi)$ is the contact set of ξ. In low dimensional cases, the calibration sets M are manifolds or manifolds with singularities. We also use Clifford algebra to study the isotropy groups of calibrations.

§1. Introduction

In [9], we gave a new treatment of the Clifford algebras. We represented the pinor and spinor spaces as subspaces of the Clifford algebras, then we used these pinors to construct isomorphisms between the Clifford algebras and the matrix algebras. In doing these, we have developed some spinor calculus. In this paper, we use Clifford algebra and the results of [9] to study the calibrations and the Grassmann manifolds.

Let ξ be a closed k-form on a Riemannian manifold M. If for every point p of M and every orthonormal vectors $e_1, \dots, e_k \in T_pM$, we have $\xi_p(e_1 \wedge \cdots \wedge e_k) \leq 1$ and there are $\bar{e}_1, \dots, \bar{e}_k$ such that $\xi_p(\bar{e}_1 \wedge \cdots \wedge \bar{e}_k) = 1$, then ξ is called a calibration. The set of such $\bar{e}_1 \wedge \cdots \wedge \bar{e}_k$ is called a contact set of the calibration ξ. The importance of the calibration is that the calibrations are closely related with the homologically volume minimizing submanifolds (see [5], [6]). Dadok and Harvey [2] have shown that from squares of spinors in dimension $8k$, one can get calibrations. Their method can be phrased as follows.

Under the canonical isomorphism between the Clifford algebras and the exterior algebras, the oriented Grassmannian $G(k, \boldsymbol{R}^m)$ can be looked as a subset

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of Clifford algebra, especially, $G(k, \boldsymbol{R}^m) \subset \mathrm{Pin}(m)$. For any $\xi, \eta \in C\ell_m$ with unit norm, $x_0, x \in G(k, \boldsymbol{R}^m)$, we have

$$\langle x_0 \xi \alpha(\eta^t), x \rangle = \langle x_0 \xi, x\eta \rangle \leq 1.$$

These shows the k-form part of $x_0 \xi \cdot \alpha(\eta^t)$ is a calibration on $\boldsymbol{R}^m$ and the element x is in the contact set of this calibration if and only if $x\xi = x_0 \eta$.

In this paper, we study the cases of $\xi = \eta$ being the generators of the Clifford modules constructed in [9], see §2 of this paper. We shall also see that all homogenous parts of $\xi \cdot \alpha(\xi^t)$ are calibrations which includes many known important calibrations. Let M be the set of calibrations defined by k-form part of $x\xi \cdot \alpha(\xi^t)$, $x \in G(k, \boldsymbol{R}^m)$. There is a map $\pi : G(k, \boldsymbol{R}^m) \to M$ defined naturally and $\pi^{-1}(\phi)$ is the contact set of $\phi \in M$. Then every Grassmann manifold can be represented as a disjoint union of the contact sets of some calibrations. In low dimensional cases, we can show that $\pi^{-1}(\phi)$ is a totally geodesic submanifold of $G(k, \boldsymbol{R}^m)$. In some cases, $\pi : G(k, \boldsymbol{R}^m) \to M$ define fibre bundles. These maps are useful for our understanding the Grassmann manifolds. In many cases, the calibration sets M are manifolds or manifolds with singularities. We call M calibration manifolds.

For example, as shown in [9], $A_8(1+\beta_8)$ generates a left irreducible module space $V_8 = C\ell_8 \cdot A_8(1+\beta_8)$ over $C\ell_8$. We shall see that $A_8(1+\beta_8) \cdot \alpha(A_8(1+\beta_8))^t = A_8(1+\beta_8)$ and $\|A_8(1+\beta_8)\| = 1/4$. Let φ_i be the 4-form part of $16 e_1 e_i A_8(1+\beta_8)$, $i = 1, \ldots, 8$. φ_1 is the Cayley calibration, the other φ_i are special Lagrangian calibrations. The calibration manifold M defined by 4-form part of $16 x A_8(1+\beta_8)$ is diffeomorphic to the unit sphere S^7, that is, $M = \{\sum v_i \varphi_i \mid v = (v_1, \ldots, v_8) \in S^7\}$. This is the content of Proposition 3.1.

For another example, let $\tilde{M}$ be the calibration manifold defined by 2-form part of $16 x A_8(1+\beta_8)$, $x \in G(2, \boldsymbol{R}^8)$. In this case, $\tilde{M}$ is diffeomorphic to S^6 and $\pi : G(2, \boldsymbol{R}^8) \to S^6$ is a fibre bundle with fibre $\boldsymbol{CP}^3$.

The paper is organized as follows. In §2, we study the calibrations defined by $x\xi \cdot \alpha(\xi^t)$ for the case of ξ being the generators of the Clifford modules. The results are in Theorem 2.4 and 2.5. In §3, we study the calibration sets in low dimensional cases.

In §4, we study the isotropy groups of calibrations. We show that in many cases, the contact sets of the calibrations can be viewed as subsets of the isotropy groups of the calibrations. In §5, we study the calibrations on Riemannian manifolds. Combining with a result of Lawson and Michelsohn [8], we show that there is a Cayley calibration on a 8-dimensional Riemannian manifold if and only if the manifold is spin and there is a parallel pinor or spinor field on this manifold (Theorem 5.3).

Many notations used in this paper have been used in [9]. We write here for easy reference. Let $e_1, \dots, e_m$ be an orthonormal basis of $\boldsymbol{R}^m$, then $C\ell_m$ is generated by $\{e_i\}$ with the relations: $e_i e_j + e_j e_i = -2\delta_{ij}$. The homomorphism $\alpha : C\ell_m \to C\ell_m$ is defined by

$$\alpha(\xi) = \xi, \quad \text{if } \xi \in C\ell_m^{even}; \quad \alpha(\eta) = -\eta, \quad \text{if } \eta \in C\ell_m^{odd}.$$

When $m = 2n$, let $g_i = \frac{1}{2}(e_{2i-1} - \sqrt{-1}e_{2i})$, $\bar{g}_i = \frac{1}{2}(e_{2i-1} + \sqrt{-1}e_{2i})$, $i = 1, \dots, n$. Denote $A_{2n} = \mathrm{Re}(\bar{g}_1 \cdots \bar{g}_n)$ and $B_{2n} = \mathrm{Im}(\bar{g}_1 \cdots \bar{g}_n)$. Let $\omega_m = e_1 e_2 \cdots e_m$ be the volume element of $C\ell_m$. The element $\beta_m \in C\ell_m$ is defined by

$$\beta_m = \begin{cases} e_1 e_3 \cdots e_{m-3} e_{m-1}, & m \text{ even;} \\ e_1 e_3 \cdots e_{m-2} e_m, & m \text{ odd.} \end{cases}$$

§ 2. Pinors and Calibrations

We have shown in [9] that every pinor or spinor space can be realized as a subspace of Clifford algebra. Under the canonical isomorphism $\rho : C\ell_m \to \bigwedge(\boldsymbol{R}^m)$ defined by $\rho(e_{i_1} \cdots e_{i_k}) = e_{i_1} \wedge \cdots \wedge e_{i_k}$, $i_1 < \cdots < i_k$, the pinor or spinor spaces can be looked as subspaces of the exterior algebra. The inner products $\langle\, , \rangle$ on $C\ell_m$ and $\bigwedge(\boldsymbol{R}^m)$ are defined as usual. The norms of ξ and $\rho(\xi)$ are

$$\|\xi\| = \|\rho(\xi)\| = \sqrt{\langle \xi, \xi \rangle}.$$

For any $x \in G(k, \boldsymbol{R}^m)$, we can write $x = e_1 \wedge \cdots \wedge e_k$, where $\{e_1, \dots, e_k\}$ is an oriented orthonormal basis of x. As noted above, if $\xi \in \Gamma(\bigwedge(\boldsymbol{R}^m))$ is a calibration, we call

$$G(\xi) = \{x \in G(k, \boldsymbol{R}^m) \mid \xi(x) = 1\}$$

the contact set of ξ.

As shown in [9], the irreducible modules over $C\ell_m$ can be generated by the one of the following elements:

(1) A_m, when $m \equiv 2, 4 \pmod 8$;
(2) $A_m(1 + \beta_m)$, when $m \equiv 0, 6 \pmod 8$;
(3) $A_{m-1}(1 + \beta_{m-1})$, when $m \equiv 1 \pmod 8$;
(4) $A_{m-1}(1 \pm \omega_m)$, when $m \equiv 3 \pmod 8$;
(5) $A_{m-1}(1 + \beta_m)$, when $m \equiv 5 \pmod 8$;
(6) $A_{m-1}(1 + \beta_{m-1})(1 \pm \omega_m)$, when $m \equiv 7 \pmod 8$.

We shall see that every homogenous part of the above generators (under the map ρ) is a calibration. It is well-known that the n-form $2^n A_{2n}$ is a calibration on $\boldsymbol{R}^{2n}$ called the special Lagrangian calibration; the 4-form part of $2^4 A_8(1+\beta_8)$ is Cayley calibration; the 3 or 4 form parts of $2^3 A_6(1+\beta_6)(1+\omega_7)$ are associative and coassociative calibrations respectively. It is easy to see that we need only to study the cases of (1), (2) and (6). First we prove several lemmas.

LEMMA 2.1. *For any n, we have* $4A_{2n}^3 = (-1)^{(n/2)(n+1)} A_{2n}$, $\|A_{2n}\|^2 = 4\|A_{2n}^2\|^2 = \frac{1}{2^{n+1}}$, *$\rho(A_{2n})$ is an n-form.*

PROOF. From $\bar{g}_1 \cdots \bar{g}_n g_1 \cdots g_n \bar{g}_n \cdots \bar{g}_1 = (-1)^n \bar{g}_1 \cdots \bar{g}_n$ and $\bar{g}_1 \cdots \bar{g}_n = A_{2n}(1-\sqrt{-1}e_1e_2)$, one has

$$(-1)^{(n/2)(n-1)} 4A_{2n}^3(1-\sqrt{-1}e_1e_2) = (-1)^n A_{2n}(1-\sqrt{-1}e_1e_2).$$

This shows $4A_{2n}^3 = (-1)^{(n/2)(n+1)} A_{2n}$. $\square$

LEMMA 2.2. *When $2n \equiv 0, 6 \pmod 8$, we have* $(A_{2n}(1+\beta_{2n}))^2 = A_{2n}(1+\beta_{2n})$, $\|A_{2n}(1+\beta_{2n})\|^2 = \frac{1}{2^n}$, *and* $\rho(A_{2n}(1+\beta_{2n})) = \psi_n + \sum_{k\geq 0} \psi_{4k}$, *where ψ_j is a j-form.*

PROOF. From $\bar{g}_1 \cdots \bar{g}_n = A_{2n}(1-\sqrt{-1}e_1e_2)$ and

$$\bar{g}_1 \cdots \bar{g}_n \beta_{2n} = \bar{g}_1 \cdots \bar{g}_n g_1 \cdots g_n = 2A_{2n}^3(1+\sqrt{-1}e_1e_2),$$

one has $A_{2n}\beta_{2n} = 2A_{2n}^2$. Then

$$(A_{2n}(1+\beta_{2n}))^2 = A_{2n}(1+\beta_{2n}), \quad (A_{2n}(1-\beta_{2n}))^2 = -A_{2n}(1-\beta_{2n}).$$

Hence

$$\|A_{2n}(1+\beta_{2n})\|^2 = \langle A_{2n}(1+\beta_{2n}), 1\rangle = \frac{1}{2^n}.$$

The representative $\rho(A_{2n}(1+\beta_{2n})) = \psi_n + \sum \psi_{4k}$ follows from

$$A_{2n}\beta_{2n} = (-1)^{(n/2)(n-1)}\ Re(\bar{g}_1 g_1 \cdots \bar{g}_n g_n). \qquad \square$$

The proof of the next lemma is easy (ω_{8k+7} in the center of $C\ell_{8k+7}$).

LEMMA 2.3. $[A_{8k+6}(1+\beta_{8k+6})(1+\omega_{8k+7})]^2 = 2A_{8k+6}(1+\beta_{8k+6})(1+\omega_{8k+7})$, $\|A_{8k+6}(1+\beta_{8k+6})(1+\omega_{8k+7})\|^2 = \frac{1}{2^{4k+2}}$, $\rho(A_{8k+6}(1+\beta_{8k+6})(1+\omega_{8k+7})) = \sum \psi_{4i} + \sum \psi_{4j+3}$.

The following theorem generalizes some important known calibrations. The methods used in the following proof were first established by Dadok and Harvey [2].

THEOREM 2.4. *Each homogeneous part of the following differential forms is a calibration*:

(1) $2^n A_{2n}$;

(2) $2^n A_{2n}(1+\beta_{2n})$, $2n \equiv 0, 6 \pmod 8$;

(3) $2^{4k+3} A_{8k+6}(1+\beta_{8k+6})(1+\omega_{8k+7})$.

Furthermore,

(1)$'$ $x \in G(2^n A_{2n})$ *if and only if* $xA_{2n} = A_{2n}\beta_{2n}$;

(2)$'$ $x \in G(2^n A_{2n}(1+\beta_{2n}))$ *if and only if* $xA_{2n}(1+\beta_{2n}) = A_{2n}(1+\beta_{2n})$, $2n \equiv 0, 6 \pmod 8$;

(3)$'$ $x \in G(2^{4k+3} A_{8k+6}(1+\beta_{8k+6})(1+\omega_{8k+7}))$ *if and only if* $xA_{8k+6}(1+\beta_{8k+6}) \cdot (1+\omega_{8k+7}) = A_{8k+6}(1+\beta_{8k+6})(1+\omega_{8k+7})$,

where the Clifford product is used.

PROOF. The theorem is an easy consequence of Lemma 2.1–2.3, we prove (1) and (1)$'$ for examples. By Lemma 2.1, for any $x \in \mathrm{Pin}(2n)$, we have

$$\langle \rho(2^n A_{2n}), \rho(x) \rangle = 2^{n+2} \langle A_{2n}^2, xA_{2n} \rangle \leq 2^{n+2} \|A_{2n}^2\| \, \|xA_{2n}\| = 1,$$

and $\langle \rho(2^n A_{2n}), \rho(x) \rangle = 1$ if and only if $xA_{2n} = 2A_{2n}^2$. It is easy to see that β_{2n} is in the contact set of the special Lagrangian calibration $\rho(2^n A_{2n})$ and $\beta_{2n} A_{2n} = A_{2n}\beta_{2n}$. $\square$

By $2^{4k} A_{8k} \alpha(\beta_{8k}^t) = Re(1+\sqrt{-1}e_1e_2)\cdots(1+\sqrt{-1}e_{8k-1}e_{8k})$, we have

$$2^{4k} A_{8k}(1+\beta_{8k}) = 2^{4k} A_{8k} + \sum_{l=0}^{2k} (-1)^l \frac{1}{(2l)!} \Omega^{2l},$$

where $\Omega = e_1e_2 + \cdots + e_{8k-1}e_{8k}$ is the kaehler form on $\boldsymbol{R}^{8k} \cong \boldsymbol{C}^{4k}$. One can show that

$$G(2^{4k} A_{8k}) \cap G\left((-1)^k \frac{1}{(2k)!} \Omega^{2k}\right) = \varnothing,$$

$$G(2^{4k} A_{8k}) \cup G\left((-1)^k \frac{1}{(2k)!} \Omega^{2k}\right) \subset G\left(2^{4k} A_{8k} + (-1)^k \frac{1}{(2k)!} \Omega^{2k}\right).$$

Thus the calibration defined by $4k$-form part of $2^{4k} A_{8k}(1+\beta_{8k})$ can be viewed as a naturally generalization of the Cayley calibration.

The proof of following theorem is easy.

THEOREM 2.5. *For any* $x_0 \in G(r, \boldsymbol{R}^m) \subset \mathrm{Pin}(m)$, *the r-form parts of the following differential forms are calibrations*:

(1) $2^{n+1}x_0A_{2n}\alpha(A_{2n}^t) = 2^n x_0 A_{2n}\alpha(\beta_{2n}^t)$, $m = 2n$;

(2) $2^n x_0 A_{2n}(1 + \beta_{2n})$, $m = 2n \equiv 0, 6 \pmod 8$;

(3) $2^{4k+3}x_0A_{8k+6}(1 + \beta_{8k+6})(1 + \omega_{8k+7})$, $m = 8k + 7$.

Furthermore,

(1)′ $x \in G(2^n x_0 A_{2n}\alpha(\beta_{2n}^t)) \cap G(r, \boldsymbol{R}^{2n})$ *if and only if* $xA_{2n} = x_0A_{2n}$;

(2)′ $x \in G(2^n x_0 A_{2n}(1 + \beta_{2n})) \cap G(r, \boldsymbol{R}^{2n})$ *if and only if* $xA_{2n}(1 + \beta_{2n}) = x_0A_{2n}(1 + \beta_{2n})$, $2n \equiv 0, 6 \pmod 8$;

(3)′ $x \in G(2^{4k+3}A_{8k+6}(1 + \beta_{8k+6})(1 + \omega_{8k+7})) \cap G(r, \boldsymbol{R}^{8k+7})$, *if and only if*, $xA_{8k+6}(1 + \beta_{8k+6})(1 + \omega_{8k+7}) = x_0A_{8k+6}(1 + \beta_{8k+6})(1 + \omega_{8k+7})$.

Let $U(n)$ be the unitary group on $\boldsymbol{C}^n \cong \boldsymbol{R}^{2n}$. The complex structure J is defined by $g_1, \ldots, g_n$. The element A_{2n} is invariant under the action of $SU(n) \subset SO(2n)$. From $\bar{g}_1 \cdots \bar{g}_n g_1 \cdots g_n = 2A_{2n}^2(1 + \sqrt{-1}e_1e_2)$, we know that $2A_{2n}^2 + A_{2n}\beta_{2n}$ is invariant under the action of $U(n)$. In the remained of this section we study the calibrations defined by $2^n xA_{2n}\alpha(\beta_{2n}^t)$. For more results, see § 3.

PROPOSITION 2.6. *For any* $x \in G(2, \boldsymbol{R}^{2n})$, *represent* x *as* $v \wedge (aJv + bw)$, *where* v, w *are unit vectors and* $w \perp v$, Jv, $a^2 + b^2 = 1$. *Then the* 2*-form part of* $2^n xA_{2n}\alpha(\beta_{2n}^t)$ *is one of the following*,

(1) $v \wedge (aJv + bw) + (aw - bJv) \wedge Jw + \cdots$, *if* $|a| < 1$, *where* $\cdots$ *denotes the* 2*-forms which can be omitted as a calibration*;

(2) $\pm(e_1e_2 + e_3e_4 + \cdots + e_{2n-1}e_{2n})$, *if* $a = \pm 1$.

PROOF. Since $A_{2n}\alpha(\beta_{2n}^t)$ is invariant under the action of $U(n)$, we can assume that $v = e_1$, $w = e_3$. The 2-form part of

$$\begin{aligned} &2^n xA_{2n}\alpha(\beta_{2n}^t) \\ &\quad = a\,Re(e_1e_2 - \sqrt{-1})(1 + \sqrt{-1}e_3e_4) \cdots (1 + \sqrt{-1}e_{2n-1}e_{2n}) \\ &\qquad + b\,Re(e_1 - \sqrt{-1}e_2)(e_3 - \sqrt{-1}e_4)(1 + \sqrt{-1}e_5e_6) \cdots (1 + \sqrt{-1}e_{2n-1}e_{2n}) \end{aligned}$$

is

$$e_1(ae_2 + be_3) + (ae_3 - be_2)e_4 + a(e_5e_6 + \cdots + e_{2n-1}e_{2n}).$$

If $|a| < 1$, the contact sets of above 2-form and $e_1(ae_2 + be_3) + (ae_3 - be_2)e_4$ are the same. This proves the proposition. □

Now we compute the calibrations defined by 4-forms of $2^n x A_{2n}\alpha(\beta_{2n}^t)$, $x \in G(4, \boldsymbol{R}^{2n})$. As pointed out in [5, p. 129], every $x \in G(4, \boldsymbol{R}^{2n})$ can be represented by

$$x = e_1(\cos\alpha e_2 + \sin\alpha e_3)e_5(\cos\beta e_6 + \sin\beta e_7),$$

where $e_1, e_2 = Je_1, \dots, e_{2n-1}, e_{2n} = Je_{2n-1}$ is some orthonormal basis on $\boldsymbol{R}^{2n}$ and J is the complex structure defined above, $0 \le \alpha \le \pi/2$, $\alpha \le \beta \le \pi$.

Proposition 2.7. *For any $x \in G(4, \boldsymbol{R}^{2n})$ represented as above, let ϕ be the calibration defined by x as in Theorem* 2.5 (1). *With some new orthonormal bases $e_1, e_2, \dots, e_{2n}$, as a calibration, ϕ or $-\phi$ can be represented by the one of the following forms,*

(1) *Kaehler calibration* $\phi_1 = \frac{1}{2}(e_1e_2 + \cdots + e_{2n-1}e_{2n})^2$, *when* $\alpha = \beta = 0$ *or* $\alpha = 0$, $\beta = \pi$;

(2) *Special Lagrangian calibration* $\phi_2 = 2^4 A_8$, *when* $\alpha = \beta = \pi/2$;

(3) $\phi_3 = (e_5e_7 - e_6e_8) \wedge (e_1e_2 + e_3e_4 + e_9e_{10} + \cdots + e_{2n-1}e_{2n})$, *when* $\alpha = 0$, $\beta = \pi/2$;

(4) $\phi_4 = (e_1e_2 + e_3e_4)(e_5e_6 + e_7e_8) \pm \cos^2\alpha(e_1e_2e_3e_4 + e_5e_6e_7e_8) - \sin^2\alpha(e_1e_4 + e_2e_3)(e_5e_8 + e_6e_7)$, *when* $\alpha = \beta \ne 0, \pi/2$ *or* $\alpha + \beta = \pi$, $\alpha \ne 0, \pi/2$;

(5) $\phi_5 = (e_1e_2 + e_3e_4)(e_5e_6 + e_7e_8)$, *for all other cases.*

Proof. By simple computation, the 4-form part of $2^n x A_{2n}\alpha(\beta_{2n}^t)$ can be written as $\phi + \psi$, with

$$\begin{aligned}\phi &= (e_1e_2 + e_3e_4)(e_5e_6 + e_7e_8) + \cos\alpha\cos\beta(e_1e_2e_3e_4 + e_5e_6e_7e_8) \\ &\quad - \sin\alpha\sin\beta(e_1e_4 + e_2e_3)(e_5e_8 + e_6e_7); \\ \psi &= \frac{1}{2}\cos\alpha\cos\beta(e_9e_{10} + \cdots + e_{2n-1}e_{2n})^2 \\ &\quad + \cos\alpha(e_5e_6 + e_7e_8)(e_9e_{10} + \cdots + e_{2n-1}e_{2n}) \\ &\quad + \cos\beta(e_1e_2 + e_3e_4)(e_9e_{10} + \cdots + e_{2n-1}e_{2n}).\end{aligned}$$

We have replaced $\cos\alpha e_2 + \sin\alpha e_3$ and $\cos\alpha e_3 - \sin\alpha e_2$ by e_2 and e_3; $\cos\beta e_6 + \sin\beta e_7$ and $\cos\beta e_7 - \sin\beta e_6$ by e_6 and e_7 respectively. Then the cases of (1), (2), (3) follow directly.

Now assuming $\alpha, \beta \ne 0, \pi/2, \pi$, we first show that for any $y \in G(4, \boldsymbol{R}^{2n})$ with $\langle \phi + \psi, y \rangle = 1$, then $\langle \psi, y \rangle = 0$. Hence as a calibration we need only consider the 4-form ϕ. Rewrite $\phi + \psi = e_9e_{10} \wedge \psi' + \chi$, where ψ' and χ are the forms in orthogonal complement of span $\{e_9, e_{10}\}$. For any $y \in G(\phi + \psi)$, as in the proof of Lemma 2.1 of [3], y can be written as

$$y = (\cos \gamma e_9 + \sin \gamma v)(\cos \gamma e_{10} + \sin \gamma w)\eta,$$

where η, v, w are orthogonal to e_9, e_{10}. Then at least one of the following holds:

$$\langle y, e_9 e_{10} \psi' \rangle = 1, \quad \langle y, \chi \rangle = 1$$

$$\langle \eta, \psi' \rangle = \langle vw\eta, \chi \rangle = 1.$$

Since $e_9 e_{10}(e_1 e_2 + \cdots + e_7 e_8 + e_{11} e_{12} + \cdots + e_{2n-1} e_{2n})$ is a calibration and $|\cos \alpha| < 1$, $|\cos \beta| < 1$ by assumption, then $\langle y, e_9 e_{10} \psi' \rangle = 1$ and $\langle \eta, \psi' \rangle = 1$ can not hold. Then we have

$$\langle y, \chi \rangle = 1 \quad \text{and} \quad \cos \gamma = 0.$$

In this way we can show that if $y \in G(\phi + \psi)$, then $\langle \psi, y \rangle = 0$. Thus we need only to consider the 4-form ϕ. Note that ϕ is calibration for any α, β, the following hold,

$$\frac{\partial}{\partial \alpha} \langle y, \phi \rangle = \frac{\partial}{\partial \beta} \langle y, \phi \rangle = 0.$$

Then

$$\sin \alpha \cos \beta a + \cos \alpha \sin \beta b = 0,$$

$$\cos \alpha \sin \beta a + \sin \alpha \cos \beta b = 0,$$

where $a = \langle e_1 e_2 e_3 e_4 + e_5 e_6 e_7 e_8, y \rangle$, $b = \langle (e_1 e_4 + e_2 e_3)(e_5 e_8 + e_6 e_7), y \rangle$. Then $a = b = 0$ if $\sin^2 \alpha \cos^2 \beta - \cos^2 \alpha \sin^2 \beta = \sin(\alpha + \beta) \sin(\alpha - \beta) \neq 0$. In this case, as calibrations, ϕ and ϕ_5 are the same. If $\alpha = \beta$ or $\alpha + \beta = \pi$, ϕ has the form ϕ_4. $\square$

§ 3. Calibration Manifolds

In this section we study the calibration sets defined by Theorem 2.5 in low dimensional cases. By Theorem 2.5, we know that $8xA_4\alpha(A_4^t) = x(1 - \omega_4)$ is a calibration for every $x \in G(2, \boldsymbol{R}^4)$. $x' \in G(2, \boldsymbol{R}^4)$ is in the contact set of this calibration if and only if $xA_4 = x'A_4$, this is equivalent to $x(1 - \omega_4) = x'(1 - \omega_4)$. On the other hand, $x = \frac{1}{2}x(1 - \omega_4) + \frac{1}{2}x(1 + \omega_4)$ and there are unit vectors $v, w \perp e_1$ such that $x(1 - \omega_4) = e_1 v(1 - \omega_4)$, $x(1 + \omega_4) = e_1 w(1 + \omega_4)$. The map $G(2, \boldsymbol{R}^4) \to S^2 \times S^2$ defined by sending x to (v, w) is a diffeomorphism. Then the map $\pi : G(2, \boldsymbol{R}^4) \to S^2$, $x \to v$, defines a fibre bundle. The contact set of the calibration $e_1v(1-\omega_4)$ is $\pi^{-1}(v)$ which is a totally geodesic submanifold of $G(2,\boldsymbol{R}^4)$.

This construction can be generalized to many cases. Let $M_1(r,2n)$ be the set of calibrations defined by r-form parts of $2^n xA_{2n}\alpha(\beta_{2n}^t)$, $x\in G(r,\boldsymbol{R}^{2n})$. When $2n\equiv 0,6 \pmod 8$, let $M_2(r,2n)$ be the set of calibrations defined by r-form parts of $2^n xA_{2n}(1+\beta_{2n})$. By Theorem 2.5, there are two maps:

$$\pi_1 : G(r,\boldsymbol{R}^{2n}) \to M_1(r,2n);$$

$$\pi_2 : G(r,\boldsymbol{R}^{2n}) \to M_2(r,2n), \quad 2n\equiv 0,6 \pmod 8.$$

For any $\varphi\in M_i(r,2n)$, $\pi_i^{-1}(\varphi)\subset G(r,\boldsymbol{R}^{2n})$ is the contact set of φ. If $xA_{2n}=x'A_{2n}$, we have $xA_{2n}(1+\beta_{2n})=x'A_{2n}(1+\beta_{2n})$. Then there is a map $\pi' : M_1(r,2n)\to M_2(r,2n)$, if $2n\equiv 0,6 \pmod 8$. This map is nontrivial in some cases. Obviously, $\pi'\pi_1=\pi_2$.

We shall see that, in low dimensional cases, the calibration set $M_i(r,m)$ are manifolds or manifolds with singularities. We call them the calibration manifolds.

We first study the sets $M_2(r,8)$. As shown in [9], there is a unit vector $v\in\boldsymbol{R}^8$, such that $xA_8(1+\beta_8)=e_1vA_8(1+\beta_8)$ or $xA_8(1+\beta_8)=vA_8(1+\beta_8)$ for any $x\in G(r,\boldsymbol{R}^8)$, according to r being even or odd. As exterior forms, $xA_8(1+\beta_8)$ is selfdual for any $x\in G(2r,\boldsymbol{R}^8)$, anti-self dual for $x\in G(2r-1,\boldsymbol{R}^8)$. Then we need only to study $M_2(r,8)$ for $r=2,3,4$. As is well known the isotropy group of $A_8(1+\beta_8)$ is $\mathrm{Spin}_7\subset SO(8)$ which acts transitively on the unit sphere S^7 in $\boldsymbol{R}^8$. The exceptional Lie group G_2 is a subgroup of Spin_7 which acts transitively on the sphere $S^6=\{v\in S^7 \mid v\perp e_1\}$. These observations are useful for the study of $M_2(r,8)$.

In the following, we shall often use A_8 and $A_8\beta_8$. By simple computation, we have

$$\begin{aligned}16A_8 &= e_1e_3e_5e_7+e_2e_4e_6e_8-e_1e_3e_6e_8-e_2e_4e_5e_7\\ &\quad -e_1e_4e_5e_8-e_1e_4e_6e_7-e_2e_3e_5e_8-e_2e_3e_6e_7,\\ 16A_8\beta_8 &= 1+\omega_8-e_5e_6e_7e_8-e_1e_2e_3e_4\\ &\quad -e_3e_4e_7e_8-e_1e_2e_5e_6-e_1e_2e_7e_8-e_3e_4e_5e_6.\end{aligned}$$

Proposition 3.1. *The calibration set $M_2(4,8)$ is a manifold diffeomorphic to S^7, the diffeomorphism is defined by sending $v\in S^7$ to 4-form part of $16e_1vA_8(1+\beta_8)$. Furthermore, $v=\pm e_1$ corresponds to the Cayley calibration, the others are the special Lagrangian calibrations.*

PROOF. We need only to show that the 4-form part of $2^4 e_1 v A_8(1+\beta_8)$ is a calibration for any $v \in S^7$. As noted above, there is an element $G \in \mathrm{Spin}_7$ such that $G(e_1) = e_1$, $G(v) = ae_1 + be_2$, then $G(e_1 v A_8(1+\beta_8)) = e_1(ae_1 + be_2) \cdot A_8(1+\beta_8)$. By $-e_1e_4e_6e_7A_8(1+\beta_8) = A_8(1+\beta_8)$, we have

$$e_1(ae_1 + be_2)A_8(1+\beta_8) = (ae_1 - be_2)e_4e_6e_7A_8(1+\beta_8).$$

By Theorem 2.5, the 4-form part of $2^4 e_1 v A_8(1+\beta_8)$ is in $M_2(4,8)$. Thus $M_2(4,8)$ is a manifold diffeomorphic to S^7. The 4-form parts of $\pm 2^4 A_8(1+\beta_8)$ are the Cayley calibrations. If $v \neq \pm e_1$, replace $ae_1 - be_2$, $be_1 + ae_2$ in $G(e_1 v A_8(1+\beta_8)) = (ae_1 - be_2)e_4e_6e_7A_8(1+\beta_8)$ by e_1, e_2, one can show that the 4-form part of $2^4 e_1 v A_8(1+\beta_8)$ is a special Lagrangian calibration. $\square$

For any unit vector $v \perp e_1$, we can define a map $J_v : \boldsymbol{R}^8 \to \boldsymbol{R}^8$, $J_v(e_1) = v$, $J_v(v) = -e_1$; for any $w \perp e_1$, v, $J_v(w)$ is determined by $J_v(w)A_8(1+\beta_8) = -e_1 v w A_8(1+\beta_8)$. It is easy to see that J_v is a complex structure on $\boldsymbol{R}^8$ and $J_{e_2} = J$. On the other hand, for any such v, there is $G \in G_2$ such that $G(e_1) = e_1$, $G(v) = e_2$. It is easy to see that $J_v = G^{-1}JG$.

By Proposition 3.1, there are calibrations φ_i defined by $2^4 e_1 e_i A_8(1+\beta_8)$, $i = 1, \dots, 8$. For any $x = (x_1, x_2, \dots, x_8) \in S^7$, $\varphi = \sum_{i=1}^{8} x_i \varphi_i$ is also a calibration. φ_1 is Cayley calibration, the other φ_i are special Lagrangian calibrations with the complex structures J_{e_i}. Then for every element of V_8^+ with norm 4 determines a calibration.

It is interesting to note that the differential equations of φ_i-submanifolds can be determined by φ_j, where $j = 1, \dots, i-1, i+1, \dots, 8$ (cf. Dodak and Harvey [2]). This method of determine the differential equations for calibrations can be applied to all calibrations studied in this paper.

PROPOSITION 3.2. *The calibration set $M_2(2,8)$ is a manifold diffeomorphic to $S^6 = \{v \in S^7 \mid v \perp e_1\}$. The two form of $16e_1 v A_8(1+\beta_8)$ is a kaehler calibration on $R^8 \cong \boldsymbol{C}^4$ with respect to the complex structure J_v. The map $\pi_2 : G(2, \boldsymbol{R}^8) \to S^6$ defines a fibre bundle with fibre $\boldsymbol{C}P^3$.*

PROOF. Let $xA_8(1+\beta_8) = e_1 v A_8(1+\beta_8)$ and $v = ae_1 + bv'$, $v' \perp e_1$, where $x \in G(2, \boldsymbol{R}^8)$. From

$$\langle xA_8(1+\beta_8), A_8(1+\beta_8)\rangle = \langle x, A_8(1+\beta_8)\rangle = 0$$

and

$$\frac{1}{16} = \langle xA_8(1+\beta_8), e_1 vA_8(1+\beta_8)\rangle = \langle xA_8(1+\beta_8), be_1 v' A_8(1+\beta_8)\rangle \le \frac{1}{16}|b|,$$

we get $v \perp e_1$. The two form of $16e_1 vA_8(1+\beta_8)$ is a kaehler calibration on $R^8 = \mathbf{C}^4$ with respect to the complex structure J_v.

Then there is a map $\pi_2 : G(2, \boldsymbol{R}^8) \to S^6$, $\pi_2(x) = u$, if $xA_8(1+\beta_8) = e_1 uA_8(1+\beta_8)$. By Theorem 2.5, it is not difficult to show that for any $u \in S^6$,

$$\pi_2^{-1}(u) = \{vJ_u v \in G(2, \boldsymbol{R}^8) \mid v \in S^7\}.$$

Then $\pi_2^{-1}(u)$ is diffeomorphic to the complex projective space $\boldsymbol{CP}^3$. These complete the proof of the proposition. $\square$

For any $G \in \mathrm{Spin}_7$, we have the following commutative diagram:

$$\begin{array}{ccc} G(2,\boldsymbol{R}^8) & \xrightarrow{G} & G(2,\boldsymbol{R}^8) \\ \Big\downarrow{\scriptstyle \pi_2} & & \Big\downarrow{\scriptstyle \pi_2} \\ S^6 & \xrightarrow{\bar{G}} & S^6, \end{array}$$

where $\bar{G}$ is defined by $G(x)A_8(1+\beta_8) = e_1 \bar{G}(u) A_8(1+\beta_8)$, $u = \pi_2(x)$. It is easy to show that $\pi_2^{-1}(u)$ is a totally geodesic submanifold of $G(2, \boldsymbol{R}^8)$ for any $u \in S^6$.

The proof of the following proposition is similar to that of Proposition 3.1 and 3.2.

Proposition 3.3. *The calibration sets $M_2(r, 8)$ are all diffeomorphic to S^7 for $r = 1, 3, 5, 7$. Every element in $M_2(3, 8)$ is essentially an associative calibration on some 7-dimensional subspace of $\boldsymbol{R}^8$. For each $r = 1, 3, 5, 7$, $\pi_2 : G(r, \boldsymbol{R}^8) \to M_2(r, 8)$ defines a fibre bundle.*

Similar to the case of $G(2, \boldsymbol{R}^8)$, for any $G \in \mathrm{Spin}_7$, we have the following commutative diagram:

$$\begin{array}{ccc} G(3,\boldsymbol{R}^8) & \xrightarrow{G} & G(3,\boldsymbol{R}^8) \\ \Big\downarrow{\scriptstyle \pi_2} & & \Big\downarrow{\scriptstyle \pi_2} \\ S^7 & \xrightarrow{G} & S^7. \end{array}$$

The group Spin_7 acts transitively on $G(3, \boldsymbol{R}^8)$ and every fibre of $G(3, \boldsymbol{R}^8) \to S^7$ is a totally geodesic submanifold of $G(3, \boldsymbol{R}^8)$ and is denoted by ASSOC.

Remark. In [4], Gluck, Mackenzie and Morgan studied the volume-minimizing cycles in Grassmann manifolds. We can show that $\pi_2^{-1}(u)$ is a calibrated submanifold of calibration $\frac{1}{6}\omega^3$ on $G(2, \boldsymbol{R}^8)$ for any $u \in S^6$, where ω is the kaehler form on $G(2, \boldsymbol{R}^8)$ (cf. [4]) and $\pi_2 : G(2, \boldsymbol{R}^8) \to S^6$ is defined in Proposition 3.2. Let E be a vector bundle on $G(2, \boldsymbol{R}^8)$, the fibre on $e_1 \wedge e_2 \in G(2, \boldsymbol{R}^8)$ is $\{v \perp e_1, e_2 \mid v \in \boldsymbol{R}^8\}$. Then the Euler class of E defines a calibration on $G(2, \boldsymbol{R}^8)$ and the sphere $S^6 \subset G(2, \boldsymbol{R}^8)$ is a calibrated submanifold of this calibration. These gives an answer to the problem (5) of [4]. Let dv_{S^6} be the volume element of S^6. It is interesting to note that the 6-form $\pi_2^*(dv_{S^6})$ is also a calibration on $G(2, \boldsymbol{R}^8)$ and can be represented as a summand of $\frac{1}{6}\omega^3$. There is no calibrated submanifold of $\pi_2^*(dv_{S^6})$ even locally.

As is well-known, there is a Hopf fibration $S^7 \to S^4$ defined by quaternions. Combining this with Proposition 3.3, we have a map $\tau : G(3, \boldsymbol{R}^8) \to S^4$. We can show that every fibre of the map $\tau : G(3, \boldsymbol{R}^8) \to S^4$ is a calibrated submanifold of $\star p_1$ the dual of the Pontryagin form p_1 on $G(3, 8)$ (cf. [4]). Then $\tau^{-1}(v)$ is volume-minimizing in the holomogy class defined by $\tau^{-1}(v)$, any $v \in S^4$. This gives a partial answer to the problem (2) in [4].

PROPOSITION 3.4. *The map* $G(r, \boldsymbol{R}^8) \to M_2(r, 8)$ *defines a fibre bundle for each* $r \neq 4, 8$. *The bundles* $\pi_2 : G(r, \boldsymbol{R}^8) \to M_2(r, 8)$ *and* $\pi_2 : G(8-r, \boldsymbol{R}^8) \to M_2(8-r, 8)$ *are dual in the sense of the following commutative diagram*:

$$\begin{tikzcd} G(r, \boldsymbol{R}^8) \arrow[r, "\star"] \arrow[d, "\pi_2"] & G(8-r, \boldsymbol{R}^8) \arrow[d, "\pi_2"] \\ M_2(r, 8) \arrow[r, "(-1)^r"] & M_2(8-r, 8) \end{tikzcd}$$

where $\star$ *is the Hodge star operator.*

PROOF. The Hodge star operator $\star$ can also be defined by $\star\xi = \omega_8 \cdot \xi$, for $\xi \in C\ell_8 \cong \bigwedge(\boldsymbol{R}^8)$. Hence

$$\star x A_8(1+\beta_8) = (-1)^r x \omega_8 A_8(1+\beta_8) = (-1)^r x A_8(1+\beta_8)$$

for any $x \in G(r, \boldsymbol{R}^8)$. This proves Proposition. □

Now we turn to study the calibrations defined by $xA_6(1+\beta_6)(1+\omega_7)$. Let $C\ell_6$ and $C\ell_7$ be generated by $e_3, \dots, e_8$ and $e_2, e_3, \dots, e_8$ respectively. The isomorphism $\Psi : C\ell_7 \to C\ell_8^{even}$ is defined by $\Psi(\xi) = \xi$ for $\xi \in C\ell_7^{even}$, $\Psi(\psi) = e_1\psi$, for $\psi \in C\ell_7^{odd}$. It is easy to see that $2A_8 = e_1 A_6 + A_6\omega_7$, then

$$\Psi(A_6(1+\beta_6)(1+\omega_7)) = 2A_8(1+\beta_8).$$

Then for any $\xi \in C\ell_7$,

$$\Psi(\xi A_6(1+\beta_6)(1+\omega_7)) = 2e_1 v A_8(1+\beta_8)$$

for some $v \in \boldsymbol{R}^8$. Since the exceptional Lie group G_2 fixes $A_6(1+\beta_6)(1+\omega_7)$ and $A_8(1+\beta_8)$, we can assume that $v = ae_1 + be_2$. Replace $-ae_3 + be_4$, $-ae_4 - be_3$ by e_3, e_4 in $2^4(-a + be_1e_2)A_8(1+\beta_8)$, we get

$$\begin{aligned} e_1(e_{357} - e_{368} - e_{458} - e_{467}) &+ e_2(e_{468} - e_{457} - e_{358} - e_{367}) \\ &- a(1 - e_{5678} - e_{3478} - e_{3456}) - ae_1(\omega_7 - e_{234} - e_{256} - e_{278}) \\ &+ b(e_1e_2 + \cdots + e_7e_8 - e_{345678}) - be_1e_2(e_{3456} + e_{3478} + e_{5678}), \end{aligned}$$

where $e_{ij\cdots k} = e_ie_j\cdots e_k$. This shows

Proposition 3.5. *Denote $M(r,7)$ the calibration sets defined by the r-form part of $2^3 xA_6(1+\beta_6)(1+\omega_7)$, $x \in G(r, \boldsymbol{R}^7)$. Then*

(1) *There are two forms in $M(3,7)$ defined by 3-form parts of $\pm 2^3 A_6(1+\beta_6)\cdot(1+\omega_7)$ which are associative calibrations. The others are special Lagrangian calibrations. $M(3,7)$ is a manifold diffeomorphic to S^7;*

(2) *$M(2,7) \approx S^6$ is a set of kaehler calibrations and $\pi : G(2,\boldsymbol{R}^7) \to S^6$ is a fibre bundle with fibre $\boldsymbol{C}P^2$;*

(3) *The calibration manifolds $M(3,7)$ and $M(4,7)$; $M(2,7)$ and $M(5,7)$ are diffeomorphic respectively.*

Now we turn to study the calibration sets $M_1(r,8)$. Recall that, for any $x \in C\ell_8^{even}$, there are unit vectors v, w, such that $xA_8(1+\beta_8) = e_1 v A_8(1+\beta_8)$ and $xA_8(1-\beta_8) = e_1 w A_8(1-\beta_8)$. Then

$$2xA_8\beta_8 = e_1(v-w)A_8 + e_1(v+w)A_8\beta_8.$$

The following lemma gives the necessary conditions of for which $v, w \in \boldsymbol{R}^8$ there exists $x \in G(r, \boldsymbol{R}^8)$ such that $e_1 v A_8 + e_1 w A_8 \beta_8 = xA_8\beta_8$ or $vA_8 + wA_8\beta_8 = xA_8\beta_8$.

Lemma 3.6. *For any $v, w \in \boldsymbol{R}^8$, if there is some $x \in G(r, \boldsymbol{R}^8)$, such that $e_1 v A_8 + e_1 w A_8 \beta_8 = xA_8\beta_8$, for r even; $vA_8 + wA_8\beta_8 = xA_8\beta_8$, for r odd. Then the vectors v, w satisfy the following conditions:*

$$|v|^2 + |w|^2 = 1, \quad \langle v, w\rangle = \langle Jv, w\rangle = 0,$$

where J is the complex structure defined as above.

PROOF. Let r be an even number, $x \in G(r, \boldsymbol{R}^8)$, $xA_8\beta_8 = e_1vA_8 + e_1wA_8\beta_8$. The equality $|v|^2 + |w|^2 = 1$ follows from $2A_8^2\beta_8 = A_8$ and $\langle e_1vA_8, e_1wA_8\beta_8\rangle = \frac{1}{2}\langle v, wA_8\rangle = 0$. By $A_8^2 = \alpha(xA_8\beta_8)^t \cdot xA_8\beta_8$, we have $A_8(\beta_8vw\beta_8 + wv)A_8 = 0$. Since A_8 and $A_8\beta_8$ are invariant under the action of $SU(4)$, choose $G \in SU(4)$ such that

$$G(v) = a_1e_1, \quad G(w) = b_1e_1 + b_2e_2 + b_3e_3.$$

Then

$$\begin{aligned} &G(A_8(\beta_8vw\beta_8 + wv)A_8) \\ &\quad = A_8(-2a_1b_1 - 2a_1b_2e_1e_2)A_8 \\ &\quad = -2a_1b_1A_8^2 + 2a_1b_2e_1e_2A_8^2. \end{aligned}$$

This shows

$$a_1b_1 = \langle G(v), G(w)\rangle = \langle v, w\rangle = 0,$$

$$a_1b_2 = \langle JG(v), G(w)\rangle = \langle Jv, w\rangle = 0.$$

The case of r being odd can be proved similarly. □

PROPOSITION 3.7. *The calibration set $M_1(2,8)$ is a manifold with two singularities. Any element of $M_1(2,8)$ can be represented by 2-form of $e_1vA_8 + e_1wA_8\beta_8$, where $v = \sum_{i=3}^{8} a_ie_i$, $w = \sum_{j=2}^{8} b_je_j$ satisfy the conditions of Lemma 3.6. The singularities correspond to $v = 0$, $w = \pm e_2$.*

PROOF. For any $x \in G(2, \boldsymbol{R}^8)$, $xA_8\beta_8 = e_1vA_8 + e_1wA_8\beta_8$, we have

$$2\langle e_1vA_8, A_8\rangle = 2\langle xA_8\beta_8, A_8\rangle = \langle x, A_8\rangle = 0,$$

$$2\langle e_1vA_8, e_1e_2A_8\rangle = 2\langle xA_8\beta_8, e_1e_2A_8\rangle = \langle x, e_1e_2A_8\rangle = \langle x, B_8\rangle = 0.$$

These show that $v \perp e_1, e_2$. Similarly we can show that $w \perp e_1$. This shows that the conditions of the proposition are necessary. On the contrary, suppose that the vectors v, w satisfy the conditions of the proposition. We can assume that

$$v = a_3e_3, \quad w = b_2e_2 + b_5e_5, \quad a_3^2 + b_2^2 + b_5^2 = 1.$$

Then

$$e_1vA_8 + e_1wA_8\beta_8 = (a_3e_7 - b_2e_6 + b_5e_1)e_5A_8\beta_8$$

and the 2-form part of $e_1vA_8 + e_1wA_8\beta_8$ is in $M_1(2,8)$.

Finally, we show that omit two pints of $M_1(2,8)$, the remained set is a manifold of dimension 10. Let

$$F_1 = |v|^2 + |w|^2, \quad F_2 = \langle v, w\rangle, \quad F_3 = \langle Jv, w\rangle.$$

The Jacobian matrix $\frac{\partial(F_1, F_2, F_3)}{\partial(a_3, a_5, a_6, b_2, b_3, b_4, b_5)}$ at point $v = a_3 e_3$, $w = b_2 e_2 + b_5 e_5$ is

$$\begin{pmatrix} a_3 & & & b_2 & & & b_5 \\ & b_5 & & & a_3 & & \\ & & -b_5 & & & a_3 & \end{pmatrix}.$$

The rank of this matrix is 3 if and only if $a_3 \neq 0$ or $b_5 \neq 0$. This completes the proof of proposition. □

By Proposition 2.6, we know that the singularities of $M_1(2,8)$ are kaehler calibrations on $\boldsymbol{R}^8$, the other elements of $M_1(2,8)$ are kaehler calibrations on some 4-dimensional subspaces of $\boldsymbol{R}^8$. Then the fibres of the map $\pi_1 : G(2,\boldsymbol{R}^8) \to M_1(2,8)$ are all diffeomorphic to $\boldsymbol{CP}^1 \approx S^2$ expect two points. One can also show that $\pi_1^{-1}(p)$ is geodesic submanifold of $G(2,\boldsymbol{R}^8)$ for any $p \in M_1(2,8)$.

As noted above, we have the following commutative diagram:

$$\begin{tikzcd} G(2,\boldsymbol{R}^8) \arrow[r, "\pi_1"] \arrow[dr, "\pi_2"'] & M_1(2,8) \arrow[d, "\pi'"] \\ & M_2(2,8) \end{tikzcd}$$

For any $u \in S^6 \approx M_2(2,8)$, $u \neq e_2$, $\pi_2^{-1}(u)$ is differemorphic to $\boldsymbol{CP}^3$. We can show that $M = \pi'^{-1}(u)$ is differemorphic to S^4. Restrict the map π_1 on $\pi_2^{-1}(u)$, we get a fibre bundle

$$\pi_1 : \boldsymbol{CP}^3 \to S^4.$$

Every fibre of this map is S^2. The map $\pi_1 : \boldsymbol{CP}^3 \to S^4$ is just the well-known map $\boldsymbol{CP}^3 \to \boldsymbol{HP}^1$.

By Hodge star operator, we know that $M_1(6,8)$ is also a manifold with singularities. Now we turn to study $M_1(4,8)$.

Proposition 3.8. *The calibration set $M_1(4,8)$ is defined by these $v, w \in \boldsymbol{R}^8$ which satisfy the conditions of Lemma* 3.6 *and $\langle w, e_2\rangle = 0$. $M_1(4,8)$ is a manifold with singularities and the singularities correspond to $v = a_1 e_1 + a_2 e_2$, $w = 0$, $a_1^2 + a_2^2 = 1$.*

Proof. Let $v, w \in \boldsymbol{R}^8$ satisfy above conditions. We show that 4-form part of $e_1 v A_8 + e_1 w A_8 \beta_8$ is in $M_1(4,8)$. Choose $G \in SU(4)$, such that

$$G(v) = a_1 e_1, \quad G(w) = b_3 e_3.$$

We can also assume that $G(e_1)$ is a linear combination of the vectors e_1, e_2, e_3, e_5. Then

$$\begin{aligned}
&G(e_1 v A_8 + e_1 w A_8 \beta_8) \\
&\quad = G(e_1)(a_1 e_1 \beta_8 + b_3 e_3) A_8 \beta_8 \\
&\quad = G(e_1)(-a_1 e_1 e_2 e_3 e_6 e_7 - b_3 e_3 e_1 e_2 e_7 e_8) A_8 \beta_8 \\
&\quad = G(e_1) e_1 e_2 e_3 e_7 (a_1 e_6 - b_3 e_8) A_8 \beta_8 \\
&\quad = -G(e_1) e_4 e_7 (a_1 e_6 - b_3 e_8) A_8 \beta_8.
\end{aligned}$$

These shows that there is $x \in G(4, \boldsymbol{R}^8)$ such that

$$x A_8 \beta_8 = e_1 v A_8 + e_1 w A_8 \beta_8.$$

As in the proof of Proposition 3.7, we can show that if $e_1 v A_8 + e_1 w A_8 \beta_8 = x A_8 \beta_8$, for some $x \in G(4, \boldsymbol{R}^8)$, then $w \perp e_2$. $\square$

It is easy to see that the singularities of $M_1(4,8)$ defined by $\pm e_1 e_2 A_8 \beta_8$ are kaehler calibrations and the other singularities are special Lagrangian calibrations (cf. Proposition 2.6).

The proof of the following proposition is similar to that of Proposition 3.7 and 3.8.

Proposition 3.9. *The calibration sets $M_1(3,8)$ and $M_1(5,8)$ are both diffeomorphic to a submanifold of $S^{15} \subset \boldsymbol{R}^{16}$ defined by*

$$|v|^2 + |w|^2 = 1, \quad \langle v, w \rangle = \langle Jv, w \rangle = 0.$$

It is not difficult to show that $M_1(3,8)$ is a minimal submanifold of the sphere S^{15} with second fundamental form of constant length 24. There is a natural action of $SU(4)$ on $M_1(3,8)$ and $M_1(5,8)$, defined by $(v, w) \to (Gv, Gw)$, for any $G \in SU(4)$.

The calibrations defined by $2^3 x A_6(1+\beta_6)$ or $2^3 A_6 \beta_6$ can be studied similarly. By [9, §3.1], any elements in $V_6 = C\ell_6 A_6(1+\beta_6)$ can be represented as $(a + v + c\omega_6) A_6 (1+\beta_6)$. With the action of $SU(3)$, this can be changed into

$$
\begin{aligned}
(a + be_1 + c\omega_6)A_6(1 + \beta_6) \\
&= (-ae_3 + be_5 + ce_4)e_3A_6(1 + \beta_6) \\
&= (ae_3 - be_5 - ce_4)e_2e_6A_6(1 + \beta_6).
\end{aligned}
$$

It is easy to see that the 2-form part of $2^3(-ae_3 + be_5 + ce_4)e_3A_6(1 + \beta_6)$ is a calibration if and only if $a = 0$ and $b^2 + c^2 = 1$; the 3-form part is a calibration for all a, b, c with $a^2 + b^2 + c^2 = 1$. These prove

Proposition 3.10. *The calibration sets $M_1(2,6) = M_2(2,6)$ and $M_1(3,6) = M_2(3,6)$ are diffeomorphic to S^6 and S^7 respectively.*

§4. Isotropy Groups of Calibrations

In this section, we study the group action on the calibrations. For any $G \in SO(m)$, G can be extended to automorphisms $G : C\ell_m \to C\ell_m$ and $G : \bigwedge(\boldsymbol{R}^m) \to \bigwedge(\boldsymbol{R}^m)$, $\rho(G(\xi)) = G(\rho(\xi))$ for any $\xi \in C\ell_m$. Let ϕ be a calibration on $\boldsymbol{R}^m$. The subgroup of $SO(m)$ defined by $\{G \in SO(m) \mid G(\phi) = \phi\}$ is called the isotropy group of ϕ. As is well known, the special Lagrangian calibration A_{2n} is fixed by the action of elements of $SU(n) \subset SO(2n)$. Moreover, we have

Proposition 4.1. *The isotropy group of special Lagrangian calibration A_{2n} is $SU(n)$, when $2n \equiv 2, 6 \pmod 8$.*

Proof. Assuming $2n \equiv 2, 6 \pmod 8$, from $\bar{g}_1 \cdots \bar{g}_n \omega_{2n} = (\sqrt{-1})^n \bar{g}_1 \cdots \bar{g}_n$, we have $A_{2n}\omega_{2n} = (\sqrt{-1})^{n+1} B_{2n}$. If $G(A_{2n}) = A_{2n}$ for some $G \in SO(2n)$, let $g \in \mathrm{Spin}(2n)$ be a lift of G. Then $G(B_{2n}) = gB_{2n}g^t = B_{2n}$, hence $G(\bar{g}_1 \cdots \bar{g}_n) = \bar{g}_1 \cdots \bar{g}_n$. Write

$$G(\bar{g}_i) = \sum_j C_{i\bar{j}} \bar{g}_j + \sum_j D_{ij} g_j.$$

Denote $C = (C_{i\bar{j}})$, $D = (D_{ij})$, they satisfy

$$C\bar{C}^t + D\bar{D}^t = I.$$

From

$$G(\bar{g}_1 \cdots \bar{g}_n) = \det(C)\bar{g}_1 \cdots \bar{g}_n + \cdots + \det(D) g_1 \cdots g_n = \bar{g}_1 \cdots \bar{g}_n,$$

we have $\det C = 1$, hence $D = 0$. This shows $G \in SU(n)$. □

Proposition 4.2. *$SU(4k)$ is a subgroup of the isotropy group of the calibration $A_{8k}(1 + \beta_{8k})$.*

The proposition follows from $A_{2n}\beta_{2n} = 2A_{2n}^2$.

Lemma 4.3. *When* $2n \equiv 0, 6 \pmod 8$, *G is in the isotropy group of* $A_{2n}(1+\beta_{2n})$, *if and only if, G can be lifted to* $g \in \mathrm{Spin}(2n)$ *such that* $gA_{2n}(1+\beta_{2n}) = A_{2n}(1+\beta_{2n})$.

Proof. By Proposition 3.1.5 of [9], the equalities

$$G(A_{2n}(1+\beta_{2n})) = gA_{2n}(1+\beta_{2n}) \cdot \alpha([A_{2n}(1+\beta_{2n})]^t)g^t$$
$$= A_{2n}(1+\beta_{2n}) \cdot \alpha([A_{2n}(1+\beta_{2n})]^t)$$

hold if and only if $gA_{2n}(1+\beta_{2n}) = \pm A_{2n}(1+\beta_{2n})$. $\square$

The next lemma can be proved by using Proposition 3.2.4 of [9].

Lemma 4.4. *The element* $G \in SO(8k+7)$ *is in the isotropy group of* $A_{8k+6}(1+\beta_{8k+6})(1+\omega_{8k+7})$ *if and only if G can be lifted to* $g \in \mathrm{Spin}(8k+7)$, *such that* $gA_{8k+6}(1+\beta_{8k+6})(1+\omega_{8k+7}) = A_{8k+6}(1+\beta_{8k+6})(1+\omega_{8k+7})$, *or equivalently*, $gA_{8k+6}(1+\beta_{8k+6}\omega_{8k+7}) = A_{8k+6}(1+\beta_{8k+6}\omega_{8k+7})$.

Combining Lemma 4.3, 4.4 with Theorem 2.4, we know that the contact sets of some calibrations can be viewed as subsets of the isotropy groups of the calibrations. For example, let $P \subset SO(8k)$ be the isotropy group of $A_{8k}(1+\beta_{8k})$ and $\tilde{P} \subset \mathrm{Spin}(8k)$ is the lift of P such that $\tilde{P}|_{A_{8k}(1+\beta_{8k})} = 1$. Hence the contact set of $4l$-form part of $A_{8k}(1+\beta_{8k})$ are

$$\rho(\tilde{P}) \cap G(4l, \boldsymbol{R}^{8k}).$$

In general, the contact set of r-form part of $x_0A_{8k}(1+\beta_{8k})$, $x_0 \in G(r, \boldsymbol{R}^{8k})$, is

$$\rho(x_0\tilde{P}) \cap G(r, \boldsymbol{R}^{8k}).$$

From $A_{8k}(1+\beta_{8k})e_1e_2 = -e_1e_2A_{8k}(1-\beta_{8k})$, we know that the lift of the isotropy group of $A_{8k}(1-\beta_{8k})$ to $\mathrm{Spin}(8k)$ is $Ad(e_1e_2)\tilde{P}$. Since the isotropy group of A_{8k} is a subgroup of the isotropy group of $A_{8k}(1+\beta_{8k})$, the lift of the isotropy group of A_{8k} is $\tilde{P} \cap Ad(e_1e_2)\tilde{P}$ and the contact set of r-form of $x_0A_{8k}\beta_{8k}$ is

$$\rho[x_0(\tilde{P} \cap Ad(e_1e_2)\tilde{P})] \cap G(r, \boldsymbol{R}^{8k}),$$

where $x_0 \in G(r, \boldsymbol{R}^{8k})$.

§5. Calibrations on Manifolds

In this section, we study calibrations on Riemannian manifolds. First we have

THEOREM 5.1. *Let M be a spin manifold with dimension $m \leq 9$. If there is a parallel real pinor or spinor field σ on M with unit norm. Then $\rho(\sigma \cdot \alpha(\sigma^t))$ is a harmonic form. Furthermore, the homogeneous parts of $\rho(\sigma \cdot \alpha(\sigma^t))$ are calibrations on M. If $m = 7$, the 3 or 4-form parts of $\rho(\sigma \cdot \alpha(\sigma^t))$ are associative or co-associative calibrations respectively. If $m = 8$, the 4-form part of $\rho(\sigma \cdot \alpha(\sigma^t))$ is a Cayley calibration.*

PROOF. Let σ be a parallel real spinor field on M with $\|\sigma\| \equiv 1$. As shown in [5], spin group $\mathrm{Spin}(m)$ acts on the unit sphere in spinor spaces transitively if $m \leq 9$. Then the theorem follows from Theorem 2.4. □

The following theorems concern the conditions of existence calibrations on Riemannian manifolds.

THEOREM 5.2. *Let M be a compact Riemannian manifold with dimension 8. If there is a Cayley calibration or a special Lagrangian calibration on M, then*

(1) $H^4(M) \neq 0$;

(2) *The structure group of M can be reduced to* Spin_7 *the isotropy group of Cayley form*;

(3) *M has a spin structure*;

(4) $p_1(M)^2 - 4p_2(M) + 8\chi(M) = 0$, *where $p_i(M)$ are Pontrjagin forms on M.*

PROOF. It is easy to see that $16A_8(1 + \beta_8)$ is a sum of Cayley form and $1 + \omega_8$. Let ϕ be a Cayley calibration on M. Then $\phi = \star\phi$ is a harmonic form and there is a pinor bundle defined by

$$S = \{\xi_q(1 + \omega_8 + \phi) \mid \xi_q \in C\ell_q(M), q \in M\},$$

where ω_8 is the volume element on M and ϕ is viewed as a section of $C\ell(M)$. Let P be the frame bundle over M formed by all frames on M with which ϕ can be represented in canonical Cayley form. Obviously, the structure group of P is Spin_7. The existence of spin structure follows from Lemma 4.3. For (4), see Theorem 10.7 on p. 349 in [8].

On the other hand, let ψ be a special Lagrangian calibration on M. Also denote the correspond element in $\Gamma(C\ell(M))$ by ψ. Since $A_8 + 2A_8 \cdot A_8 =$

$A_8(1+\beta_8)$, from $\psi + \frac{1}{8}\psi \cdot \psi$, we can get a Cayley form on M. Notice also that ψ is a harmonic form $(\omega_8 \cdot A_8 = A_8)$. □

Combining Theorem 5.1, 5.2 with Theorem 10.20 of [8, p. 356], we have

THEOREM 5.3. *Let M be a Riemannian manifold with dimension* 8. *Then there is a Cayley calibration on M if and only if M is spin and there is a parallel pinor or spinor field on M.*

PROPOSITION 5.4. *Let M be an oriented Riemannian manifold. If there is a special Lagrangian calibration ψ on M, then*

(1) *When* $2n \equiv 2$ *or* 6 (mod 8), *the structure group of M can be reduced to $SU(n)$, hence there is a complex structure on M. Moreover, M is spin and there is a pinor bundle generated by $\psi - (\sqrt{-1})^n \psi \cdot \omega_{2n}$ as subbundle of $\boldsymbol{C}\ell(M)$;*

(2) *When* $2n \equiv 0$ *or* 6 (mod 8), *there is a real pinor bundle over M generated by $\psi + \frac{1}{2^{n-1}}\psi \cdot \psi$.*

PROOF. With the notations used in previous sections, we have

$$A_{2n} + \sqrt{-1}B_{2n} = A_{2n} - (\sqrt{-1})^n A_{2n} \cdot \omega_{2n}, \quad \text{if } 2n \equiv 2, 6 \pmod 8,$$

$$A_{2n}(1+\beta_{2n}) = A_{2n} + 2A_{2n} \cdot A_{2n}, \quad \text{if } 2n \equiv 0, 6 \pmod 8.$$

Then the proposition follows from Proposition 4.1, 4.2 and the results of §3 in [9]. □

PROPOSITION 5.5. *If there is an associative or coassociative calibration on a Riemannian manifold with dimension* 7. *Then*

(1) *The structure group of the manifold can be reduced to exceptional Lie group G_2;*

(2) *The manifold has a spin structure and there is a pinor bundle on it.*

The proof of the proposition is similar to that of Proposition 5.4, so we omit it.

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