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IRREDUCIBLE CLIFFORD MODULES

By

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Abstract. This paper gives a new treatment of the Clifford algebras. We represent the pinor and spinor spaces as subspaces of the Clifford algebras and use these pinors to construct isomorphisms between the Clifford algebras and the matrix algebras. In doing these we develop some spinor calculus.

1. Introduction

The Clifford algebras and the related topics have come to play important role in mathematics and in mathematical physics. In the area of differential geometry and topology they have become fundamental.

As is well-known, the structures of Clifford algebras can be studied by induction. This paper gives a new treatment. We construct the real and complex pinor spaces as subspaces of Clifford algebras and use them to construct isomorphisms between the Clifford algebras and matrix algebras. H. B. Lawson and M. Michelsohn pointed out that there must exist a local spinor calculus, like the tensor calculus, which should be an important component of local Riemannian geometry (see [5], Introduction). In a subsequent paper [10], we shall use spinor calculus developed in this paper to study the Grassmann manifolds and the calibrations.

The methods used in this paper are simple. Let $\triangle_{2n}$ be an irreducible module over complex Clifford algebra $\boldsymbol{C}\ell_{2n}$, $\triangle_{2n}$ can be generated by one element of $\boldsymbol{C}\ell_{2n}$ as a left ideal. We show in §2 that, there is a decomposition $\boldsymbol{C}\ell_{2n} = \triangle \cdot \bar{\triangle}^t \cong \triangle \otimes \triangle$ as bimodules over $\boldsymbol{C}\ell_{2n}$. Thus we can construct explicit isomorphisms between complex Clifford algebras and matrix algebras. The subspace S_{2n} of $C\ell_{2n}$ generated by $x + \bar{x}$, $\sqrt{-1}(x - \bar{x})$ is a module over $C\ell_{2n}$, $x \in \triangle_{2n}$. If S_{2n} is

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reducible, it can be decomposed into a direct sum of irreducible modules. In this way we show in §3 that the irreducible modules over $C\ell_m$ can be generated by the one of the following elements (for the notations see the end of this section):

(1) A_m, when $m \equiv 2, 4 \pmod 8$;
(2) $A_m(1 + \beta_m)$, when $m \equiv 0, 6 \pmod 8$;
(3) $A_{m-1}(1 + \beta_{m-1})$, when $m \equiv 1 \pmod 8$;
(4) $A_{m-1}(1 \pm \omega_m)$, when $m \equiv 3 \pmod 8$;
(5) $A_{m-1}(1 + \beta_m)$, when $m \equiv 5 \pmod 8$;
(6) $A_{m-1}(1 + \beta_{m-1})(1 \pm \omega_m)$, when $m \equiv 7 \pmod 8$.

It is interesting to note that the number of the summand in $A_m(1 + \beta_m)$ is $2^{m/2}$ which is also the dimension of the irreducible modules over $C\ell_m$, when $m \equiv 0, 6 \pmod 8$. The numbers of the summand in other generators listed above are also closely related to the dimensions of the correspond irreducible Clifford modules.

These pinors have many interesting properties and applications. With an orthonormal basis for every irreducible module over $C\ell_m$, we construct isomorphism between the Clifford algebra and the matrix algebra. In §4, we study $C\ell_8$ in some details. As is well known (see for example [2] or [3]), the octonians can be used to study Clifford algebra and spin group. In an appendix we show that the octonians can also be defined by Clifford algebra.

In the following, we give some notations used in this paper. Let $\boldsymbol{R}^m$ be an Euclidean space and $C\ell_m$ be its associated Clifford algebra, $\boldsymbol{C\ell}_m = C\ell_m \otimes \boldsymbol{C}$ be the corresponding complex Clifford algebra. Let $e_1, \ldots, e_m$ be an orthonormal basis of $\boldsymbol{R}^m$, then $C\ell_m$ is generated by $\{e_i\}$ with the relations: $e_ie_j + e_je_i = -2\delta_{ij}$. The homomorphism $\alpha : C\ell_m \to C\ell_m$ is defined by

$$\alpha(\xi) = \xi, \quad \text{if } \xi \in C\ell_m^{even}; \quad \alpha(\eta) = -\eta, \quad \text{if } \eta \in C\ell_m^{odd}.$$

Let $\omega_m = e_1e_2\cdots e_m$ be the volume element of $C\ell_m$. Then $\omega_m^2 = 1$ if $m \equiv 0, 3 \pmod 4$. The element $\beta_m \in C\ell_m$ is defined by

$$\beta_m = \begin{cases} e_1e_3\cdots e_{m-3}e_{m-1}, & m \text{ even;} \\ e_1e_3\cdots e_{m-2}e_m, & m \text{ odd.} \end{cases}$$

Note that $\beta_m^2 = 1$ if and only if $m \equiv 0, 5, 6, 7 \pmod 8$, otherwise $\beta_m^2 = -1$.

If $m = 2n$ being even, let $g_i = \frac{1}{2}(e_{2i-1} - \sqrt{-1}e_{2i})$, $\bar{g}_i = \frac{1}{2}(e_{2i-1} + \sqrt{-1}e_{2i})$, $i = 1, \ldots, n$. It is easy to see that $g_ig_i = \bar{g}_i\bar{g}_i = 0$ and $g_i\bar{g}_ig_i = -g_i$. Denote $A_{2n} = \mathrm{Re}(\bar{g}_1\cdots\bar{g}_n)$ and $B_{2n} = \mathrm{Im}(\bar{g}_1\cdots\bar{g}_n)$.

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2. Complex Clifford Algebras

It is well known that the complex Clifford algebra $\boldsymbol{C}\ell_{2n}$ decomposes into a direct sum of irreducible left modules over $\boldsymbol{C}\ell_{2n}$. Denote one of these modules by $\triangle$, $\triangle$ is a spin representation space of spin group $Spin(2n)$. If $\xi = \lambda v_1 \cdots v_k \in \boldsymbol{C}\ell_{2n}$, $\lambda \in \boldsymbol{C}$, $v_i \in \boldsymbol{R}^{2n}$, we set $\bar{\xi}^t = \bar{\lambda} v_k \cdots v_1$. The space $\triangle' = \{\bar{\xi}^t \mid \xi \in \triangle\}$ is an irreducible right module over $\boldsymbol{C}\ell_{2n}$.

Lemma 2.1. *Let $\triangle$ be a spin representation space described as above, then $\boldsymbol{C}\ell_{2n} = \triangle \cdot \triangle'$, where $\cdot$ denote the Clifford product. Hence there is an isomorphism of bimodules*:

$$\boldsymbol{C}\ell_{2n} \cong \triangle \otimes \triangle'.$$

Proof. Let $\alpha_i = \sqrt{-1} e_{2i-1} e_{2i}$, $g_i = \frac{1}{2}(e_{2i-1} - \sqrt{-1} e_{2i})$, $\bar{g}_i = \frac{1}{2}(e_{2i-1} + \sqrt{-1} e_{2i})$, $i = 1, \dots, n$. Let the $\{\alpha_i\}$ act on $\boldsymbol{C}\ell_{2n}$ from the right and decompose $\boldsymbol{C}\ell_{2n}$ into 2^n simultaneous eigenspaces of this action (cf. [6]). Let $\triangle(\varepsilon_1, \dots, \varepsilon_{2n})$ be one of such space, $\triangle(\varepsilon_1, \dots, \varepsilon_{2n})$ be eigenspace of α_i with eigenvalue ε_i for $1 \le i \le n$, $\varepsilon_i = 1$ or -1. Then $\triangle = \triangle(-1, \dots, -1)$ is generated by $\bar{g}_1 \cdots \bar{g}_n$ as left ideal of $\boldsymbol{C}\ell_{2n}$. It is easy to see that $\triangle \cdot g_1 \cdots g_n \bar{g}_{i_1} \cdots \bar{g}_{i_k} = \triangle(\tau_1, \dots, \tau_n)$ with $\tau_{i_1} = \cdots = \tau_{i_k} = -1$, $\tau_p = 1$ for $p \ne i_1, \dots, i_k$. These prove that

$$\boldsymbol{C}\ell_{2n} = \bigoplus \triangle(\varepsilon_1, \dots, \varepsilon_n) = \triangle \cdot \triangle'.$$

Since $\dim \triangle \cdot \dim \triangle' = \dim \boldsymbol{C}\ell_{2n}$, there is a natural isomorphism: $\boldsymbol{C}\ell_{2n} = \triangle \cdot \triangle' \to \triangle \otimes \triangle'$ and the decomposition is invariant under the right and the left action of the elements of $\boldsymbol{C}\ell_{2n}$. The lemma has been proved. □

Let $\underline{so}(2n)$ and $\underline{spin}(2n)$ be the Lie algebras of special orthonormal group $SO(2n)$ and spin group $Spin(2n)$ respectively. There is an isomorphism $\Xi : \underline{spin}(2n) \to \underline{so}(2n)$ defined as usual. The following lemma is well known, its proof can be reduced to the case of $n = 1$.

Lemma 2.2. *The exponential maps* $\exp : \underline{spin}(2n) \to Spin(2n)$ *and* $\exp : \underline{so}(2n) \to SO(2n)$ *are epimorphisms and* $\exp \circ \Xi = Ad \circ \exp : \underline{spin}(2n) \to SO(2n)$, *where* $Ad : Spin(2n) \to SO(2n)$ *is the covering map.*

From this lemma we know that for any $g = \exp(h_{ij}) \in SO(2n)$, the lift of g to $Spin(2n)$ is given by

$$\pm\exp\left(\frac{1}{4}\sum h_{ij}e_ie_j\right).$$

Let $\bigwedge(\boldsymbol{R}^{2n})$ be the exterior algebra of $\boldsymbol{R}^{2n}$ and $\bigwedge_{\boldsymbol{c}}(\boldsymbol{R}^{2n}) = \bigwedge(\boldsymbol{R}^{2n}) \otimes \boldsymbol{C}$. There is an isomorphism $\rho : \bigwedge_{\boldsymbol{c}}(\boldsymbol{R}^{2n}) \to \boldsymbol{C}\ell_{2n}$, given by $\rho(e_{i_1} \wedge \cdots \wedge e_{i_k}) = e_{i_1}\cdots e_{i_k}$, $i_1 < \cdots < i_k$.

LEMMA 2.3. *For any $g \in SO(2n)$, $\tilde{g}$ is a lift of g to $Spin(2n)$, then we have $\rho \circ g = Ad(\tilde{g}) \circ \rho : \bigwedge_{\boldsymbol{c}}(\boldsymbol{R}^{2n}) \to \boldsymbol{C}\ell_{2n}$, where $g : \bigwedge_{\boldsymbol{c}}(\boldsymbol{R}^{2n}) \to \bigwedge_{\boldsymbol{c}}(\boldsymbol{R}^{2n})$ and $Ad(\tilde{g}) = g : \boldsymbol{C}\ell_{2n} \to \boldsymbol{C}\ell_{2n}$ are the induced maps.*

By Lemma 2.1–2.3, we can show that the de Rham-Hodge and Signature operators on a Riemannian manifold are essentially the twisted Atiyah-Singer operators. If the manifold has a spin structure, the de Rham-Hodge and the Signature operators are twisted Atiyah-Singer operators in usual sense (see [7, 8]).

Using Lemma 2.1, we can construct isomorphism between $\boldsymbol{C}\ell_{2n}$ and matrix algebra $\boldsymbol{C}(2^n)$. Denote $\tau = (\sqrt{-1})^n e_1 \cdots e_{2n}$. We have half spinor spaces $\triangle^{\pm} = (1 \pm \tau)\triangle$. Let $\lambda = (\lambda_1, \ldots, \lambda_k)$ be multiindex and $\beta_\lambda = g_{\lambda_1} \cdots g_{\lambda_k} \bar{g}_1 \cdots \bar{g}_n \in \triangle = \triangle^+ \oplus \triangle^-$, $\lambda_1 < \cdots < \lambda_k$. Then β_λ and $\beta_\lambda \cdot \alpha(\bar{\beta}_\mu^t)$ form vector bases of $\triangle$ and $\boldsymbol{C}\ell_{2n} = \triangle \cdot \bar{\triangle}^t$ respectively.

LEMMA 2.4. $\beta_\lambda \cdot \alpha(\bar{\beta}_\mu^t) \cdot \beta_\tau = \begin{cases} 0, & \mu \neq \tau, \\ \beta_\lambda, & \mu = \tau. \end{cases}$

PROOF. Since $g_i g_i = \bar{g}_i \bar{g}_i = 0$ and $\bar{g}_i \cdot g_i \cdot \bar{g}_i = -\bar{g}_i$, the lemma can be verified easily. □

Given an order of $\lambda = (\lambda_1, \ldots, \lambda_k)$, $\lambda_1 < \cdots < \lambda_k$, $k = 0, \ldots, n$. Let $E_{\lambda\mu}$ be the standard elements of matrix algebra $\boldsymbol{C}(2^n)$. Then we have

PROPOSITION 2.5. *The homomorphism $\Phi : \boldsymbol{C}\ell_{2n} \to \boldsymbol{C}(2^n)$ defined by*

$$\Phi(\beta_\lambda \cdot \alpha(\bar{\beta}_\mu^t)) = E_{\lambda\mu}$$

is an algebraic isomorphism.

As $\boldsymbol{C}\ell_{2n+1} \cong \boldsymbol{C}\ell_{2n+2}^{even}$, there is an isomorphism $\boldsymbol{C}\ell_{2n+1} \cong \boldsymbol{C}(2^n) \oplus \boldsymbol{C}(2^n)$. In what follows, we construct such an isomorphism directly.

Let $h = \frac{1}{2}(1 + \sqrt{-1}e_{2n+1})$ *and* $\bar{h} = \frac{1}{2}(1 - \sqrt{-1}e_{2n+1})$, *then we have*

$$\boldsymbol{C}\ell_{2n+1} = \triangle \cdot h \cdot \triangle' \oplus \triangle \cdot \bar{h} \cdot \triangle'.$$

Proposition 2.6. *There is a natural isomorphism between two algebras*:

$$\Psi : \boldsymbol{C}\ell_{2n+1} \to \begin{pmatrix} \boldsymbol{C}(2^n) & \\ & \boldsymbol{C}(2^n) \end{pmatrix},$$

which is given by $\triangle \cdot h \cdot \triangle' \cong \boldsymbol{C}\ell_{2n}$ *and* $\triangle \cdot \bar{h} \cdot \triangle' \cong \boldsymbol{C}\ell_{2n}$.

Proof. We need only to show that

$$\triangle \cdot h \cdot \triangle' \cdot \triangle \cdot \bar{h} \cdot \triangle' = \triangle \cdot \bar{h} \cdot \triangle' \cdot \triangle \cdot h \cdot \triangle' = 0,$$

and both $\triangle \cdot h \cdot \triangle'$ and $\triangle \cdot \bar{h} \cdot \triangle'$ are isomorphic to the Clifford algebra $\boldsymbol{C}\ell_{2n}$. These can be proved by using Lemma 2.4 and the fact that $\xi h = \bar{h}\xi$ for $\xi \in \boldsymbol{C}\ell_{2n}^{odd}$, $\eta h = h\eta$ for $\eta \in \boldsymbol{C}\ell_{2n}^{even}$. $\square$

The spaces $\triangle \cdot h$ and $\triangle \cdot \bar{h}$ are irreducible modules over $\boldsymbol{C}\ell_{2n+1}$.

Remark. As more application, we consider Seiberg-Witten monopole equations:

$$D_A\Phi = 0, \quad \sigma^+(F_A) = (\Phi\Phi^*)_0.$$

For notations see D. Salamon: Spin geometry and Seiberg-Witten invariants. By Lemma 2.1, 2.2 and 2.3 above, see also [7], we know that there is a natural isomorphism between $S \otimes S^*$ and $\boldsymbol{C}\ell(X) \cong \bigwedge_c(X)$, where $S = S^+ \oplus S^-$ is a *spin*^{c} bundle on a 4-manifold X. Then for any $\Phi \in \Gamma(S^+)$, $\Phi \cdot \overline{\Phi}^t$ can be viewed as a section of $\boldsymbol{C}\ell^+(X) \cap \boldsymbol{C}\ell^{even}(X) \cong \bigwedge_c^+(X) \cap \bigwedge_c^{even}(X)$. Locally, $\Phi, \tau \in \Gamma(S^+)$ can be represented by

$$\Phi = a\bar{g}_1\bar{g}_2 + bg_1g_2\bar{g}_1\bar{g}_2, \quad \tau = c\bar{g}_1\bar{g}_2 + dg_1g_2\bar{g}_1\bar{g}_2.$$

By Lemma 2.4, $\Phi\Phi^*\tau = \Phi\langle\Phi, \tau\rangle = (\bar{a}c + \bar{b}d)\Phi = \Phi \cdot \overline{\Phi}^t \cdot \tau$, where $\cdot$ is the Clifford product. Then as a section of $\boldsymbol{C}\ell^+(X) \cap \boldsymbol{C}\ell^{even}(X)$, we have

$$\begin{aligned} \Phi\Phi^* &= \Phi \cdot \overline{\Phi}^t \\ &= a\bar{a}\bar{g}_1\bar{g}_2g_2g_1 + b\bar{b}g_1g_2\bar{g}_2\bar{g}_1 + a\bar{b}\bar{g}_2\bar{g}_1 + \bar{a}bg_1g_2 \\ &= \frac{1}{4}(a\bar{a} + b\bar{b})(1 - e_1e_2e_3e_4) + \frac{\sqrt{-1}}{4}(a\bar{a} - b\bar{b})(e_1e_2 + e_3e_4) \\ &\quad + \frac{1}{4}(\bar{a}b - a\bar{b})(e_1e_3 - e_2e_4) - \frac{\sqrt{-1}}{4}(\bar{a}b + a\bar{b})(e_2e_3 + e_1e_4). \end{aligned}$$

3. Irreducible Modules over $C\ell_m$

As an algebra, $C\ell_m$ is isomorphic to some matrix algebra, then any irreducible module over $C\ell_m$ is generated by one element. In this section we construct such elements and study their properties.

3.1. Even Dimensional Case

As in §2, let $\triangle_{2n}$ be an irreducible module over $\boldsymbol{C\ell}_{2n}$ generated by $\bar{g}_1 \cdots \bar{g}_n$. Let S_{2n} be the realization space of $\triangle_{2n}$, that is, S_{2n} is a subspace of $C\ell_{2n}$ generated by $x + \bar{x}$, $\sqrt{-1}(x - \bar{x})$, $x \in \triangle_{2n}$. Denote

$$\bar{g}_1 \cdots \bar{g}_n = A_{2n} + \sqrt{-1}B_{2n}, \quad A_{2n}, B_{2n} \in S_{2n}.$$

LEMMA 3.1.1. *We have* (1) $B_{2n} = e_1e_2A_{2n} = -A_{2n}e_1e_2$, (2) $A_{2n}\beta_{2n} = \beta_{2n}A_{2n}$, (3) $4A_{2n}\alpha(A_{2n}^t)A_{2n} = A_{2n}$.

PROOF. By $\bar{g}_1 \cdots \bar{g}_n\beta_{2n} = \bar{g}_1 \cdots \bar{g}_n g_1 \cdots g_n$ and $\bar{g}_1 \cdots \bar{g}_n e_1e_2 = -e_1e_2\bar{g}_1 \cdots \bar{g}_n = \sqrt{-1}\bar{g}_1 \cdots \bar{g}_n$, we have $A_{2n}\beta_{2n} = \beta_{2n}A_{2n}$, $B_{2n} = e_1e_2A_{2n} = -A_{2n}e_1e_2$. Then

$$\bar{g}_1 \cdots \bar{g}_n = A_{2n}(1 - \sqrt{-1}e_1e_2).$$

The equality $\bar{g}_1 \cdots \bar{g}_n g_n \cdots g_1 \bar{g}_1 \cdots \bar{g}_n = (-1)^n \bar{g}_1 \cdots \bar{g}_n$ yields

$$4A_{2n}A_{2n}^tA_{2n}(1 - \sqrt{-1}e_1e_2) = (-1)^n A_{2n}(1 - \sqrt{-1}e_1e_2).$$

This shows $4A_{2n}\alpha(A_{2n}^t)A_{2n} = A_{2n}$. □

This lemma is important in our study. Equation (3) is used in the isomorphism between the Clifford algebras and the matrix algebras.

LEMMA 3.1.2. *The space S_{2n} is a left module over $C\ell_{2n}$ generated by A_{2n} or B_{2n}. This space is invariant under the action of $e_1e_2, \beta_{2n}, \omega_{2n}$ on the right of it. Moreover,* $C\ell_{2n} = S_{2n} \cdot \alpha(S_{2n}^t)$.

PROOF. Obviously, S_{2n} is a left module over $C\ell_{2n} \subset \boldsymbol{C\ell}_{2n}$ with $\dim S_{2n} = 2^{n+1}$ and is generated by A_{2n} or B_{2n}. The equality $C\ell_{2n} = S_{2n} \cdot \alpha(S_{2n}^t)$ follows form $\boldsymbol{C\ell}_{2n} = \triangle_{2n} \cdot \triangle'_{2n}$. □

PROPOSITION 3.1.3. *The spaces $V_{8k+2} = S_{8k+2}$ and $V_{8k+4} = S_{8k+4}$ are irreducible and they are also the right $C\ell_2$-modules. Then there are quaternion structures on V_{8k+2} and V_{8k+4} respectively.*

Proof. For dimensional reasons, $V_{2n} = S_{2n}$ is irreducible if $2n \equiv 2$ or 4 (mod 8). These can also be proved by using Proposition 3.1.10. The subspace of $C\ell_{2n}$ generated by e_1e_2 and β_{2n} is isomorphic to $C\ell_2$. This shows that the pinor spaces V_{8k+2} and V_{8k+4} are right modules of $C\ell_2$. □

Proposition 3.1.4. *If* $2n \equiv 0, 6 \pmod 8$, *the space* $V_{2n} = S_{2n}(1+\beta_{2n})$ *is an irreducible module over* $C\ell_{2n}$.

Proof. When $2n \equiv 0, 6 \pmod 8$, $\beta_{2n}^2 = 1$, the module S_{2n} can be decomposed into

$$S_{2n} = S_{2n}(1+\beta_{2n}) \oplus S_{2n}(1-\beta_{2n}).$$

As $\beta_{2n}e_1e_2 = -e_1e_2\beta_{2n}$, acting e_1e_2 on the right of $S_{2n}(1+\beta_{2n})$ induces a module isomorphism between $S_{2n}(1+\beta_{2n})$ and $S_{2n}(1-\beta_{2n})$. □

Now we turn to study the isomorphisms between the Clifford algebras and the matrix algebras. Let $c_\lambda = e_{2\lambda_1-1}e_{2\lambda_2-1}\cdots e_{2\lambda_k-1}$. With the notations used in §2, we have

$$\beta_\lambda = c_\lambda \bar{g}_1 \cdots \bar{g}_n = c_\lambda(1+\sqrt{-1}e_1e_2)A_{2n}.$$

Then the elements

$$c_\lambda A_{2n}, \quad c_\lambda A_{2n}e_1e_2,$$

$$\lambda = (\lambda_1, \ldots, \lambda_k), \quad \lambda_1 < \cdots < \lambda_k, \quad k = 0, 1, \ldots, n,$$

form a basis of S_{2n}.

Lemma 3.1.5. (1) *If* $\lambda_1 = \mu_1 = 1$, *we have*

$$\alpha(c_\lambda A_{2n})^t \cdot c_\mu A_{2n} = \delta_\mu^\lambda \alpha(A_{2n}^t)A_{2n},$$

where $\delta_\mu^\lambda = \delta_{\mu_1,\ldots,\mu_l}^{\lambda_1,\ldots,\lambda_k}$ *is the Kronecker delta.*

(2) $c_\lambda A_{2n}\beta_{2n} = (-1)^{k(k+1)/2}\delta(\lambda,\mu)c_\mu A_{2n}$, *where* $\mu = (\mu_1, \ldots, \mu_{n-k})$ *and* $\delta(\lambda,\mu) = \delta_{1,2,\ldots,n}^{\lambda_1,\ldots,\lambda_k,\mu_1,\ldots,\mu_{n-k}}$.

Proof. From $\alpha(\bar{\beta}_\lambda^t)\beta_\mu = (-1)^n\delta_\mu^\lambda g_n \cdots g_1\bar{g}_1\cdots\bar{g}_n$, we have

$$(-1)^k A_{2n}^t[c_\lambda^t c_\mu + e_2e_1c_\lambda^t c_\mu e_1e_2]A_{2n} = 2\delta_\mu^\lambda A_{2n}^t A_{2n}.$$

As $\lambda_1 = \mu_1 = 1$, $e_2e_1c_\lambda^t c_\mu e_1e_2 = c_\lambda^t c_\mu$. These prove (1). The equation (2) follows from

$$
\begin{aligned}
&c_\lambda(1+\sqrt{-1}e_1e_2)A_{2n}\beta_{2n}\\
&\quad = g_{\lambda_1}\cdots g_{\lambda_k}\bar{g}_1\cdots\bar{g}_n g_1\cdots g_n\\
&\quad = (-1)^{k(k-1)/2}\delta(\lambda,\mu)g_{\lambda_k}\cdots g_{\lambda_1}\bar{g}_{\lambda_1}\cdots\bar{g}_{\lambda_k}\bar{g}_{\mu_1}\cdots\bar{g}_{\mu_{n-k}}g_1\cdots g_n\\
&\quad = (-1)^{k(k+1)/2}\delta(\lambda,\mu)\bar{g}_{\mu_1}\cdots\bar{g}_{\mu_{n-k}}g_1\cdots g_n\\
&\quad = (-1)^{k(k+1)/2}\delta(\lambda,\mu)c_\mu(1-\sqrt{-1}e_1e_2)A_{2n}. \qquad \square
\end{aligned}
$$

PROPOSITION 3.1.6. *There is an isomorphism of bimodules over* $C\ell_{2n}$:

$$C\ell_{2n} = V_{2n}\cdot\alpha(V_{2n}^t) \cong V_{2n}\otimes\alpha(V_{2n}^t),$$

when $2n \equiv 0$ *or* $6 \pmod 8$.

PROOF. When $2n \equiv 0$ or $6 \pmod 8$, by Lemma 3.1.2 and $A_{2n}(1+\beta_{2n})\cdot\alpha(A_{2n}^t) = B_{2n}(1-\beta_{2n})\alpha(B_{2n}^t)$, we have

$$
\begin{aligned}
C\ell_{2n} &= S_{2n}(1+\beta_{2n})\alpha(S_{2n}^t) + S_{2n}(1-\beta_{2n})\alpha(S_{2n}^t)\\
&= S_{2n}(1+\beta_{2n})\alpha(S_{2n}^t) = [S_{2n}(1+\beta_{2n})]\cdot\alpha([S_{2n}(1+\beta_{2n})]^t). \qquad \square
\end{aligned}
$$

PROPOSITION 3.1.7. *In the cases of* $2n \equiv 0$ *or* $6 \pmod 8$, *the algebraic isomorphism* $\Phi: C\ell_{2n} \to \boldsymbol{R}(2^n)$ *can be defined by*

$$\Phi[f_\alpha A_{2n}(1+\beta_{2n})\cdot\alpha(f_\beta A_{2n}(1+\beta_{2n}))^t] = E_{\alpha\beta},$$

where $f_\alpha, f_\beta = c_\lambda$ *or* $c_\lambda e_1e_2$, $\lambda_1 = 1$.

PROOF. By Proposition 3.1.4 and Lemma 3.1.5, the following pinors form a basis of V_{2n} if $2n \equiv 0$ or $6 \pmod 8$,

$$c_\lambda A_{2n}(1+\beta_{2n}), \quad c_\lambda e_1e_2A_{2n}(1+\beta_{2n}),$$

where $\lambda_1 = 1$.

By Lemma 3.1.5, if $\lambda_1 = \mu_1 = 1$, we have

$$(-1)^{k+1}A_{2n}^tc_\lambda^tc_\mu e_1e_2A_{2n} = (-1)^kA_{2n}^tc_\lambda^tc_\mu A_{2n}e_1e_2 = \delta_\mu^\lambda A_{2n}^tA_{2n}e_1e_2.$$

Combining this with $e_1e_2(1+\beta_{2n}) = (1-\beta_{2n})e_1e_2$ shows

$$\alpha[c_\lambda A_{2n}(1+\beta_{2n})]^t\cdot c_\mu e_1e_2A_{2n}(1+\beta_{2n}) = 0.$$

Then the proposition follows from Lemma 3.1.1, 3.1.5 and Proposition 3.1.6. $\square$

In the rest of this paper we always assume that $\lambda_1 = \mu_1 = 1$ in $\lambda = (\lambda_1, \dots, \lambda_k)$ and $\mu = (\mu_1, \dots, \mu_l)$.

In case of $C\ell_6$, V_6 is generated by $A = A_6(1+\beta_6)$. Let

$$\alpha_1 = \omega_6 A, \quad \alpha_2 = e_1 A, \quad \alpha_3 = e_3 A, \quad \alpha_4 = e_5 A,$$

$$\alpha_5 = A, \quad \alpha_6 = e_2 A, \quad \alpha_7 = e_4 A, \quad \alpha_8 = e_6 A.$$

Similar to Lemma 4.1, we can show that the equalities $\alpha(\alpha_k^t) \cdot \alpha_l = \delta_{kl} A$ hold for all k, l. Thus $\{\alpha_k\}$ forms a basis of V_6. The algebraic isomorphism $\Phi : C\ell_6 \to \boldsymbol{R}(8)$ can also be given by

$$\Phi(\alpha_k \cdot \alpha(\alpha_l^t)) = E_{kl}, \quad k, l = 1, 2, \dots, 8$$

One can compute,

$$\Phi\Big(\sum v_i e_i\Big) = \begin{pmatrix} 0 & v_2 & v_4 & v_6 & 0 & -v_1 & -v_3 & -v_5 \\ -v_2 & 0 & v_5 & -v_3 & v_1 & 0 & -v_6 & v_4 \\ -v_4 & -v_5 & 0 & v_1 & v_3 & v_6 & 0 & -v_2 \\ -v_6 & v_3 & -v_1 & 0 & v_5 & -v_4 & v_2 & 0 \\ 0 & -v_1 & -v_3 & -v_5 & 0 & -v_2 & -v_4 & -v_6 \\ v_1 & 0 & -v_6 & v_4 & v_2 & 0 & -v_5 & v_3 \\ v_3 & v_6 & 0 & -v_2 & v_4 & v_5 & 0 & -v_1 \\ v_5 & -v_4 & v_2 & 0 & v_6 & -v_3 & v_1 & 0 \end{pmatrix}.$$

Denote $\Phi(\sum v_i e_i) = E$. If $\sum v_i^2 = 1$, then $E^2 = -I$, $EE^t = I$.

Notice that

$$\Phi(C\ell_6^{even}) = \left\{ \begin{pmatrix} C & D \\ -D & C \end{pmatrix} \mid C, D \in \boldsymbol{R}(4) \right\},$$

$$\Phi(C\ell_6^{odd}) = \left\{ \begin{pmatrix} C & D \\ D & -C \end{pmatrix} \mid C, D \in \boldsymbol{R}(4) \right\},$$

$$\begin{aligned}\Phi(Spin(6)) &= \{E \in \Phi(C\ell_6^{even}) \mid EE^t = I\} \\ &= SU(4) \subset SO(8).\end{aligned}$$

By definition, $\Phi(\alpha(\xi^t)) = (\Phi(\xi))^t$ for any $\xi \in C\ell_6$, these also shows

$$Spin(6) = \{\xi \in C\ell_6^{even} \mid \xi \cdot \xi^t = 1\}.$$

PROPOSITION 3.1.8. (1) *There is a* $C\ell_{8k}$-$C\ell_{8k+2}$-*bimodule isomorphism*

$$C\ell_{8k+2} \cong V_{8k+2} \otimes \alpha(V_{8k}^t);$$

(2) *There is an isomorphism of bimodules over* $C\ell_{8k+2}$,

$$C\ell_{8k+2} \cong V_{8k+2} \otimes_{C\ell_2} \alpha(V^t_{8k+2}).$$

PROOF. It is easy to see that $-e_{8k+1}A_{8k+2} = \frac{1}{2}A_{8k}(1 - e_1e_2e_{8k+1}e_{8k+2})$. The map $C\ell_2 \cdot V_{8k} \to V_{8k+2}$ given by $x \to \frac{1}{2}x(1 - e_1e_2e_{8k+1}e_{8k+2})$ is an isomorphism. Combining this with $C\ell_{8k+2} = C\ell_2 \cdot V_{8k} \cdot \alpha(V^t_{8k})$ gives (1). (2) follows from Lemma 3.1.2 and Proposition 3.1.3 and the dimensions of $C\ell_{8k+2}$ and $V_{8k+2} \otimes_{C\ell_2} \alpha(V^t_{8k+2})$. □

PROPOSITION 3.1.9. (1) *There is a* $C\ell_{8k+1}$-$C\ell_{8k+4}$-*bimodule isomorphism*

$$C\ell_{8k+4} \cong V_{8k+4} \otimes \alpha(V^t_{8k+1}).$$

(2) *As bimodules over* $C\ell_{8k+4}$,

$$C\ell_{8k+4} \cong V_{8k+4} \otimes_{C\ell_2} \alpha(V^t_{8k+4}).$$

PROOF. Let $\widehat{C\ell_4}$ and $\widehat{C\ell_{8k}}$ be subspaces of $C\ell_{8k+4}$ generated by $e_{8k+1}, \dots, e_{8k+4}$ and $e_1, \dots, e_{8k}$ respectively. Let $\hat{V}_4$ and $\hat{V}_{8k}$ be irreducible modules over $\widehat{C\ell_4}$ and $\widehat{C\ell_{8k}}$ respectively. Denote $\hat{\omega}_4 = e_{8k+1} \cdots e_{8k+4}$. It is easy to see that

$$\hat{V}_4 = \widehat{C\ell_4}(1 - \hat{\omega}_4), \quad (1 + \hat{\omega}_4)e_{8k+1} = e_{8k+1}(1 - \hat{\omega}_4).$$

These shows $\widehat{C\ell_4} = \hat{V}_4 \cdot \widehat{C\ell_1}$, where $\widehat{C\ell_1}$ is generated by e_{8k+1}. Then

$$\begin{aligned} C\ell_{8k+4} &= \widehat{C\ell_4} \cdot \widehat{C\ell_{8k}} = \hat{V}_4 \cdot \widehat{C\ell_1} \cdot \hat{V}_{8k}\alpha(\hat{V}^t_{8k}) \\ &= \hat{V}_4 \cdot \hat{V}_{8k}\alpha(\hat{V}^t_{8k}) \cdot \widehat{C\ell_1} \end{aligned}$$

As modules over $C\ell_{8k+4}$, $V_{8k+4} \cong \hat{V}_4 \cdot \hat{V}_{8k}$, we have proved

$$C\ell_{8k+4} \cong V_{8k+4} \otimes \alpha(V^t_{8k+1}).$$

The proof of (2) is similar to that of Proposition 3.1.8. □

Let $\boldsymbol{H}$ be the field of quaternions and $i, j, k = ij$ be defined as usual. Notice that e_1e_2 and β_{2n} commute with $A_{2n}\alpha(A^t_{2n})$. The following proposition can be proved by using Lemma 3.1.5.

PROPOSITION 3.1.10. *When* $2n \equiv 2$ *or* $4 \pmod 8$, *the map* $\Phi : C\ell_{2n} \to \boldsymbol{H}(2^{n-1})$ *defined by*

$$\Phi[4c_\lambda A_{2n}(x_1 + x_2e_1e_2 + x_3\beta_{2n} + x_4e_1e_2\beta_{2n}) \cdot \alpha(c_\mu A_{2n})^t]$$
$$= (x_1 + x_2 i + x_3 j + x_4 k)E_{\lambda\mu},$$

is an isomorphism, where $x_i \in R$.

The case of $C\ell_4$ is easy. The isomorphism $\Phi : C\ell_4 \to \boldsymbol{H}(\boldsymbol{2})$ can be defined by

$$\Phi\Big(\sum v_l e_l\Big) = \begin{pmatrix} & v_1 + v_2 i + v_3 j + v_4 k \\ -v_1 + v_2 i + v_3 j + v_4 k & \end{pmatrix}.$$

For Proposition 3.1.6, 3.1.8 and 3.1.9, see also [5], II, §7.

3.2. Odd Dimensional Case

Let V_{2n} be an irreducible module defined as in §§3.1. Let

$$T_{2n+1} = \{(a + be_{2n+1})v \mid a, b \in R, v \in V_{2n}\}.$$

Then T_{2n+1} is a left module over $C\ell_{2n+1}$. For dimensional reasons we have

Proposition 3.2.1. *The space* $V_{8k+1} = T_{8k+1}$ *is a left irreducible module over* $C\ell_{8k+1}$ *and a right* $C\ell_1$ *module* (*where* $C\ell_1$ *can be generated by* e_{8k+1}).

Proposition 3.2.2. *If* $2n + 1 \equiv 3$ *or* $7 \pmod 8$, *the spaces* $V_{2n+1} = T_{2n+1}(1 + \omega_{2n+1})$ *and* $V'_{2n+1} = T_{2n+1}(1 - \omega_{2n+1})$ *are two different irreducible modules over* $C\ell_{2n+1}$. *Furthermore,* V_{8k+3} *and* V'_{8k+3} *are also the right* $C\ell_2$ *modules.*

Proof. Since ω_{2n+1} is in the center of $C\ell_{2n+1}$, for any $v \in V_{2n+1}$, one has

$$\omega_{2n+1}v(1 + \omega_{2n+1}) = v(1 + \omega_{2n+1})\omega_{2n+1} = v(1 + \omega_{2n+1}),$$
$$\omega_{2n+1}v(1 - \omega_{2n+1}) = v(1 - \omega_{2n+1})\omega_{2n+1} = -v(1 - \omega_{2n+1}).$$

This shows, V_{2n+1} and V'_{2n+1} are not isomorphic as left modules. Since V_{8k+2} is a right $C\ell_2$ module and ω_{8k+3} is in the center of $C\ell_{8k+3}$, the spaces V_{8k+3} and V'_{8k+3} are right $C\ell_2$ modules. □

Proposition 3.2.3. *The space* $V_{8k+5} = T_{8k+5}(1 + \beta_{8k+5})$ *is an irreducible module over* $C\ell_{8k+5}$. *There is a complex structure on* V_{8k+5} *defined by acting* ω_{8k+5} *on the right of it.*

The proof is similar to that of Proposition 3.1.4 (note that ω_{8k+5} is in the center of $C\ell_{8k+5}$).

PROPOSITION 3.2.4. (1) $C\ell_{8k+1} \cong V_{8k+1} \otimes \alpha(V_{8k}^t) \cong V_{8k+1} \otimes_{C\ell_1} \alpha(V_{8k+1}^t)$;
(2) $C\ell_{8k+3} \cong V_{8k+3} \otimes \alpha(V_{8k}^t) \oplus V'_{8k+3} \otimes \alpha(V_{8k}^t)$
$\cong V_{8k+3} \otimes_{C\ell_2} \alpha(V_{8k+3}^t) \oplus V'_{8k+3} \otimes_{C\ell_2} \alpha(V_{8k+3}^{\prime t})$;
(3) $C\ell_{8k+5} \cong V_{8k+5} \otimes \alpha(V_{8k+2}^t) \cong V_{8k+5} \otimes_{C\ell_1} \alpha(V_{8k+5}^t)$;
(4) $C\ell_{8k+7} \cong V_{8k+7} \otimes \alpha(V_{8k+6}^t) \oplus V'_{8k+7} \otimes \alpha(V_{8k+6}^t)$
$\cong V_{8k+7} \otimes \alpha(V_{8k+7}^t) \oplus V'_{8k+7} \otimes \alpha(V_{8k+7}^{\prime t})$.

PROOF. The proof of (1) is easy. Equation (2) follows from Proposition 3.1.8 and 3.2.2.

By $e_{8k+2}(1+\beta_{8k+5}) = (1-\beta_{8k+5})e_{8k+2}$, we have

$$T_{8k+5} = T_{8k+5}(1+\beta_{8k+5}) \oplus T_{8k+5}(1-\beta_{8k+5}) = T_{8k+5}(1+\beta_{8k+5})\widehat{C\ell}_1,$$

where $\widehat{C\ell}_1$ is generated by e_{8k+2}. By Proposition 3.1.9,

$$C\ell_{8k+5} \cong T_{8k+5} \cdot \alpha(V_{8k+1}^t) \cong V_{8k+5}\widehat{C\ell}_1 \cdot \alpha(V_{8k+1}^t) \cong V_{8k+5} \cdot \alpha(V_{8k+2}^t).$$

As $A_{8k+4}(1-\beta_{8k+5})e_1e_2 = -e_1e_2A_{8k+4}(1+\beta_{8k+5})$ and $\alpha(\beta_{8k+5}^t) = \beta_{8k+5}$, we have

$$\begin{aligned} C\ell_{8k+5} &= \widetilde{C\ell}_1 V_{8k+4} \cdot \alpha(V_{8k+4}^t)\widetilde{C\ell}_1 \\ &= T_{8k+5}(1+\beta_{8k+5})\alpha(T_{8k+5}^t) + T_{8k+5}(1-\beta_{8k+5})\alpha(T_{8k+5}^t) \\ &= T_{8k+5}(1+\beta_{8k+5})\alpha(T_{8k+5}^t) \\ &= [T_{8k+5}(1+\beta_{8k+5})] \cdot \alpha([T_{8k+5}(1+\beta_{8k+5})]^t) \\ &\cong V_{8k+5} \otimes_{\widehat{C\ell}_1} \alpha(V_{8k+5}^t), \end{aligned}$$

where $\widetilde{C\ell}_1$ and $\widehat{C\ell}_1$ are generated by e_{8k+5} and ω_{8k+5} respectively. These prove (3).

In case of $C\ell_{8k+7}$, one has

$$C\ell_{8k+7} = C\ell_1 \cdot V_{8k+6} \cdot \alpha(V_{8k+6}^t) \cong V_{8k+7} \otimes \alpha(V_{8k+6}^t) \oplus V'_{8k+7} \otimes \alpha(V_{8k+6}^t),$$

and

$$C\ell_{8k+7} = C\ell_1 V_{8k+6} \cdot \alpha(V_{8k+6}^t) C\ell_1 \cong (V_{8k+7} \oplus V'_{8k+7}) \cdot \alpha(V_{8k+7}^t \oplus V_{8k+7}^{\prime t}).$$

(4) follows from the facts that $V_{8k+7} \cdot \alpha(V_{8k+7}^{\prime t}) = 0$ and $V'_{8k+7} \cdot \alpha(V_{8k+7}^t) = 0$. □

In the following, we construct the isomorphisms between Clifford algebras and matrix algebras for odd dimensional case.

PROPOSITION 3.2.5. *The isomorphism* $\Phi : C\ell_{8k+1} \to \boldsymbol{C}(2^{4k})$ *can be given by*

$$\Phi(f_\alpha A(x_1 + x_2\omega_{8k+1})\alpha(f_\beta A)^t) = (x_1 + x_2 i)E_{\alpha\beta},$$

where $f_\alpha, f_\beta = c_\lambda,\ A = A_{8k}(1 + \beta_{8k})$.

Denote $A = A_{8k+2}(1 + \omega_{8k+3})$. The elements

$$c_\lambda A, \quad c_\lambda A e_1 e_2, \quad c_\lambda A\beta_{8k+2}, \quad c_\lambda A e_1 e_2 \beta_{8k+2},$$

form a basis of V_{8k+3}, where c_λ are defined as in §3.1, $\lambda_1 = 1$.

PROPOSITION 3.2.6. *The isomorphism* $\Phi : C\ell_{8k+3} \to \boldsymbol{H}(2^{4k}) \oplus \boldsymbol{H}(2^{4k})$ *can be given by*

$$\begin{aligned}&\Phi[f_\alpha A(x_1 + x_2 e_1 e_2 + x_3\beta_{8k+2} + x_4 e_1 e_2 \beta_{8k+2})\alpha(f_\beta A)^t \\ &\qquad + f_{\alpha'} A'(x_1' + x_2' e_1 e_2 + x_3'\beta_{8k+2} + x_4' e_1 e_2\beta_{8k+2})\alpha(f_{\beta'} A')^t] \\ &= \begin{pmatrix} (x_1 + x_2 i + x_3 j + x_4 k)E_{\alpha\beta} & \\ & (x_1' + x_2' i + x_3' j + x_4' k)E_{\alpha'\beta'} \end{pmatrix},\end{aligned}$$

where $f_\alpha, f_{\alpha'} = c_\lambda,\ A = A_{8k+2}(1 + \omega_{8k+3})$ *and* $A' = A_{8k+2}(1 - \omega_{8k+3})$.

PROPOSITION 3.2.7. *The isomorphism* $\Phi : C\ell_{8k+5} \to \boldsymbol{C}(2^{4k+2})$ *can be defined by*

$$\Phi[(x_1 + x_2\omega_{8k+5})f_\alpha A\alpha(f_\beta A)^t] = (x_1 + x_2 i)E_{\lambda\mu},$$

where $A = A_{8k+4}(1 + \beta_{8k+5}),\ x_i \in R$ *and* $f_\alpha, f_\beta = c_\lambda$ *or* $c_\lambda e_1 e_2$.

By Proposition 3.2.3, V_5 is generated by $A = \frac{1}{4}(1 - \omega_4)(1 + \beta_5)$ as left module of $C\ell_5$. Let

$$\alpha_1 = A, \quad \alpha_2 = e_1 A, \quad \alpha_3 = e_1 e_4 A, \quad \alpha_4 = e_4 A.$$

It is easy to see that each α_i commutes with ω_5 and $\alpha(\alpha_j^t)\cdot\alpha_k = \delta_{jk}A$. Then α_i, $\omega_5\alpha_i$, $i = 1, \ldots, 4$, form a basis of V_5 and ω_5 defines a complex structure on V_5. The isomorphism $\Phi : C\ell_5 \to \boldsymbol{C}(4)$ can be defined by

$$\Phi((x + y\omega_5)\alpha_i \cdot \alpha(\alpha_j^t)) = (x + \sqrt{-1}y)E_{ij}.$$

By easy computation, for any $v \in \boldsymbol{R}^5$, we have

$$\Phi\left(\sum_{j=1}^{5} v_j e_j\right) = \begin{pmatrix} -v_5 i & -v_1 - v_3 i & 0 & -v_4 - v_2 i \\ v_1 - v_3 i & v_5 i & v_4 + v_2 i & 0 \\ 0 & -v_4 + v_2 i & -v_5 i & v_1 - v_3 i \\ v_4 - v_2 i & 0 & -v_1 - v_3 i & v_5 i \end{pmatrix}.$$

Notice that $\Phi(\sum v_j e_j) \in SU(4)$ if $\sum_{j=1}^{5} v_j^2 = 1$, also

$$\Phi(C\ell_5^{even}) = \left\{ \begin{pmatrix} C & D \\ -\bar{D} & \bar{C} \end{pmatrix} \mid C, D \in \boldsymbol{C}(2) \right\},$$

$$\Phi(C\ell_5^{odd}) = \left\{ \begin{pmatrix} C & D \\ \bar{D} & -\bar{C} \end{pmatrix} \mid C, D \in \boldsymbol{C}(2) \right\}.$$

PROPOSITION 3.2.8. *The isomorphism* $\Phi : C\ell_{8k+7} \to \boldsymbol{R}(2^{4k-1}) \oplus \boldsymbol{R}(2^{4k-1})$ *can be defined by*

$$\Phi\left[\frac{1}{4} f_\alpha A \cdot \alpha(f_\beta A)^t + \frac{1}{4} f_{\alpha'} A' \cdot \alpha(f_{\beta'} A')^t\right] = \begin{pmatrix} E_{\alpha\beta} & \\ & E_{\alpha'\beta'} \end{pmatrix},$$

where $A = A_{8k+6}(1 + \beta_{8k+6})(1 + \omega_{8k+7})$, $A' = A_{8k+6}(1 + \beta_{8k+6})(1 - \omega_{8k+7})$, f_α, f_β *as in the case of* $C\ell_{8k+6}$.

4. The Clifford Algebra $C\ell_8$

The Clifford algebra $C\ell_8$ is important in the theory and application. In this section we study $C\ell_8$ in some details. By Proposition 3.1.6, $C\ell_8 = V_8 \cdot \alpha(V_8^t)$ and V_8 is generated by $A = A_8(1 + \beta_8)$. Let $\alpha_i = e_1 e_i A$, $\alpha_{i+8} = e_i A$, $i = 1, 2, \ldots, 8$.

LEMMA 4.1. $\alpha(\alpha_k^t) \cdot \alpha_l = \delta_{kl} A$, $k, l = 1, 2, \ldots, 16$.

PROOF. $16A$ is invariant by acting every summand of $16A$ on itself, then $A \cdot A = A$. Also note that for any $1 \le i < j < k \le 8$, there is a unique summand in A which contains $e_i e_j e_k$. We need only to show $\alpha(\alpha_k^t) \cdot \alpha_l = 0$ for $k \ne l$. This can be proved as follows.

$$\begin{aligned} \alpha(\alpha_7^t) \cdot \alpha_{10} &= \frac{1}{2} A e_7 e_1 e_2 (1 - e_1 e_2 e_7 e_8) A \\ &= \frac{1}{2} A (1 + e_1 e_2 e_7 e_8) e_7 e_1 e_2 A = 0. \end{aligned}$$

The other cases can be proved similarly. □

The elements $\{\alpha_i\}_{1 \le i \le 8} (\{\alpha_{i+8}\}_{1 \le i \le 8})$ form bases of V_8^+ (V_8^-) respectively, where $V_8^\pm = (1 \pm \omega_8) V_8$.

PROPOSITION 4.2. *The algebraic isomorphism* $\Phi : C\ell_8 \cong \boldsymbol{R}(16)$ *can be defined by*

$$\Phi(\alpha_k \cdot \alpha(\alpha_j^t)) = E_{kj}, \quad k, j = 1, \ldots, 16.$$

It is easy to see that $\Phi(\xi) = T^t$ for any $\xi \in C\ell_8$, where $T \in \boldsymbol{R}(16)$ are determined by

$$\xi\begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_{16} \end{pmatrix} = T\begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_{16} \end{pmatrix}.$$

Then $\Phi(\omega_8) = \begin{pmatrix} I & \\ & -I \end{pmatrix}$. By simple computation, we have $\Phi(\sum v_i e_i) = \begin{pmatrix} & P_v \\ -P_v^t & \end{pmatrix}$, where

$$P_v = \begin{pmatrix} v_1 & v_2 & v_3 & v_4 & v_5 & v_6 & v_7 & v_8 \\ -v_2 & v_1 & -v_4 & v_3 & -v_6 & v_5 & -v_8 & v_7 \\ -v_3 & v_4 & v_1 & -v_2 & v_7 & -v_8 & -v_5 & v_6 \\ -v_4 & -v_3 & v_2 & v_1 & -v_8 & -v_7 & v_6 & v_5 \\ -v_5 & v_6 & -v_7 & v_8 & v_1 & -v_2 & v_3 & -v_4 \\ -v_6 & -v_5 & v_8 & v_7 & v_2 & v_1 & -v_4 & -v_3 \\ -v_7 & v_8 & v_5 & -v_6 & -v_3 & v_4 & v_1 & -v_2 \\ -v_8 & -v_7 & -v_6 & -v_5 & v_4 & v_3 & v_2 & v_1 \end{pmatrix}.$$

If $\xi, \eta \in C\ell_7 \subset C\ell_8$ generated by $e_2, e_3, \ldots, e_8$, then

$$\Phi(\xi) = \begin{pmatrix} C_1 & D_1 \\ D_1 & C_1 \end{pmatrix}, \quad \Phi(e_1\eta) = \begin{pmatrix} C_2 & D_2 \\ -D_2 & -C_2 \end{pmatrix}.$$

Thus $\Phi : C\ell_8 \to \boldsymbol{R}(16)$ induces an isomorphism

$$\Phi' : C\ell_7 \to \boldsymbol{R}(8) \oplus \boldsymbol{R}(8).$$

The algebraic structure on $\boldsymbol{R}(8) \oplus \boldsymbol{R}(8)$ is defined by

$$(A, B)(C, D) = (AC + BD, AD + BC), \quad (A, B), (C, D) \in \boldsymbol{R}(8) \oplus \boldsymbol{R}(8).$$

The following proposition is helpful for our understanding the spin group (see also [3], p. 273).

Proposition 4.3. *For any $G \in SO(V_8^+)$, there are two elements $g_1, g_2 \in Spin(8)$, such that $g_1 = g_2\omega_8$, $g_1|_{V_8^+} = G$. The elements g_1 and g_2 can be constructed from G.*

Proof. We first show that, for any $g_1, g_2 \in Spin(8)$, $\Phi(g_i) = \begin{pmatrix} A_i & \\ & B_i \end{pmatrix}$, $i = 1, 2$, if $A_1 = A_2$, then $B_1 = \pm B_2$. That is, $g_1 = g_2$ or $g_1 = g_2\omega_8$. We need only to show that if $\Phi(g) = \begin{pmatrix} I & \\ & B \end{pmatrix}$, $g \in Spin(8)$, then $g = 1$ or $g = \omega_8$. From $\Phi(w) =$

$\Phi(g)\Phi(v)\Phi(g^t)$ for $v, w = gvg^t \in \boldsymbol{R}^8$, we have $P_w = P_v B^t$ and $B^t = P_{e_1} B^t = P_{v_0}$ with $v_0 = ge_1g^t = \sum a_i e_i$. Set $v = e_2$ in $P_w = P_v B^t = P_v P_{v_0}$, we can show that $a_3 = a_4 = \cdots = a_8 = 0$. In this way we can show $v_0 = \pm e_1$, hence $B = P^t_{\pm e_1} = \pm I$, $g = 1$ or $g = \omega_8$.

Next we show that the matrices $T_{ij} = P_{e_i} P_{e_j}$, $1 \le i < j \le 8$, are linearly independent. If there are real numbers b_{ij} such that $\sum_{i<j} b_{ij} T_{ij} = 0$. Then from $\Phi(e_ie_j) = \begin{pmatrix} -P_{e_i}P^t_{e_j} & \\ & -P^t_{e_i}P_{e_j} \end{pmatrix}$ and $-\sum_{i<j} b_{ij} P^t_{e_i} P_{e_j} = -2\sum_j b_{1j} P_{e_j}$, we have

$$\Phi\Big(\exp \sum b_{ij} e_i e_j\Big) = \begin{pmatrix} I & \\ & \exp\Big(-2\sum_j b_{1j} P_{e_j}\Big) \end{pmatrix} \in \Phi(Spin(8)).$$

This shows that $b_{1j} = 0$ for $j = 2, \dots, 8$, then $b_{ij} = 0$ for all $i < j$. Thus $\{T_{ij}\}$ is a basis of $\underline{so}(8)$, the Lie algebra of $SO(8)$. Notice that $T_{ij} \in SO(8)$.

For any $A \in SO(8) \cong SO(V_8^+)$, we can write $A^t = \exp\Big(\sum_{i<j} b_{ij} T_{ij}\Big)$. Let $g_1 = \exp \sum b_{ij} e_i e_j$ and $g_2 = g_1 \omega_8$. Then $g_1, g_2 \in Spin(8)$ as claimed. □

From this proposition, we know that if $g \in Spin(8)$, $\Phi(g) = \begin{pmatrix} A & \\ & B \end{pmatrix}$, and $A = \exp(\sum b_{ij} T_{ij})$, then $g = \exp(\sum b_{ij} e_i e_j)$ or $\exp(\sum b_{ij} e_i e_j)\omega_8$. Hence

$$B = \pm\exp\Big(-\sum b_{1j} T_{1j} + \sum_{i \ne 1} b_{ij} T_{ij}\Big).$$

Let $C\ell_{8k}$ and $C\ell_l$ $(1 \le l \le 8)$ be subalgebras of $C\ell_{8k+l}$ generated by $e_1, \dots, e_{8k}$ and $e_{8k+1}, \dots, e_{8k+l}$ respectively, where $e_1, e_2, \dots, e_{8k+l}$ is an orthonormal basis of $\boldsymbol{R}^{8k+l}$. Let ω_{8k} be the volume element of $C\ell_{8k}$ and $\widehat{C\ell_l}$ be a subalgebra of $C\ell_{8k+l}$ generated by $e_{8k+i}\omega_{8k}$, $i = 1, \dots, l$. Then $uv = vu$ for any $u \in C\ell_{8k}$, $v \in \widehat{C\ell_l}$.

Lemma 4.4. $\widehat{C\ell_l} \cong C\ell_l$, $C\ell_{8k+l} = C\ell_{8k} \cdot \widehat{C\ell_l} \cong C\ell_{8k} \otimes \widehat{C\ell_l}$.

Remark. The Clifford algebra $C\ell(m,n)$ is generated by $e_1, \dots, e_{m+n}$ with the relations:

$$e_ie_j + e_je_i = \begin{cases} -2 & \text{if } i = j = 1, \dots, m; \\ 2 & \text{if } i = j = m+1, \dots, m+n; \\ 0 & \text{if } i \ne j. \end{cases}$$

In the case of $m = n$, let $h_i = \frac{1}{2}(e_i - e_{m+i})$, $\bar{h}_i = \frac{1}{2}(e_i + e_{m+i})$, $i = 1, \dots, m$, then $h_i \cdot h_i = \bar{h}_i \cdot \bar{h}_i = 0$, $\bar{h}_i h_i \bar{h}_i = -h_i$, $h_i \bar{h}_i h_i = -h_i$. The pinor space of $C\ell(m,m)$ can be generated by $\bar{h}_1 \cdots \bar{h}_m$. As Proposition 2.5, one can use elements $h_i, \bar{h}_i$ to construct isomorphism between $C\ell(m,m)$ and $\boldsymbol{R}(2^m)$.

For the general case, as Lemma 4.4, one can show that $C\ell(m,n) \cong C\ell(m-n,0) \otimes C\ell(n,n)$ if $m > n$; $C\ell(m,n) \cong C\ell(0,n-m) \otimes C\ell(m,m)$ if $m < n$.

Appendix

As is well-known (see for example [2] or [3]), the octonians can be used to study Clifford algebra and spin group. In this appendix we show that the octonians can also be defined by Clifford algebra. For notations see §4.

Fix an isomorphism $\phi : V_8^- \to \boldsymbol{R}^8$ defined by $\phi(e_i A) = e_i$.

DEFINITION. Define a product $\circ$ on $\boldsymbol{R}^8$ by

$$x \circ y = -\phi(y e_1 x A)$$

for all $x, y \in \boldsymbol{R}^8$.

LEMMA 1. *For any* $x = \sum_{i=1}^{8} x_i e_i$, $y = \sum_{j=1}^{8} y_j e_j \in \boldsymbol{R}^8$, *we have*

$$x \circ y = (x_1, \dots, x_8) P_y \begin{pmatrix} e_1 \\ \vdots \\ e_8 \end{pmatrix}.$$

PROOF. By definition,

$$\begin{aligned} x \circ y &= -\sum \phi(x_i y_j e_j \alpha_i) \\ &= \sum_i x_i y_1 e_i + \sum_{j>1} \phi((x_1, \dots, x_8) y_j P_{e_j} \begin{pmatrix} \alpha_9 \\ \vdots \\ \alpha_{16} \end{pmatrix}) \\ &= (x_1, \dots, x_8) P_y \begin{pmatrix} e_1 \\ \vdots \\ e_8 \end{pmatrix}. \qquad \square \end{aligned}$$

The conjugate $\bar{x}$ for any $x \in \boldsymbol{R}^8$ is defined by $\bar{x} = -e_1 x e_1$. Then $x \circ \bar{x} = -\phi(\bar{x} e_1 x A) = -\phi(e_1 x x A) = |x|^2 e_1$.

PROPOSITION 2. *The algebra* $(\boldsymbol{R}^8, \circ)$ *is isomorphic to the octonians.*

PROOF. For any $x = \sum_{i=1}^{4} x_i e_i$, $y = \sum_{j=1}^{4} y_j e_j \in \boldsymbol{R}^8$, by Lemma 1 it is easy to see that $x \circ y$ is just the quaternionic product of x and y. Then we can identify $(\boldsymbol{R}^4, \circ)$ with quaternions $\boldsymbol{H}$. Hence we need only to show that for any $a, b, c, d \in \boldsymbol{H} = (\boldsymbol{R}^4, \circ)$ (see [3], p. 105)

$$(a + b \circ e_5) \circ (c + d \circ e_5) = a \circ c - \bar{d} \circ b + (d \circ a + b \circ \bar{c}) \circ e_5.$$

We verify $(b \circ e_5) \circ (d \circ e_5) = -\bar{d} \circ b$ for instance.

Let $d = \sum_{i=1}^{4} d_i e_i$, $d \circ e_5 = d_1 e_5 + d_2 e_6 - d_3 e_7 + d_4 e_8$. Then $(d \circ e_5) e_5 b = -e_5 b (\bar{d} \circ e_5)$. By definition

$$\begin{aligned} (b \circ e_5) \circ (d \circ e_5) &= \phi((d \circ e_5) e_1 e_5 e_1 b A) \\ &= -\phi(e_5 b (\bar{d} \circ e_5) A) \\ &= \phi(e_5 b e_5 e_1 \bar{d} A) \\ &= \phi(e_1 \bar{b} \bar{d} A). \end{aligned}$$

On the other hand

$$\bar{d} \circ b = -\phi(b e_1 \bar{d} A) = -\phi(e_1 \bar{b} \bar{d} A).$$

Let $\boldsymbol{O} = \boldsymbol{H}(+)$ be the octonians defined as in [3], p. 107. Then the isomorphism between $(\boldsymbol{R}^8, \circ)$ and $\boldsymbol{O}$ is defined by

$$\sum_{i=1}^{8} x_i e_i \to (x_1 + x_2 i + x_3 j + x_4 k, x_5 + x_6 i - x_7 j + x_8 k). \qquad \square$$

References

[1] Atiyah, M. F., Bott, R. and Shapiro, A., Clifford modules, Topology **3** (1964), 3–38.

[2] Dadok, J. and Harvey, F. R., Calibrations and spinors, Acta Math., **170** (1993), 83–120.

[3] Harvey, F. R., Spinors and calibrations, Perspectives in Math., 9. Academic Press, New York, 1990.

[4] Harvey, F. R. and Lawson, H. B., Jr., Calibrated geometries, Acta Math., **148** (1982), 47–157.

[5] Lawson, H. B., Jr. and Michelsohn, M., Spin geometry, Princeton University Press, Princeton, New Jersey, 1989.

[6] Wu, H., The Bochner techniques, Proceedings of 1980 Beijing symposium on differential geometry and differential equations (S. S. Chern and Wu Wen-tsun eds.), Science Press China and Gordon and Breach, New York **2** (1982), 929–1072.

[7] Zhou, J. W., On twisted Atiyah-Singer operators, (I), J. of Math. Res. & Exp., **19** (1999), 44–48. MR 2000f: 58046, Zbl 943: 58015.
[8] Zhou, J. W., The twisted Atiyah-Singer operators, (II), J. of Math. Res. & Exp., **19** (1999), 349–354. MR 2000f: 58047, Zbl 948: 58019.
[9] Zhou, J. W., The upper bounds for eigenvalues of Dirac operators, Tsukuba J. Math. **23** (1999), 447–463.
[10] Zhou, J. W., Spinors, calibrations and Grassmannians, Tsukuba J. Math. **27** (2003), 77–97.

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