

ANALYTIC REPRESENTATION OF GENERALIZED TEMPERED DISTRIBUTIONS OF EXPONENTIAL GROWTH BY WAVELETS

By

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Abstract. The analytic representation of the generalized tempered distributions of $e^{M(kx)}$ -growth with restricted order, $\mathcal{K}'_M(\mathcal{R})$, is given in terms of series of analytic wavelets. These series converge uniformly on compact subsets of the upper and lower half planes.

1. Introduction

The analytic representation of functions or distributions on the real line \mathcal{R} is usually given by a Cauchy type formula, but in some cases may also be given by an orthogonal series. It is well-known that trigonometric series may be used for the analytic representation of periodic functions and distributions. Also, Hermite series and Legendre polynomials can be used for the representation of non-periodic functions and functions with compact support, respectively. Recently a new category of orthogonal systems has been introduced in [1]. These systems are composed of wavelets, i.e., orthogonal functions on \mathcal{R} consisting of dilations and translations of a fixed function. G. G. Walter has found an expansion in orthogonal wavelets and pointwise convergence of that expansion from $L^2(\mathcal{R})$ to the tempered distributions with restricted order of derivative, $\mathcal{S}'_r(\mathcal{R})$, in [6] and [8] and has showed an analytic representation of $\mathcal{S}'_r(\mathcal{R})$ in terms of series of analytic wavelets in [7]. These two results were extended by us to the case of the tempered distributions of exponential growth with restricted order in [3], [5]. Also, we have found the wavelet expansion of the tempered distributions of $e^{M(kx)}$ -growth

2000 Mathematics Subject Classification: 41A15, 46F20.

Key words and phrases: generalized tempered distribution, wavelet, analytic representation.

The first author was supported by Inje Research and Scholarship Foundation 2000-20001 and the second author was supported by KRF 99-005-D00009.

Received June 5, 2001.

Revised July 8, 2002.

with restricted order and the pointwise convergence of the wavelet expansion of $\mathcal{K}_M'(R)$ in [4].

In this paper, we will present an analytic representation of $\mathcal{K}_M'(R)$ in terms of series of analytic wavelets. These series converge uniformly on compact subsets of the upper and lower half planes.

2. The Generalized Tempered Distributions Space $\mathcal{K}_M'(R)$

Let $\mu(\xi)$ ($0 \leq \xi \leq \infty$) denote a continuous increasing function such that $\mu(0) = 0$, $\mu(\infty) = \infty$. For $x \geq 0$, we define

$$M(x) = \int_0^x \mu(\xi) d\xi.$$

The function $M(x)$ is an increasing, convex and continuous function with $M(0) = 0$, $M(\infty) = \infty$ and satisfies the fundamental convexity inequality $M(x_1) + M(x_2) \leq M(x_1 + x_2)$. Further we define $M(x)$ for negative x by means of the equality $M(-x) = M(x)$. Note that since the derivative $\mu(x)$ of $M(x)$ is unbounded in R , the function $M(x)$ will grow faster than any linear function as $|x| \rightarrow \infty$. Now we list some properties of $M(x)$ which will be frequently used in this paper.

$$M(x) + M(y) \leq M(x + y) \quad \text{for all } x, y \geq 0. \quad (1)$$

$$M(x + y) \leq M(2x) + M(2y) \quad \text{for all } x, y \geq 0. \quad (2)$$

Using the function $M(x)$ we define the space $\mathcal{K}_M(R)$ as the space of all functions $\phi \in C^\infty(R)$ such that

$$v_k(\phi) = \sup_{x \in R, \alpha \leq k} e^{M(kx)} |D^\alpha \phi(x)| < \infty, \quad k = 1, 2, \dots, \quad (3)$$

where $D^\alpha = d^\alpha/dx^\alpha$. The topology in $\mathcal{K}_M(R)$ is defined by the family of the seminorms v_k . Then $\mathcal{K}_M(R)$ becomes a Fréchet space and the embeddings $\mathcal{D} \hookrightarrow \mathcal{K}_M \hookrightarrow \mathcal{S} \hookrightarrow \mathcal{E}$ are continuous; here \mathcal{E} denotes the space of all C^∞ -functions, \mathcal{S} the space of the tempered distributions of polynomial growth and \mathcal{D} the space of C^∞ -functions with compact supports. By $\mathcal{K}_M'(R)$, we mean the space of continuous linear functionals on $\mathcal{K}_M(R)$. Pakh characterized the distributions in $\mathcal{K}_M'(R)$ by the growth at infinity [2, Theorem 2.3]; a distribution $T \in \mathcal{D}'$ is in $\mathcal{K}_M'(R)$ if and only if there exist positive integers α, k_0 and a bounded continuous function $f(x)$ on R such that

$$T = D^\alpha [e^{M(k_0 x)} f(x)].$$

DEFINITION 1. For a natural number r , we denote by $\mathcal{K}_M^r(R)$ the space of all functions $\phi \in C^r(R)$ such that

$$v_k^r(\phi) = \sup_{x \in R, \alpha \leq r} e^{M(kx)} |D^\alpha \phi(x)| < \infty, \quad k = 1, 2, 3, \dots$$

The topology of $\mathcal{K}_M^r(R)$ is defined by the family of semi-norms $\{v_k^r\}_{k=1,2,\dots}$. By $\mathcal{K}_M^r(R)$, we mean the space of continuous linear functionals on $\mathcal{K}_M^r(R)$. Each $S \in \mathcal{K}_M^r(R)$ is characterized by

$$S = D^r[e^{M(k_0x)}f(x)], \quad (4)$$

where $f(x)$ is a bounded continuous function on R and $r, k_0 \in N$, the set of natural numbers, by the same method of the above \mathcal{K}_M^r -case in [2, Theorem 2.3]. Similarly, we can define

$$\mathcal{S}_r(R) = \{\theta(t) \in C^r(R); |D^k \theta(t)| \leq C_{pk}(1 + |t|)^{-p}, p \in N, k = 0, 1, \dots, r\}$$

and its dual $\mathcal{S}_r'(R)$. For further details, we refer to [2].

3. Multiresolution Analysis of $L^2(R)$ Associated with $\phi \in \mathcal{K}_M^r(R)$

Let $\phi \in \mathcal{K}_M^r(R)$. In order for it to qualify as a scaling function, there must be associated with ϕ a multiresolution analysis of $L^2(R)$, i.e., a nested sequence of closed subspaces $\{V_m\}_{m \in Z}$ for the set of integers Z such that

- (i) $\{\phi(\cdot - n)\}$ is an orthonormal basis of V_0 ,
- (ii) $\dots \subset V_{-1} \subset V_0 \subset V_1 \subset \dots \subset L^2(R)$,
- (iii) $f(\cdot) \in V_m \Leftrightarrow f(2\cdot) \in V_{m+1}$,
- (iv) $\bigcap_m V_m = \{0\}$, $\overline{\bigcap_m V_m} = L^2(R)$.

Then ϕ has an expansion

$$\phi(t) = \sum_n c_n \sqrt{2} \phi(2t - n), \quad \{c_n\} \in l^2, t \in R, \quad (5)$$

where $l^2 = \{\{c_n\}; \sum_n |c_n|^2 < \infty\}$. Once we have the scaling function $\phi \in \mathcal{K}_M^r(R)$, we can obtain a mother wavelet ψ such that $\{\psi(t - n)\}$ is an orthogonal basis of the space W_0 , given by the orthogonal complement of V_0 in V_1 . Also, ψ has an expansion

$$\psi(t) = \sum_n d_n \sqrt{2} \phi(2t - n), \quad \{d_n\} \in l^2, \quad (6)$$

for d_n corresponding to c_n in (5). We will adopt the construction of a mother wavelet defined by $d_n = (-1)^n \overline{c_{1-n}}$. If such a $\psi(t)$ can be found, then $\psi_{mn}(t) = 2^{m/2} \psi(2^m t - n)$ is an orthogonal basis of W_m which is the orthogonal complement of V_m in V_{m+1} .

EXAMPLE. In [1], Corollary 5.5.3 states that it is impossible that ψ has exponential decay and that $\psi \in C^\infty$, with all derivatives bounded, unless $\psi = 0$. Hence there is no mother wavelet $\psi \in \mathcal{K}_M(R)$. So we will restrict our attention to $\mathcal{K}_M^r(R)$. Daubechies' compactly supported wavelets are examples of $\mathcal{K}_M^r(R)$, but Battle-Lemarié's wavelets (in the page 152 of [1]) are not $\mathcal{K}_M^r(R)$ wavelets even if they have exponential decay and smoothness.

The reproducing kernel of V_0 is given by

$$q(x, t) = \sum_n \overline{\phi(x-n)} \phi(t-n),$$

where $\phi(x)$ is the scaling function. The series and its derivatives with respect to t of order $\leq r$ converge uniformly on $x \in R$ because of the regularity of $\phi \in \mathcal{K}_M^r(R)$, i.e.,

$$|\phi^{(\alpha)}(x)| \leq C_{\alpha k} e^{-M(kx)}, \quad \alpha = 0, 1, \dots, r; \quad k = 1, 2, \dots \quad (7)$$

The reproducing kernel for V_m is given by

$$q_m(x, t) = 2^m q(2^m x, 2^m t).$$

Similarly, we can define the reproducing kernel $r_m(x, t)$ for W_m by

$$r_m(x, t) = 2^m \sum_n \overline{\psi(2^m x - n)} \psi(2^m t - n),$$

where $\psi(t)$ is the mother wavelet.

The sequence $\{q_m(x, t)\}$ is a delta sequence in $\mathcal{S}'(R) \subset \mathcal{K}_M^r(R)$, i.e., $q_m(x, t) \rightarrow \delta(x-t)$. This follows from the fact that

$$\int_{-\infty}^{\infty} q_m(x, t) \theta(t) dt \rightarrow \theta(x) \quad \text{as } m \rightarrow \infty,$$

for each $\theta \in \mathcal{K}_M^r(R) \subset \mathcal{S}'(R)$, where the convergence is in the L^2 -sense. These kernels have a number of interesting properties, some of which come out of the wavelet moment theorem. Since $\mathcal{K}_M^r(R) \subset \mathcal{S}'(R)$, we have by [1],

LEMMA 2. Let $\psi \in \mathcal{K}_M^r(\mathbb{R})$ with $\psi_{nm}(x) = 2^{m/2}\psi(2^m x - n)$ an orthogonal system in $L^2(\mathbb{R})$. Then

$$\int_{-\infty}^{\infty} x^k \psi(x) dx = 0, \quad k = 0, 1, \dots, r.$$

DEFINITION 3. We define the spaces T_0 and U_0 by $T_0 = \{f; f(t) = \sum_n a_n \phi(t - n) \text{ for some sequence of complex numbers with } a_n = \mathcal{O}(e^{M(k_1 n)}) \text{ for some } k_1 \in N\}$ and $U_0 = \{g; g(t) = \sum_n a_n \psi(t - n) \text{ for some sequence of complex numbers with } a_n = \mathcal{O}(e^{M(k_1 n)}) \text{ for some } k_1 \in N\}$. We denote by T_m and U_m their corresponding dilation spaces, i.e., $f \in T_0 \Leftrightarrow f(2^m t) \in T_m$ and $g \in U_0 \Leftrightarrow g(2^m t) \in U_m$.

We may expect that a multiresolution analysis of $\mathcal{K}_M^r(\mathbb{R})$ exists, namely,

$$\dots \subset T_{-m} \dots \subset T_{-1} \subset T_0 \subset T_1 \dots \subset T_m \subset \dots \subset \mathcal{K}_M^r(\mathbb{R}) \quad (8)$$

and

$$\overline{\bigcup_m T_m} = \mathcal{K}_M^r(\mathbb{R}),$$

where the closure is in the topology of $\mathcal{K}_M^r(\mathbb{R})$.

Now in [3], we have found the expansion in orthogonal wavelets from $L^2(\mathbb{R})$ to $\mathcal{K}_M^r(\mathbb{R})$.

THEOREM 4. Let the scaling function $\phi \in \mathcal{K}_M^r(\mathbb{R})$ satisfy the dilation equation (5) with $c_k = \mathcal{O}(e^{-M(lk)})$ for all $l \in N$, and have an associated multiresolution analysis in $L^2(\mathbb{R})$; let $\psi \in \mathcal{K}_M^r(\mathbb{R})$ be the mother wavelet given in (6). Then there exists a multiresolution analysis (8) of closed dilation subspaces $\{T_m\}$ whose union is dense in $\mathcal{K}_M^r(\mathbb{R})$; the closed subspace U_m in Definition 3 is a complementary subspace of T_m in T_{m+1} and

$$T_m = U_0 \oplus U_1 \oplus \dots \oplus U_m \oplus T_0,$$

where \oplus denotes the nonorthogonal direct sum.

4. Analytic Representation of Distributions of \mathcal{K}_M^r by Wavelets

A quasi-positive delta sequence is a sequence $\{\delta_m(\cdot, y)\}$ of functions in $L^1(\mathbb{R})$ with a parameter $y \in \mathbb{R}$ which satisfies the following:

(a) there is a $C > 0$ such that

$$\int_{-\infty}^{\infty} |\delta_m(x, y)| dx \leq C, \quad y \in \mathbb{R}, m \in N;$$

(b) there is a $c > 0$ such that

$$\int_{y-c}^{y+c} \delta_m(x, y) dx \rightarrow 1$$

uniformly on compact subsets of R , as $m \rightarrow \infty$;

(c) for each $\gamma > 0$,

$$\sup_{|x-y| \leq \gamma} |\delta_m(x, y)| \rightarrow 0 \quad \text{as } m \rightarrow \infty.$$

Then since $\mathcal{K}_M^r(R) \subset \mathcal{G}_r(R)$, we have the following important lemmas as in [8]:

LEMMA 5. *Let $\{\delta_m(x, y)\}$ be a quasi-positive delta sequence and let $f \in L^1(R)$ be continuous on (a, b) . Then*

$$f_m(y) = \int_{-\infty}^{\infty} \delta_m(x, y) f(x) dx \rightarrow f(y) \quad \text{as } m \rightarrow \infty$$

uniformly on compact subsets of (a, b) .

LEMMA 6. *If the scaling function $\phi \in \mathcal{K}_M^r(R)$, then the reproducing kernel $q_m(x, y)$ and $K_m(x, t) = \frac{(x-t)}{\alpha!} \frac{\partial^\alpha}{\partial t^\alpha} q_m(x, t)$ for $\alpha \in N$, $0 \leq \alpha \leq r$, are quasi-positive delta sequences on R .*

In order to represent an element of $\mathcal{K}_M^{r'}(R)$ by series of analytic wavelets, we impose conditions on the scaling function ϕ again. Since $\mathcal{K}_M^r(R) \subset L^2(R)$, an analytic representation of ϕ is given by

$$\phi^\pm(z) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{\phi(x)}{x-z} dx, \quad \text{Im } z \gtrless 0,$$

where ϕ^\pm are analytic in the upper half-plane and the lower half-plane, respectively. An analytic representation of the mother wavelet is also given by

$$\psi^\pm(z) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{\psi(x)}{x-z} dx, \quad \text{Im } z \gtrless 0,$$

and the analytic wavelets ψ_{mn}^\pm are obtained by dilation and translation of ψ^\pm . Now, we define $T_0^\pm = \{f(z) = \sum_n a_n \phi^\pm(z-n); a_n = \mathcal{O}(c^{l_0 M(n)}) \text{ for some } l_0 \in N\}$ and we denote by the subspaces T_m^\pm of T_0^\pm the corresponding dilation spaces. Then the spaces T_m^+ and T_m^- are composed of analytic functions in the upper and the lower half-planes, respectively, whose boundary functions are continuous

functions of $e^{M(x)}$ -growth. Since $\overline{\bigcup T_m} = \mathcal{K}_M^{r'}(\mathcal{R})$, we might expect to obtain an analytic representation of $f \in \mathcal{K}_M^{r'}(\mathcal{R})$ in terms of wavelets,

$$f^+(z) = \sum_{n=-\infty}^{\infty} a_n \phi^+(z-n) + \sum_{m=0}^{\infty} \sum_{n=-\infty}^{\infty} b_{mn} 2^{m/2} \psi^+(2^m z - n),$$

where the first series may not converge. Since an analytic representation is a continuous map from $\mathcal{K}_M^{r'}(\mathcal{R})$ to a corresponding space of analytic functions and $f_m(x) = (f, q_m(x, t)) \rightarrow f(x) = D^r F(x)$ in $\mathcal{K}_M^{r'}(\mathcal{R})$ for a continuous function of $e^{M(x)}$ -growth $F(x)$ [cf. (4)] by Lemmas 5 and 6, $f_m^+(z) \rightarrow f^+(z)$ uniformly on bounded subsets of the upper half-plane. Moreover, $f^+(z) = D_z^r F^+(z)$, where $F^+(z)$ is an analytic representation of $F(z)$, and is given by

$$F^+(z) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{F(x)}{x-z} e^{-M(kx)} e^{M(kz)} dx,$$

for a sufficiently large k such that $F(x)e^{-M(kx)} \in L^2(\mathcal{R})$. Here for $z \in C$, we define $e^{M(z)}$ as $e^{M(|z|)}$.

We may express f_m as

$$f_m = f_0 + f_m - f_0 = f_0 + \sum_{k=0}^{m-1} \sum_{n=-\infty}^{\infty} b_{kn} \psi_{kn},$$

and if the inner sum converges,

$$f_m^+(z) - f_0^+(z) = \sum_{k=0}^{m-1} \sum_{n=-\infty}^{\infty} b_{kn} \psi_{kn}^+(z) + g_m(z), \quad (9)$$

where $g_m(z)$ is an entire function.

LEMMA 7. *Let $\psi \in \mathcal{K}_M^r(\mathcal{R})$ and $b_n = \mathcal{O}(e^{M(kn)-\varepsilon})$ for any $k \in N$ and some $\varepsilon > 0$. Then*

$$\sum_{n=-\infty}^{\infty} b_n \psi^+(z-n)$$

converges uniformly on compact subsets of the upper half-plane.

PROOF. The proof is based on the moment property, Lemma 2,

$$\int_{-\infty}^{\infty} x^l \psi(x) dx = 0, \quad l = 0, 1, \dots, r.$$

Hence, for any $k \in N$ and a natural number $p \leq r+1$,

$$\begin{aligned}
e^{M(kz)}\psi^+(z) &= \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{e^{M(kz)}}{z^p} \cdot \frac{z^p}{x-z} \psi(x) dx \\
&= \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{e^{M(kz)}}{z^p} \cdot \frac{z^p - x^p}{x-z} \psi(x) dx \\
&\quad + \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{e^{M(kz)}}{z^p} \cdot \frac{x^p}{x-z} \psi(x) dx \\
&= -\frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{e^{M(kz)}}{z^p} \cdot (x^{p-1} + zx^{p-2} + \cdots + z^{p-2}x + z^{p-1}) \psi(x) dx \\
&\quad + \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{e^{M(kz)}}{z^p} \cdot \frac{x^p}{x-z} \psi(x) dx \\
&= \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{e^{M(kz)}}{z^p} \cdot \frac{x^p}{x-z} \psi(x) dx
\end{aligned}$$

holds. By the growth condition of $\psi \in \mathcal{H}_M^r(\mathbb{R})$, $|e^{M(kz)}\psi^+(z)|$ is uniformly bounded on compact subsets of the half-plane $\text{Im } z \geq \varepsilon > 0$ for any $k \in \mathbb{N}$ and a natural number $p \leq r+1$. Hence, the preceding fact holds for any $k \in \mathbb{N}$ and any $p \leq r+1$. Thus the conclusion follows.

THEOREM 8. For natural numbers s, r with $s < r$, let $f \in \mathcal{H}_M^{st}(\mathbb{R})$, $\phi, \psi \in K_M^r(\mathbb{R})$ and let $b_{mn} = \langle f, \psi_{mn} \rangle$, $m = 0, 1, 2, \dots$; $n = 0, \pm 1, \pm 2, \dots$ be the wavelet coefficients of f . Then an analytic representation of f is given by

$$f^+(z) = f_0^+(z) + \sum_{m=0}^{\infty} \sum_{n=-\infty}^{\infty} b_{mn} \psi_{mn}^+(z),$$

where the series converges uniformly on compact subsets of the half-plane $\text{Im } z \geq 1$ and $f_0^+(z)$ is an analytic representation of f_0 , the projection of f on T_0 .

PROOF. First, we will estimate $|b_{mn}|$. Each $f \in \mathcal{H}_M^{st}(\mathbb{R})$ is characterized by

$$f = D^s [e^{M(k_0 x)} \mu]$$

for some integer k_0 and finite measure μ on \mathbb{R} . Each $\psi \in \mathcal{H}_M^r(\mathbb{R})$ satisfies

$$|\psi^{(l)}(x)| \leq C_j e^{-M(jx)}, \quad l = 1, 2, \dots, r; j \geq 0.$$

If we use integration by parts s -times, we have, for $m > 1$,

$$\begin{aligned}
 |b_{mn}| &\leq \int_{-\infty}^{\infty} |D^s[e^{M(k_0x)}]\psi_{mn}(x)| d|\mu| \leq \int_{-\infty}^{\infty} e^{M(k_0x)} |\psi_{mn}^{(s)}(x)| d|\mu| \\
 &\leq \int_{-\infty}^{\infty} e^{M(k_0x)} c_{k_0} 2^{m/2+sm} e^{-M(k_0(2^m x - n))} d|\mu| \\
 &\leq \int_{-\infty}^{\infty} e^{M(2k_0(x-n2^{-m}))} e^{M(2k_0n2^{-m})} c_{k_0} 2^{m/2+sm} e^{-M(2^m k_0(x-n2^{-m}))} d|\mu| \\
 &\leq c'_{k_0} 2^{m/2+sm} e^{M(2k_0n2^{-m})}.
 \end{aligned}$$

By the fact in the proof of Lemma 7, on every compact subset K of the half-plane $\operatorname{Im} z \geq 1$, there exists a constant c such as $|\psi^+(z)| \leq ce^{-M(kz)}$ for any $k \in N$. Hence if we take k sufficiently large with $k > \sup\{k_0, \frac{3}{2} + s\}$, then for $z \in K$,

$$\begin{aligned}
 &\sum_{m=0}^{\infty} \sum_{n=-\infty}^{\infty} |b_{mn}\psi_{mn}^+(z)| \\
 &\leq \sum_{m=0}^{\infty} \sum_{n=-\infty}^{\infty} c'_{k_0} e^{(1/2+s)m} e^{M(2k_0n2^{-m})} c 2^{m/2} e^{-M(k(2^m z - n))} \\
 &\leq \sum_{m=0}^{\infty} \sum_{n=-\infty}^{\infty} cc'_{k_0} e^{(3/2+s)m} e^{M(2k_0n2^{-m})} e^{-M(k2^m(n2^{-m} - \operatorname{Re} z - i \operatorname{Im} z))} 2^{-m/2} \\
 &\leq \sum_{m=0}^{\infty} \sum_{n=-\infty}^{\infty} cc'_{k_0} e^{(3/2+s)m} e^{M(2k_0n2^{-m})} \\
 &\quad \times e^{-M(k2^m(n2^{-m} - \operatorname{Re} z))} e^{-M(2^{m-1}k)} 2^{-m/2} \\
 &\leq \left\{ \sum_{m=0}^2 \sum_{n=-\infty}^{\infty} + \sum_{m=3}^{\infty} \sum_{n=-\infty}^{\infty} \right\} cc'_{k_0} e^{(3/2+s)m} e^{M(2k_0n2^{-m})} \\
 &\quad \times e^{-M(k2^m(n2^{-m} - \operatorname{Re} z))} e^{-M(2^{m-1}k)} 2^{-m/2} \\
 &\leq \sum_{m=3}^{\infty} C_{k_0, z} 2^{-m/2} < \infty,
 \end{aligned}$$

where we use the properties (1), (2) and the inequality $\sqrt{2}|z| \geq |\operatorname{Re} z| + |\operatorname{Im} z| \geq |\operatorname{Re} z| + 1$ for $|\operatorname{Im} z| \geq 1$. Hence the series $\sum_{m=0}^{\infty} \sum_{n=-\infty}^{\infty} b_{mn}\psi_{mn}^+(z)$ converges uniformly on compact subsets of the half-plane $\operatorname{Im} z \geq 1$.

Now, by taking the limit in (9) as $m \rightarrow \infty$, we have

$$f^+(z) = f_0^+(z) + \sum_{k=0}^{\infty} \sum_{n=-\infty}^{\infty} b_{kn}\psi_{kn}^+(z) + g_{\infty}(z),$$

where $g_{\infty}(z) = \lim_{m \rightarrow \infty} g_m(z)$ is an entire function. Since an analytic representation plus an entire function is an analytic representation, we can drop g_{∞} in (9).

REMARK. We have only worked out the convergence for f^+ but proof

for f^- is parallel. Then by the same method as in the proof of Theorem 8, an analytic representation of f is given by

$$f^-(z) = f_0^-(z) + \sum_{m=0}^{\infty} \sum_{n=-\infty}^{\infty} b_{mn} \psi_{nm}^-(z),$$

where the series converges uniformly on compact subsets of the half-plane $\operatorname{Im} z \leq -1$ and $f_0^-(z)$ is an analytic representation of f_0 , the projection of f on T_0 .

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