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BLOW-UP AND LARGE TIME BEHAVIOR OF SOLUTIONS OF A WEAKLY COUPLED SYSTEM OF REACTION-DIFFUSION EQUATIONS

By

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1 Introduction

We consider nonnegative solutions of the initial value problem for a weakly coupled system

$$\begin{cases} u_{it} = \Delta u_i + u_{i+1}^{p_i}, & x \in \mathbf{R}^d,\ t > 0,\ i \in N^*, \\ u_i(x,0) = u_{i0}(x), & x \in \mathbf{R}^d,\ i \in N^*, \end{cases} \tag{1}$$

where $N \geq 1$, $N^* = \{1, 2, \dots N\}$, $u_{N+i} = u_i$, $u_{N+i,0} = u_{i0}$, $p_{N+i} = p_i$ $(i \in N^*)$, $u = (u_1, u_2, \dots, u_N)$, $u_0 = (u_{10}, u_{20}, \dots, u_{N0})$, $p = (p_1, p_2, \dots, p_N)$, $d \geq 1$, $p_i \geq 1$ $(i \in N^*)$ and $\prod_{i=1}^N p_i > 1$, u_{i0} $(i \in N^*)$ are nonnegative bounded and continuous functions.

Problem (1) has a unique, nonnegative and bounded solution at least locally in time. For given initial values u_0, let $T^* = T^*(u_0)$ be the maximal existence time of the solution. If $T^* = \infty$ the solutions are gloval. On the other hand, if $T^* < \infty$ there exists $i \in N^*$ such that

$$\limsup_{t \to T^*} \|u_i(t)\|_\infty = \infty. \tag{2}$$

When (2) holds we say that the solutions blows up in finite times.

The blow-up and the global existence of solutions are studied by Escobedo-Herrero [1] in case $N = 2$, and the following results are proved there

(I) If $2 \max\{p_1 + 1, p_2 + 1\} \geq d(p_1 p_2 - 1)$, then $T^* < \infty$ for every nontrivial solution $u(t)$ of (1);

(II) If $2 \max\{p_1 + 1, p_2 + 1\} < d(p_1 p_2 - 1)$, then there exist both nonglobal solutions and non-trivial global solutions of (1).

In this article we shall first treat blow-up solutions. We can use it to simplify the proof of (I) (Theorem 3.2 and 3.6). Moreover, requiring the polynomial decay of initial values u_0, say, $u_{0i} \sim \lambda^{\mu_i}(1 + |x|)^{-a_i}$ $(i \in N^*)$ where λ, μ_i and a_i $(i \in N^*)$ are all positive, we obtain another cutoff of $a = (a_1, a_2, \dots, a_N)$ which

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divides the blow-up case and the global existence case when $2\max_{i\in N^*}\{\sum_{j=1}^{N-1}(\prod_{k=0}^{j-1}p_{i+k})+1\}<N(\prod_{j=1}^{N}p_i-1)$ (Theorem 3.3). The new cutoff will be the vector

$$\alpha_i=\frac{2\sum_{j=1}^{N-1}\left(\prod_{k=0}^{j-1}p_{i+k}\right)+2}{\prod_{l=1}^{N}p_l-1}\quad i\in N^* \tag{3}$$

which solves

$$\begin{pmatrix} 1 & -p_1 & 0 & 0 & \cdots & 0 & 0 & 0\\ 0 & 1 & -p_2 & 0 & \cdots & 0 & 0 & 0\\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots\\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 & -p_{N-1}\\ -p_N & 0 & 0 & 0 & \cdots & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix}\alpha_1\\ \alpha_2\\ \vdots\\ \alpha_{N-1}\\ \alpha_N\end{pmatrix}=-\begin{pmatrix}2\\2\\ \vdots\\2\\2\end{pmatrix}.$$

Note that with the use of α, the first cutoff $2\max_{i\in N^*}\{\sum_{j=1}^{N-1}(\prod_{k=0}^{j-1}p_{i+k})+1\}=N(\prod_{j=1}^{N}p_i-1)$ is expressed as follows:

$$2\max_{i\in N^*}\{\alpha_i\}=N$$

In the second half of the article we consider the large time behavior of global solutions. Not only the precise decay estimate (Theorem 4.1) but also the asymptotic profile (Theorem 6.1) are obtained for a class of vector $a=(a_1,a_2,\dots,a_N)$ in the domain $\{a;a_i>\alpha_i,\ i\in N^*\}$. For these purposes a scaling argument for solutions $u(x,t)$ will play an important role.

When $N=2$, these problems have been studied by K. Mochizuki [3]. In this paper, we extended this to the case $N\geq 3$.

2 Preliminaries

We first recall the local solvability of the Cauchy problem (1). We use the notation $S(t)\xi$ to represent the solution of the heat equation with initial value $\xi(x)$:

$$S(t)\xi(x)=(4\pi t)^{-d/2}\int_{\mathbf{R}^d}e^{-|x-y|^2/4t}\xi(y)\,dy \tag{4}$$

For arbitrary $T>0$, let

$$E_T=\{u:[0,T]\to(L^\infty)^N;\|u\|_{E_T}<\infty\} \tag{5}$$

where

$$\|u\|_{E_T}=\sup_{t\in[0,T]}\left\{\sum_{i=1}^{N}\|u_i(t)\|_\infty\right\}$$

Theorem 2.1. *Assume that* u_0 *is a vector of nonnegative bounded continuous functions. Then there exists* $0 < T \le \infty$ *and a unique vector* $u(t) \in P_T = \{u \in E_T;\ \{u_i \ge 0, i \in N^*\}\}$ *which solves* (1) *in* $\mathbf{R}^d \times [0, T)$.

Proof. It is obvious (see [2], [3]). □

We consider in E_T the related integral system

$$u_i(t) = S(t)u_{i0} + \int_0^t S(t-s)|u_{i+1}(s)|^{p_i - 1}u_{i+1}(s)\, ds \tag{6}$$

where $i \in N^*$. Note that in the closed subset P_T of E_T, (1) is reduced to (6).

Next, we obtain a necessary condition for the global existence of solutions. Let $\rho_\varepsilon(x) = (\varepsilon/\pi)^{d/2}e^{-\varepsilon|x|^2}$, $\varepsilon > 0$. For a solution $u(t) \in E_T$ of (1) we put

$$F_{i\varepsilon}(t) = \int_{\mathbf{R}^d} u_i(x,t)\rho_\varepsilon(x)\, dx \quad (i \in N^*) \tag{7}$$

Since $-\Delta\rho_\varepsilon \le 2d\varepsilon\rho_\varepsilon(x)$, the pair $\{2N\varepsilon, \rho_\varepsilon(x)\}$ is regarded as an approximate principal eigensolution of $-\Delta$ in $\mathbf{R}$. With this fact and Jensen's inequality we easily have

$$F'_{i\varepsilon} \ge -2d\varepsilon F_{i\varepsilon}(t) + F_{i+1,\varepsilon}(t)^{p_i} \quad (i \in N^*) \tag{8}$$

Let us consider the system of ordinary differential equations

$$\begin{cases} f'_{i\varepsilon} = -2d\varepsilon f_{i\varepsilon}(t) + f_{i+1,\varepsilon}(t)^{p_i} & (i \in N^*) \\ f_{i\varepsilon}(0) = F_{i\varepsilon}(0), & (i \in N^*) \end{cases} \tag{9}$$

By the scaling with (3)

$$f_i(t) = (2d\varepsilon)^{-\alpha_i/2} f_{i\varepsilon}\left(\frac{t}{2d\varepsilon}\right) \quad (i \in N^*)$$

we obtain the simpler system of equations

$$f_i'(t) = -f_i(t) + f_{i+1}(t)^{p_i}, \quad (i \in N^*) \tag{10}$$

Lemma 2.2. *Let* $f(t) = (f_1(t), f_2(t), \dots, f_N(t))$ *be the solution to* (10) *with initial data*

$$f_1(0) = f_0 > 1,\ f_j(0) = 0 \quad (j \in N^* \backslash \{1\})$$

If f_0 *is sufficiently large, then* $f(t)$ *blows up in finite time. Moreover, the life span* T_0 *of* $f(t)$ *is estimated from above like*

$$T_0 \le t_0 + \int_{\prod_{i=1}^N f_i(t_0)}^{\infty} \{C_1(p)\xi^{C_2(p)+1} - N\xi\}^{-1}\, d\xi \tag{11}$$

where

$$C_1(p) = \prod_{i=1}^N \frac{1}{\beta_i^{\beta_i}}, \quad \left(\beta_i = \frac{\alpha_{i+1}}{\sum_{j=1}^N \alpha_j} \quad (i \in N^*)\right)$$

$$C_2(p) = \frac{2}{\sum_{i=1}^N \alpha_i},$$

and $0 < t_0 < T_0$ *is chosen to satisfy* $\{\prod_{i=1}^N f_i(t_0)\}^{C_2(p)} > N$.

PROOF. Multiplying e^t on the both sides of (10) and integrating it, we obtain

$$\begin{cases} f_N(t) = e^{-t}\int_0^t e^{s_1} f_1(s_1)^{p_N}\, ds_1 \\ f_{N-1}(t) = e^{-t}\int_0^t e^{(1-p_2)s_1}\left\{\int_0^{s_1} e^{s_2} f(s_2)^{p_N}\, ds_2\right\}^{p_{N-1}} ds_1, \\ \qquad \vdots \\ f_2(t) = e^{-t}\int_0^t e^{(1-p_2)s_1}\left[\int_0^{s_1} e^{(1-p_3)s_2} \times\cdots\times \left(\int_0^{s_{N-3}} e^{(1-p_{N-1})s_{N-2}}\right.\right. \\ \qquad \left.\left.\times \left\{\int_0^{s_{N-2}} e^{s_{N-1}} f_1(s_{N-1})^{p_N}\, d_{S_{N-1}}\right\}^{p_{N-1}} ds_{N-2}\right)^{p_{N-2}} \cdots ds_2\right]^{p_2} ds_1, \end{cases} \tag{12}$$

$$f_1(t) = e^t f_0 + e^{-t}\int_0^t e^{(1-p_1)s_1}\left[\int_0^{s_1} e^{(1-p_2)s_2} \times\cdots\times \left(\int_0^{s_{N-2}} e^{(1-p_{N-1})s_{N-1}}\right.\right.$$
$$\left.\left.\times \left\{\int_0^{s_{N-1}} e^{s_N} f_1(s_N)^{p_N}\, d_{s_N}\right\}^{p_{N-1}} ds_{N-1}\right)^{p_{N-2}} \cdots ds_2\right]^{p_2} ds_1. \tag{13}$$

Let $f_0 > 1$ be choosen large enough to satisfy

$$\inf_{t_0>0}\left\{e^{t_0} f_0 + 2^{p_1 p_2\cdots p_N} e^{-t_0}\int_0^{t_0} e^{(1-p_1)s_1}\left[\int_0^{s_1} e^{(1-p_2)s_2} \times\cdots\times \left(\int_0^{s_{N-3}} e^{(1-p_{N-2})s_{N-2}}\right.\right.\right.$$
$$\left.\left.\left.\times \left\{\int_0^{s_{N-2}} e^{s_{N-1}}(1 - e^{-s_{N-1}})\, d_{s_N}\right\}^{p_{N-2}} ds_{N-2}\right)^{p_{N-3}} \cdots ds_2\right]^{p_2} ds_1\right\} \tag{14}$$
$$\ge 2^{p_1 p_2\cdots p_N} - \delta$$

where $\delta > 0$ is small constant satisfying $\delta < 2^{p_1 p_2\cdots p_N} - 2$.

We shall first show that under this condition $f(t) > 2$ for any $0 < t < T_0$. Assume contrary that there exist $0 < t_1 < T_0$ such that $f(t) > 2$ in $0 \le t < t_0$ and $f_1(t_1) = 2$. Then it follows from (13) and (14) that

$$\begin{aligned}2 &= f_1(t_1)\\ &\ge e^{t_1} f_0 + 2^{p_1 p_2 \cdots p_N} e^{-t_1} \int_0^{t_1} e^{(1-p_1)s_1} \left[\int_0^{s_1} e^{(1-p_2)s_2} \times \cdots \right.\\ &\quad \left. \times \left(\int_0^{s_{N-3}} e^{(1-p_{N-2})s_{N-2}} \left\{ \int_0^{s_{N-2}} e^{s_{N-1}} (1 - e^{-s_{N-1}})\, d_{s_N} \right\}^{p_{N-2}} ds_{N-2} \right)^{p_{N-3}} \cdots ds_2 \right]^{p_2} ds_1 \Bigg\}\\ &\ge 2^{p_1 p_2 \cdots p_N} - \delta > 2\end{aligned}$$

and a contradiction occurs. Next, we shall show that $\lim_{t \to T_0} f(t) = \infty \ (T_0 \le \infty)$. Assume contrary that there exist a sequence $\{t_j\}$ such that

$$\lim_{t_j \to \infty} f_1(t_j) = M \quad \text{for some } 2 \le M < \infty.$$

We choose $\varepsilon > 0$ and $t_* > 0$ to satisfy $M < (M - \varepsilon)^{p_1 p_2 \cdots p_N}$ and $f(t) > M - \varepsilon$ in $t_* < t < T$. It then follows from (13) that

$$\begin{aligned}f_1(t_j) &\ge e^{t_1} f_0 + 2^{p_1 p_2 \cdots p_N} e^{-t_j} \int_0^{t_*} e^{(1-p_1)s_1} \left[\int_0^{s_1} e^{(1-p_2)s_2} \times \cdots \times \left(\int_0^{s_{N-3}} e^{(1-p_{N-2})s_{N-2}} \right.\right.\\ &\qquad \left.\left. \times \left\{ \int_0^{s_{N-2}} e^{s_{N-1}} (1 - e^{-s_{N-1}})\, d_{s_N} \right\}^{p_{N-2}} ds_{N-2} \right)^{p_{N-3}} \cdots ds_2 \right]^{p_2} ds_1 \Bigg\}\\ &\qquad + (M - \varepsilon)^{p_1 p_2 \cdots p_N} e^{-t_1} \int_{t_*}^{t_j} e^{(1-p_1)s_1} \left[\int_{t_*}^{s_1} e^{(1-p_2)s_2} \times \cdots \times \left(\int_{t_*}^{s_{N-3}} e^{(1-p_{N-2})s_{N-2}} \right.\right.\\ &\qquad \left.\left. \times \left\{ \int_{t_*}^{s_{N-2}} e^{s_{N-1}} (1 - e^{-s_{N-1}})\, d_{s_N} \right\}^{p_{N-2}} ds_{N-2} \right)^{p_{N-3}} \cdots ds_2 \right]^{p_2} ds_1 \Bigg\}\\ &\qquad \to (M - \varepsilon)^{p_1 p_2 \cdots p_N} > M \quad (t_j \to \infty)\end{aligned}$$

Noting (12), we now conclude

(15) $$\lim_{t \to T_0} f_1(t) = \lim_{t \to T_0} f_2(t) = \cdots = \lim_{t \to T_0} f_N(t) = \infty \quad (T_0 \le \infty)$$

To complete the assertion we put $h(t) = f_1(t) f_2(t) \cdots f_N(t)$. Then by (10) and Young's inequality,

(16) $$h'(t) \ge -N h(t) + C_1(p) h(t)^{C_2(p)+1}$$

Integrating this, we obtain

$$t - t_0 \leq \int_{h(t_0)}^{h(t)} \left\{ C_1(p)\xi^{C_2(p)+1} - N\xi \right\}^{-1} d\xi$$

Since $p_1 p_2 \cdots p_N > 1$, this and (15) show that $h(t)$ blows up in a finite time and the life span T_0 is estimated by (11). □

Let us consider the solution $f_\varepsilon(t) = (f_{1\varepsilon}(t), f_{2\varepsilon}(t), \ldots, f_{N\varepsilon}(t))$ of (9). As is shown in this above lemma, there exist $A_i > 0$ $(i \in N^*)$ such that if

$$F_{i\varepsilon}(0) > A_i(2d\varepsilon)^{\alpha_i/2} \quad (i \in N^*), \tag{17}$$

then f_ε blows up in finite time. Moreover, its life span is estimated from above by $(2d\varepsilon)^{-1}T_0$.

THEOREM 2.3. *Let $F_\varepsilon(t) = (F_{1\varepsilon}(t), F_{2\varepsilon}(t), \ldots, F_{N\varepsilon}(t))$ satisfy differential inequalities* (8). *If* (17) *is satisfied for some $\varepsilon > 0$, then $F_\varepsilon(t)$ blow up in finite time. Moreover, its life span is estimated from above by $(2d\varepsilon)^{-1}T_0$. Then, we obtain*

$$T^*(u_0) \leq (2d\varepsilon)^{-1}T_0. \tag{18}$$

3 Blow-up Conditions

In this section we summarize several blow-up condition which follow from Theorem 2.3. By BC, we denote the space of all bounded continuous functions in $\mathbf{R}^d$ and define for $a \geq 0$,

$$I^a = \{\xi \in BC; \xi(x) \geq 0 \text{ and } \limsup_{|x|\to\infty} |x|^a \xi(x) < \infty\}$$

$$I_a = \{\xi \in BC; \xi(x) \geq 0 \text{ and } \liminf_{|x|\to\infty} |x|^a \xi(x) > 0\}$$

Let L_a^∞ be the Banach space of L^∞-function such that

$$\|\xi\|_{\infty,a} = \sup_{x\in\mathbf{R}^d} \langle x\rangle^a |\xi| < \infty$$

where $\langle x\rangle = (1 + |x|^2)^{1/2}$. Obviously $I^a \subset L_a^\infty$. The letter C denotes a positive generic constant which may vary from line to line.

LEMMA 3.1. *Let $(u_{10}, u_{20}, \ldots, u_{N0}) \neq (0, 0, \ldots, 0)$ and u be solutions of* (1). *Then there exist $\tau = \tau(u_0) \geq 0$ and constants $C > 0$, $\nu > 0$ such that*

$$u_i(\tau) \geq Ce^{-\nu|x|^2} \quad (i \in N^*)$$

Proof. It is obvious (see [1; Lemma 2.4]). □

Theorem 3.2. *Assume* $\max_{i \in N^*}\{\alpha_i\} > d$. *Then* $T^* < \infty$ *for every nontrivial solution* $u(t)$ *of* (1).

Theorem 3.3. *Assume* $\max_{i \in N^*}\{\alpha_i\} < d$. *Moreover suppose also one of the following two conditions.*

(i) *There exists some* $i \in N^*$ *such that* $u_{i,0} \in I_{a_i}$ *with* $a_i < \alpha_i$.
(ii) *There exists some* $i \in N^*$ *such that* $u_{i,0} \geq Ce^{-v_0|x|^2}$ *for some* $v_0 > 0$ *and some* $C > 0$ *large enough.*

Then $T^* < \infty$ *holds for every solution* $u(t)$ *of* (1).

Proof of Theorem 3.2 and 3.3. These Theorem can be shown by the same argument the case $N = 2$ (See [2], [3], [7]). □

In the rest of this section we consider the critical case $\max_{i \in N^*}\{\alpha_i\} = d$. We suppose $\alpha_1 = d$. Let $u(t) \in E_T$ be a nontrivial solution of (1). By Lemma 3.1 we may assume

$$u_{10} \geq Ce^{-\mu|x|^2}$$

for some $C > 0$ and $\mu > 0$. Then by a semigroup property of $S(t)$ we have

$$u_1 \geq S(t)u_{10}(x) \geq C(4t + 1/\mu)^{-d/2}e^{-|x|^2/(4t+1/\mu)} \tag{19}$$

Lemma 3.4. *For* $v > 0$,

$$S(t)e^{-v|x|^2} \geq C(2vt + 1)^{-d/2}e^{-|x|^2/2t}$$

Lemma 3.5. *We assume* $\alpha_1 = d$. *Then we have*

$$u_1(x,t) \geq Ct^{-d/2}e^{-|x|^2/t}\log(t/(2a)) \quad (a \leq t < T)$$

where $a > 0$ *is a small constant.*

Proof. We shall consider the following inequalition.

$$u_N(x,t) \geq \int_0^t S(t-s)u_1(x,s)^{p_N}\,ds$$
$$\geq \int_0^t (4s + 1/\mu)^{-dp_N/2}S(t-s)e^{-p_N|x|^2/(4s+1/\mu)}\,ds.$$

Since

$$S(t)e^{-p_N|x|^2/(4s+1/\mu)} \geq C\left\{\frac{2p_N t}{4s+1/\mu}+1\right\}^{-d/2} e^{-|x|^2/2t}$$

by Lemma 3.4, we obtain

$$\begin{aligned} u_N(x,t) &\geq C\int_{t/4}^{t/2} (4s+1/\mu)^{-dp_N/2} e^{-|x|^2/2(t-s)}\, ds \\ &\geq Ct(t+1)^{dp_N/2} e^{-|x|^2/t}. \end{aligned}$$

Substitute this into $u_{N-1}(x,t) \geq \int_0^t S(t-s)u_N(x,s)^{p_{N-1}}$, then

$$\begin{aligned} u_{N-1}(x,t) &\geq C\int_0^t s^{p_{N-1}}(s+1)^{-dp_{N-1}p_N/2}\left\{\frac{2p_{N-1}(t-s)}{s}+1\right\}^{-d/2} e^{-|x|^2/(t-s)}\, ds \\ &\geq Ce^{-|x|^2/t}\int_{t/4}^{t/2} s^{-dp_{N-1}p_N/2+p_{N-1}}\, ds \\ &\geq Ct^{-dp_{N-1}p_N/2+p_{N-1}+1} e^{-|x|^2/t}\, ds \end{aligned}$$

by Lemma 3.4 again. By repeating this work,

$$u_2 \geq Ct^{-dp_2p_3\cdots p_N/2+p_2p_3\cdots p_{N-1}+\cdots+p_2p_3+p_2+1} e^{-|x|^2/t}\, ds$$

using Lemma 3.4 again, we obtain

$$\begin{aligned} u_1(x,t) &\geq C\int_0^t s^{p_1} s^{-dp_1p_2\cdots p_N/2+p_1p_2\cdots p_{N-1}+\cdots+p_1p_2+p_1} \\ &\quad \times \left\{\frac{2p_1(t-s)}{s}+1\right\}^{-d/2} e^{-|x|^2/(t-s)}\, ds \\ &\geq C(t+1)^{-d/2} e^{-|x|^2/t}\int_a^{t/2} s^{-d(p_1p_2\cdots p_N-1)/2+p_1p_2\cdots p_{N-1}+\cdots+p_1p_2+p_1}\, ds \end{aligned}$$

for small $a > 0$. Since $\alpha_1 = \dfrac{2(p_1p_2\cdots p_{N-1}+\cdots+p_1p_2+p_1+1)}{p_1p_2\cdots p_N - 1} = d$, $-d(p_1p_2\cdots p_N-1)/2 + p_1p_2\cdots p_{N-1}+\cdots+p_1p_2+p_1 = -1$, we have

$$u_1(x,t) \geq Ct^{-d/2}e^{-|x|^2/t}\log(t/2a). \qquad \square$$

THEOREM 3.6 (critical blow-up). *Assume* $\max_{i\in N^*}\{\alpha_i\} = d$. *Then* $T^* < \infty$ *for every nontrivial solution* $u(t)$ *of* (1).

Proof (See [2], [3]). For each nontrivial solution $u(t) \in E_T$ of (1), it follows from Lemma 3.5 that

$$S(t)u_1(0,t) \geq Ct^{-d/2}\log(t/2a)\int_{\mathbf{R}^d} e^{-5|x|^2/4t}\,dx \geq Ct^{-d/2}\log(t/2a) \tag{20}$$

in $a < t < T^*$. Contrary to the conclusion assume that u is global. Then by Theorem 2.3

$$F_{1,\varepsilon}(t) = (\varepsilon/\pi)^{d/2}\int_{\mathbf{R}^d} u_1(x,t)e^{-\varepsilon|x|^2}\,dx \leq A_1\varepsilon^{\alpha_1/2}$$

holds for any $t \geq 0$ and $\varepsilon > 0$. Thus, choosing $\varepsilon = (4t)^{-1}$, we obtain

$$F_{1,1/4t}(t) = S(t)u_1(0,t) \leq A_1(4t)^{-2\alpha_1/2} = A_1(4t)^{-d/2}$$

This and (20) contradict to each other if $T^* = \infty$.

The proof of Theorem 3.5 is thus complete. □

4 Global Existence and Decay Estimates

In this and next section we require $\max_{i\in n^*}\{\alpha_i\} < d$, and treat the existence and large time behavior of global solutions of (1). Note that our condition imply that there exists $i \in N^*$ such that $p_i > 1 + 2/d$. Similar results are also obtained when $p_2 > 1 + 2/d$.

Theorem 4.1. *Assume* $\max_{i\in N^*}\{\alpha_i\} < d$ *and that there exists* $i \in N^*$ *such that* $p_i > 1 + 2/d$. *Let*

$$u_{i0} \in I^{a_i} \quad \text{with } a_i > \alpha_i \tag{21}$$

If $\|u_{i,0}\|_{\infty,a_i}$ *is small enough, then* $T^* = \infty$ *and we have*

$$u_i(x,t) \leq CS(t)\langle x\rangle^{-\hat{a}_i} \tag{22}$$

in $\mathbf{R}^d \times (0,\infty)$, *where* $\hat{a}_i \leq a_i$ $(i \in N^*)$ *are chosen to satisfy*

$$p_i\hat{a}_{i+1,d} - \hat{a}_i > 2. \tag{23}$$

First note that condition (21) can be replaced by $u_{i0} \in I^{\hat{a}_i}$ $(i \in N^*)$ since we have $I^{a_i} \subset I^{\hat{a}_i}$ $(i \in N^*)$. Then, to establish Theorem 4.1, we have only to consider the special case $\hat{a}_i = a_i$ $(i \in N^*)$. As is easily seen, in this case condition (23) is equivalent to

$$p_i a_{i+1,d} - a_i > 2 \quad (i \in N^*) \tag{24}$$

We set for $\gamma > 0$

$$\eta_\gamma(x,t) = S(t)\langle x\rangle^{-\gamma} \tag{25}$$

LEMMA 4.2. *The following in equality holds*

$$\eta_\gamma(x,t) \geq C \min\{\langle x\rangle^{-\gamma}, (1+t)^{-\gamma/2}\}$$

PROOF. See [2], [3]. □

LEMMA 4.3. *Let* $\gamma > 0$, $0 \leq \delta \leq \gamma_d = \min\{d, \gamma\}$. *Then we have*

$$\|\eta_\gamma(\cdot\,,t)\|_{\infty,\delta} \leq \begin{cases} C(1+t)^{-\gamma_d+\delta} & (\gamma \neq d), \\ C(1+t)^{-d+\delta}\log(2+t) & (\gamma = d). \end{cases}$$

PROOF. See [2], [3]. □

LEMMA 4.4. *We have in* $\mathbf{R}^d \times (0,\infty)$

$$\eta_{a_{i+1}}(x,t)^{p_i} \leq \begin{cases} C(1+t)^{(a_i - a_{i+1,d}p_i)/2}\eta_{a_i}(x,t) & a_{i+1} \neq d \\ C(1+t)^{(a_i - dp_i)/2}[\log(2+t)]^{p_i}\eta_{a_i}(x,t) & a_{i+1} = d \end{cases} \tag{26}$$

PROOF. We shall consider only the case $i = 1$ because similar argument also can be applied to other cases. We have by Lemma 4.2

$$\begin{aligned} \eta_{a_2}(x,t)^{p_1} &= \eta_{a_2}(x,t)^{p_1}\eta_{a_1}(x,t)^{-1}\eta_{a_1}(x,t) \\ &\leq C\max\{\langle x\rangle^{a_1}, (1+t)^{a_1/2}\}\eta_{a_2}(x,t)^{p_1}\eta_{a_1}(x,t) \end{aligned}$$

Since $a_1 \leq p_1 a_{2d}$ from (24), we can use Lemma 4.2 to obtain (26). □

We define the Banach spaces E_η and X as

$$E_\eta = \left\{u; \|u\|_{E_\eta} \equiv \sum_{i=1}^{N}(|||u_i/\eta_{a_i}|||_\infty) < \infty\right\},$$

and

$$X = \{v; |||v/\eta_a|||_\infty < \infty\},$$

where

$$|||w|||_\infty = \sup_{(x,t)\in \mathbf{R}^d\times(0,\infty)} |w(x,t)|.$$

(6) is reduced to

$$u_N(t) = V(t)(u_0, u_N), \tag{27}$$

where

$$\begin{aligned} V(T)(u_0,v) = S(t)u_{N0} + \int_0^t S(t-s_1)\Bigg(S(s_1)u_{10} + \int_0^{s_1} S(s_1-s_2) \\ \times \Bigg\{S(s_2)u_{20} + \int_0^{s_2} S(s_2-s_3)\times\cdots\times\Bigg[S(s_{N-1})u_{N-1,0} \\ + \int_0^{s_{N-1}} S(s_{N-1}-s_N)v^{p_{N-1}}(s_N)\,ds_N\Bigg]^{p_{N-2}}\times\cdots\times ds_3\Bigg\}^{p_1} ds_2\Bigg)^{p_N} ds_1 \end{aligned}$$

Moreover, using that $(a+b)^p \le 2^{p-1}(a^p+b^p)$ for $a>0,\ b>0,\ p>1,$

$$V(T)(u_0,v) \le T(t)(u_0) + \Gamma(t)(v)$$

where

$$\begin{aligned} T(t)(u_0) = S(t)u_{N0} + 2^{p_N-1}\int_0^t S(t-s_1)(S(s_1)u_{10})^{p_N}\,ds \\ + 2^{(p_N-1)(p_1-1)}\int_0^t S(t-s_1)\left(\int_0^{s_1} S(s_1-s_2)\{S(s_2)u_{20}\}^{p_1}\,dr\right)^{p_N} ds \\ + \cdots + 2^{(p_N-1)(p_1-1)\cdots(p_{N-2}-1)}\int_0^t S(t-s_1)\left(\int_0^{s_1} S(s_1-s_2)\left\{\int_0^{s_2} S(s_2-s_3)\right.\right. \\ \times\cdots\times\left(\int_0^{s_{N-2}} S(s_{N-2}-s_{N-1})[S(s_{N-1})u_{N-1,0}]^{p_{N-2}}\,ds_{N-1}\right) \\ \left.\left.\times\cdots\times ds_3\right\}^{p_1} ds_2\right)^{p_N} ds_1, \end{aligned}$$

and

$$\begin{aligned} \Gamma(t)(v) = 2^{(p_N-1)(p_1-1)\cdots(p_{N-2}-1)}\int_0^t S(t-s_1)\left(\int_0^{s_1} S(s_1-s_2)\left\{\int_0^{s_2} S(s_2-s_3)\times\cdots\right.\right. \\ \times\left[\int_0^{s_{N-2}} S(s_{N-2}-s_{N-1})\left\{\int_0^{s_{N-1}} S(s_{N-1}-s_N)v^{p_{N-1}}(s_N)\,ds_N\right\}^{p_{N-2}} ds_{N-1}\right]^{p_{N-3}} \\ \left.\left.\times\cdots\times ds_3\right\}^{p_1} ds_2\right)^{p_N} ds_1 \end{aligned}$$

LEMMA 4.5.

(i) *Let* u_0 *satisfy* (21). *Then* $T(\cdot)(u_0) \in X$ *and*

$$|||T(\cdot)(u_0)|||_\infty \leq \|u_{N0}\|_{\infty, a_N} + C2^{p_N-1}\|u_{10}\|_{\infty, a_1}^{p_N} + C2^{(p_N-1)(p_1-1)}\|u_{20}\|_{\infty, a_2}^{p_N p_1} + \cdots + C2^{(p_N-1)(p_1-1)\cdots(p_{N-2}-1)}\|u_{N-1,0}\|_{\infty, a_{N-1}}^{p_N p_1 \cdots p_{N-2}}$$

(ii) Γ *maps* X *into itself and*

$$|||\Gamma(v)/\eta_{a_N}|||_\infty \leq C2^{(p_N-1)(p_1-1)\cdots(p_{N-2}-1)}|||v/\eta_{a_N}|||_\infty^{p_1 p_2 p_3 \cdots p_N}$$

PROOF. (i) By (25) and (26), we obtain where $T(t)(u_0) = I_1 + I_2 + \cdots + I_N$

$$I_1 \leq \|u_{N0}\|_{\infty, a_N}\eta_{a_N}(t)$$

$$I_2 \leq 2^{p_N-1}\int_0^t S(t-s)(\eta_1\|u_{10}\|_{\infty, a_1})^{p_N}\, ds \leq C2^{p_N-1}\|u_{10}\|_{\infty, a_1}^{p_N}\eta_{a_N}(t)$$

and

$$I_3 \leq C2^{(p_N-1)(p_1-1)}\|u_{20}\|_{\infty, a_1}^{p_1 p_N}\eta_{a_3}(t)$$

$$\vdots$$

$$I_N \leq C2^{(p_N-1)(p_1-1)\cdots(p_{N-2}-1)}\|u_{N-1,0}\|_{\infty, a_{N-1}}^{p_N p_1 \cdots p_{N-2}}$$

by the same reason.

(ii) By (25) and (26)

$$\begin{aligned}\Gamma(v) \leq{}& C2^{(p_N-1)(p_1-1)(p_2-1)\cdots(p_{N-2}-1)}|||v/\eta_{a_N}|||_\infty^{p_1 p_2 \cdots p_N} \\ &\times \int_0^t S(t-s_1)\int_0^s S(s_1-s_2)\left\{\int_0^{s_2} S(s_2-s_3) \times \cdots \times \left[\int_0^{s_{N-2}} S(s_{N-2}-s_{N-1})\right.\right. \\ &\left.\left.\times \left\{\int_0^{s_{N-1} S(s_{N-1}-sN)\eta_{a_N}(s_N)^{p_1}\, ds_N}\right\}^{p_N} d_{s_{N-1}}\right]^{p_{N-3}} \times \cdots \times ds_3\right\}^{p_1} ds_2\Bigg)^{p_N} ds_1 \\ \leq{}& C2^{(p_N-1)(p_1-1)(p_2-1)\cdots(p_{N-2}-1)}|||v/\eta_{a_3}|||_\infty^{p_1 p_2 \cdots p_N}\int_0^t \eta_{a_1}(s)^{p_3}\, ds \\ \leq{}& C2^{(p_N-1)(p_1-1)(p_2-1)\cdots(p_{N-2}-1)}|||v/\eta_{a_3}|||_\infty^{p_1 p_2 \cdots p_N}\eta_{a_3}\end{aligned}$$

Proof of Theorem 4.1.

Let

$$\|u_{N0}\|_{\infty,a_N} + C2^{p_N-1}\|u_{10}\|_{\infty,a_1}^{p_N} + C2^{(p_N-1)(p_1-1)}\|u_{20}\|_{\infty,a_2}^{p_Np_1} + \cdots + C2^{(p_N-1)(p_1-1)\cdots(p_{N-2}-1)}\|u_{N-1,0}\|_{\infty,a_{N-1}}^{p_Np_1\cdots p_{N-2}} \le m,$$

$\|u_i\|_{\infty,a_i} \le m$ $(i \in N^*)$, $B_m = \{v \in X : \||v/\eta_{a_3}\||_\infty \le 2m\}$ and $P = \{u \in X; u \ge 0\}$. Then we shall show that $V(u_0, v)$ is a strict contraction of $B_m \cap P$ into provided m is small enough.

It is trivial that V maps P into P. We shall show that V maps $B_m \to B_m$. If m is small enough, then

$$V(t)(u_0, v)/\eta_{a_3} \le m + C2^{p_N-1}(2m)^{p_1p_2\cdots p_N} \le 2m$$

This contradicts $B_m \to B_m$.

Using $|a^p - b^p| \le p(a+b)^{p-1}|a-b|$ for $a > 0$, $b > 0$ and $p \ge 1$, with $v = \max\{v_1, v_2\}$, we can estimate as following

$$\begin{aligned}
&|V(t)(u_0, v_1) - V(t)(u_0, v_2)| \\
&\quad \le C\int_0^t S(t-s_1)\biggl(2S(s_1)u_{10} + 2\int_0^{s_1} S(s_1-s_2)\biggl\{S(s_2)u_{20} + \int_0^{s_2} S(s_2-s_3) \\
&\qquad \times \cdots \times \biggl\{S(s_{N-2})u_{N-1,0} + \int_0^{s_{N-2}} S(s_{N-2}-s_{N-1}) \\
&\qquad \times \biggl[S(s_{N-1}u_{N0} + \int_0^{s_{N-1}} S(s_{N-1}-s_N)v^{p_{N-1}}(s_N)\,ds_N\biggr]^{p_{N-2}} ds_{N-1}\biggr\}^{p_{N-2}} \\
&\qquad \times \cdots \times ds_3\biggr\}^{p_1} ds_2\biggr)^{p_N-1} \\
&\qquad \times \cdots \times \int_0^{s_1} S(s_{N-2}-s_{N-1})\biggl\{2S(s_{N-1})u_{N-1,0} \\
&\qquad + 2\int_0^{s_{N-1}} S(s_{N-1}-s_N)v^{p_{N-1}}(s_N)\,ds_N\biggr\}^{p_{N-2}-1} \\
&\qquad \times \int_0^{s_{N-1}} S(s_{N-1}-s_N)|v_1^{p_{N-1}}(s_N) - v_2^{p_{N-1}}(s_N)|\,ds_N \cdots ds_2ds_1 \\
&\quad = C\int_0^t S(t-s_1) \times J_1 \times \int_0^{s_1} S(s_1-s_2) \times J_2 \\
&\qquad \times \cdots \times \int_0^{s_{N-2}} S(s_{N-2}-s_{N-1}) \times J_{N-1} \times J_N d_{s_{N-1}} \times \cdots \times ds_2ds_1.
\end{aligned}$$

Noting $(a+b)^p = 2^{\max\{p-1,0\}}(a^p+b^p)$ for $a>0$, $b>0$ and $p\geq 0$, we find

$$\begin{aligned} J_{N-1} &\leq C\Bigg\{ \|u_{N-1,0}\|_{\infty,a_{N-1}}^{p_{N-2}-1}\eta_{a_{N-1}}(s_{N-1})^{p_{N-2}-1} \\ &\quad + \left(\int_0^r S(s_{N-1}-s_N)|v/\eta_{a_N}|^{p_{N-1}}\eta_{a_N}^{p_{N-1}}(s_N)\,ds_N\right)^{p_{N-2}-1}\Bigg\} \\ &\leq C\{(m^{p_{N-2}-1}+C(2m)^{(p_{N-2}-1)p_{N-1}})\eta_{a_{N-1}}(s_{N-1})^{p_{N-2}-1}\} \\ &\leq Cm^{p_{N-2}-1}\eta_{a_{N-1}}(s_{N-1})^{p_{N-2}-1}. \end{aligned}$$

Similarly we have

$$\begin{aligned} J_1 &\leq Cm^{p_N-1}\eta_{a_1}^{p_N-1}(s_1) \\ J_2 &\leq Cm^{p_1-1}\eta_{a_2}^{p_1-1}(s_2) \\ J_3 &\leq Cm^{p_2-1}\eta_{a_3}^{p_2-1}(s_3) \\ &\vdots \\ J_{N-2} &\leq Cm^{p_{N-4}-1}\eta_{a_{N-4}}^{p_{N-2}-1}(s_{N-2}) \\ J_N &\leq Cm^{p_{N-2}-1}(|v_1-v_2|/\eta_{a_N})\eta_{a_{N-1}} \end{aligned}$$

Thus, we obtain

$$|V(t)(u_0,v_1)-V(t)(u_0,v_2)| \leq Cm^{p_1+p_2+\cdots+p_N-N}|v_1-v_2|$$

Since $p_2>1$, $V(t)$ is a stract contraction of $Bm\cap P$ into itself provided m is small enough. Hence, there exist a unique fixed point $u_3\in X$ which solves (27). We substitute u_3 into (6). Then the vector u solve (6). Moreover, since $u_3\in B_m$, we find

$$u_N \leq CS(t)\langle x\rangle^{-a_N}$$

By the same reason in the proof of Lemma 4.5, we have

$$\begin{aligned} |u_{N-1}(t)| &\leq \eta_{a_{N-1}}(x,t)\{\|u_{N-1,0}\|_{\infty,a_{N-1}}+C|||u_N/\eta_{a_N}|||_\infty\} \\ &\vdots \\ |u_2(t)| &\leq \eta_{a_2}(x,t)\{\|u_{20}\|_{\infty,a_2}+C|||u_3/\eta_{a_3}|||_\infty\} \\ |u_1(t)| &\leq \eta_{a_1}(x,t)\{\|u_{10}\|_{\infty,a_1}+C|||u_2/\eta_{a_2}|||_\infty\} \end{aligned}$$

Then $u_i\in B_m$ $(i\in N^*)$ and the proof of Theorem 4.1 is completed. □

5 Asymptotic Behavior of Global Solution

In this section we shall prove the following theorem for the global solution $u(t)$ of (1) constructed in the previous section.

THEOREM 5.1. (i) *If we can choose* $\hat{a}_i = a_i < d$ $(i \in N^*)$ *in* (23) *and if*

$$\lim_{|x|\to\infty} |x|^{a_i} u_{i0}(x) = A_i > 0 \tag{28}$$

then

$$t^{a_i/2} |u(x,t) - A_1 S|x|^{-a_i}| \to 0 \quad (t \to \infty) \tag{29}$$

as $t \to \infty$ *uniformly in* $\mathbf{R}^d$.

(ii) *If we can choose* $\hat{a}_j > d$ $(j \in N^*)$ *in* (23), *then*

$$t^{d/2} |u_j(x,t) - M_j (4\pi t)^{-d/2} e^{-|x|^2/4t}| \to 0 \quad (t \to \infty) \tag{30}$$

uniformly on the set $\{x \in \mathbf{R}^d; |x| \le R t^{1/2}\}$ $(R > 0)$, *where*

$$M_j = \int_{\mathbf{R}^d} u_{j0}\, dx + \int_0^t \int_{\mathbf{R}^d} u_{n(j+1)}(s)^{p_j}(s)\, dx ds < \infty. \tag{31}$$

PROOF. This theorem can be shown by same way the case $N = 2$. (See [2], [3] or [5].) □

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