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THE CAUCHY PROBLEM FOR STRICTLY HYPERBOLIC OPERATORS WITH NON-ABSOLUTELY CONTINUOUS COEFFICIENTS

By

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Introduction

Let us consider the Cauchy problem

$$(1)\qquad \begin{aligned} &Pu(t,x) = 0 \quad \text{in } [0,T]\times \boldsymbol{R}^n \\ &u(0,x) = u_0(x), \quad \partial_t u(0,x) = u_1(x) \quad \text{in } \boldsymbol{R}^n \end{aligned}$$

for a strictly hyperbolic operator

$$(2)\qquad P = \partial_t^2 - \sum_{j,k=1}^{n} a_{j,k}(t,x)\partial_{x_j}\partial_{x_k} + \sum_{j=1}^{n} b_j(t,x)\partial_{x_j} + b_{n+1}(t,x)$$

with $(a_{j,k})$ a real symmetric matrix, $b_j \in C([0,T];\mathscr{B}(\boldsymbol{R}^n))$, $\mathscr{B}(\boldsymbol{R}^n)$ the space of all C^∞ functions which are bounded together with all their derivatives in $\boldsymbol{R}^n$.

It is well known that if $\partial_t a_{j,k} \in L^1([0,T];\mathscr{B}(\boldsymbol{R}^n))$ then problem (1) is well posed in Sobolev spaces: for every $u_0 \in H^s(\boldsymbol{R}^n)$, $u_1 \in H^{s-1}(\boldsymbol{R}^n)$ there is a unique solution $u \in C([0,T];H^s(\boldsymbol{R}^n)) \cap C^1([0,T];H^{s-1}(\boldsymbol{R}^n))$ which satisfies

$$(3)\qquad \|u(t)\|_s + \|\partial_t u(t)\|_{s-1} \le C(\|u_0\|_s + \|u_1\|_{s-1}), \quad 0 \le t \le T.$$

By the finite speed of propagation one obtains the well posedness in C^∞.

Our aim is to consider non-absolutely continuous coefficients assuming $a_{j,k} \in C^1(]0,T];\mathscr{B}(\boldsymbol{R}^n))$ and

$$(4)\qquad |\partial_t a_{j,k}(t,x)| \le Ct^{-q}, \quad q \ge 1, t > 0, x \in \boldsymbol{R}^n$$

as it is done by Colombini, Del Santo and Kinoshita in [3] for coefficients of P depending only on the time variable t. Here we treat the general case and, beside (4), we permit:

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(5) $$|\partial_x^\beta a_{j,k}(t,x)| \le C_\beta t^{-p}, \quad p \in [0,1[, |\beta| > 0, t > 0, x \in \boldsymbol{R}^n.$$

For $q = 1$ in (4) and any $p \in [0,1[$ in (5), we prove the inequality

(6) $$\|u(t)\|_{s-h} + \|\partial_t u(t)\|_{s-1-h} \le C(\|u(0)\|_s + \|\partial_t u(0)\|_{s-1}), \quad C, h > 0, 0 \le t \le T$$

for every $u \in C([0,T]; H^{s+1}(\boldsymbol{R}^n)) \cap C^1([0,T]; H^s(\boldsymbol{R}^n))$ such that $Pu = 0$. In particular, we obtain the well posedness in C^∞ of the Cauchy problem (1) with a loss of h derivatives.

In the case $q > 1$ in (4), we assume boundness and Gevrey regularity $\gamma^{(s)}$ for the coefficients, that is we take $p = 0$ and $C_\beta = CA^{|\beta|}(\beta!)^s$ in (5). Then we prove the well posedness of problem (1) in $\gamma^{(s)}$ for $1 < s < q/(q-1)$.

We refer to [3] for counter examples that show the sharpness of these results; in particular C^∞ well posedness does not hold for $q > 1$.

In (4) and (5) one can subsitute t^{-q} and t^{-p} with $|T_0 - t|^{-q}$ and $|T_0 - t|^{-p}$, respectively, $T_0 \in [0,T]$, $t \ne T_0$. So inequality (6) can be applied also to the study of the blowup rate in some nonlinear equations. Consider, for istance, a smooth solution u for $t < T$ of

$$\partial_t^2 u - \alpha\left(\int_0^t \partial_x u(s,x)\, ds\right)\partial_x^2 u = 0, \quad \alpha(y) \ge \alpha_0 > 0$$

such that

$$|\partial_x^\beta u(t,x)| \le C_\beta (T-t)^{-1}, \quad t < T.$$

If α' is bounded and $|\alpha^{(k)}(y)| \le A_k e^{\mu|y|}$, $\mu < 1/C_1$, $k \ge 2$, then $a(t,x) := \alpha(\int_0^t \partial_x u(s,x)\, ds)$ satisfies (4) with $q = 1$ and (5) with $p \in]\mu C_1, 1[$, $(T-t)^{-1}$ and $(T-t)^{-p}$ in place of t^{-1} and t^{-p} respectively. So inequality (6) implies $u \in C^\infty$ also for $t = T$. This means that $(T-t)^{-1}$ is not a sufficient breakdown rate of the derivatives $\partial_x^\beta u$ to have blowup of u at $t = T$, cf. [1].

1. Main Results

Let

$$P = \partial_t^2 - \sum_{j,k=1}^n a_{j,k}(t,x)\partial_{x_j}\partial_{x_k} + \sum_{j=1}^n b_j(t,x)\partial_{x_j} + b_{n+1}(t,x)$$

be a linear differential operator in $[0,T] \times \boldsymbol{R}^n$, with $(a_{j,k})$ a symmetric matrix of real valued functions, $a_{j,k} \in C^1(]0,T]; C^\infty(\boldsymbol{R}^n))$, $b_j \in C([0,T]; C^\infty(\boldsymbol{R}^n))$. We consider the Cauchy problem for the equation

$$Pu(t,x) = 0 \quad \text{in } [0,T] \times \boldsymbol{R}^n \tag{1.1}$$

with initial data at $t = 0$

$$u(0,x) = u_0(x), \quad \partial_t u(0,x) = u_1(x) \quad \text{in } \boldsymbol{R}^n \tag{1.2}$$

under the hypothesis of strict hyperbolicity

$$a(t,x,\xi) := \sum_{j,k=1}^{n} a_{j,k}(t,x)\xi_j\xi_k \geq c_0|\xi|^2, \quad c_0 > 0 \tag{1.3}$$

and we deal with its well posedness according to the behaviour of $\partial_t a$ as $t \to 0$. Our first result is the following:

Theorem 1. *Assume that there exist* $p, r \in [0,1[$ *and positive constants* C_β *such that*

$$|\partial_x^\beta a_{j,k}(t,x)| \leq C_\beta t^{-p}, \quad |\beta| > 0; \quad |\partial_x^\beta \partial_t a_{j,k}(t,x)| \leq C_\beta t^{-1-r|\beta|}, \quad |\beta| \geq 0. \tag{1.4}$$

Then, for every $u_0, u_1 \in C^\infty(\boldsymbol{R}^n)$ *the Cauchy problem* (1.1), (1.2) *has a unique solution* $u \in C^1([0,T]; C^\infty(\boldsymbol{R}^n))$.

Remark. A consequence of (1.4) is the finite speed of propagation. So it is not restrictive to consider $u_0, u_1 \in C_0^\infty(\boldsymbol{R}^n)$ and to assume

$$|\partial_x^\beta b_j(t,x)| \leq C_\beta, \quad (t,x) \in [0,T] \times \boldsymbol{R}^n. \tag{1.5}$$

In Section 2 we shall prove an estimate in Sobolev spaces that implies Theorem 1:

Theorem 2. *Under the hypotheses of Theorem* 1 *there are positive constants* C, h *such that for every* $u \in C([0,T]; H^{s+1}(\boldsymbol{R}^n)) \cap C^1([0,T]; H^s(\boldsymbol{R}^n))$ *which satisfies* $Pu = 0$ *we have*

$$\|u(t)\|_{s-h} + \|\partial_t u(t)\|_{s-1-h} \leq C(\|u(0)\|_s + \|\partial_t u(0)\|_{s-1}), \quad 0 \leq t \leq T. \tag{1.6}$$

When $t\partial_t a(t,x,\xi|\xi|^{-2})$ is not bounded, problem (1.1), (1.2) may not be well posed in C^∞.

For $s > 1$, $A > 0$, we denote by $\gamma_A^{(s)} = \gamma_A^{(s)}(\boldsymbol{R}^n)$ the space of all functions f satisfying

$$\|f\|_{s,A} := \sup_{\beta \in \boldsymbol{Z}_+^n, x \in \boldsymbol{R}^n} A^{-|\beta|}(\beta!)^{-s}|\partial_x^\beta f(x)| < \infty$$

so $\gamma^{(s)} := \bigcup_{A>0} \gamma_A^{(s)}$ is a Gevrey space.

Theorem 3. *Assume* $a_{j,k}, b_j \in C([0,T];\gamma_A^{(s)}(\boldsymbol{R}^n))$ *and*

$$(1.7)\quad |\partial_x^\beta \partial_t a_{j,k}(t,x)| \le Ct^{-q}A^{|\beta|}(\beta!)^s, \quad (t,x) \in]0,T] \times \boldsymbol{R}^n, \quad q > 1, s < q/(q-1).$$

Then there exists $A_0 > A$ *such that for every* $u_0, u_1 \in \gamma_A^{(s)}(\boldsymbol{R}^n)$ *the Cauchy problem* (1.1), (1.2) *has a unique solution* $u \in C^1([0,T];\gamma_{A_0}^{(s)}(\boldsymbol{R}^n))$.

As Theorem 1, we shall obtain Theorem 3 from an *a priori* estimate; so we introduce Gevrey-Sobolev spaces adapted to our problem. We fix $\delta \in]0,1[$ such that $1/s = (q-1+\delta)/q$ then for $k > 0$, $t \in [0,T]$, $\mu \in \boldsymbol{R}$ we denote by $H^{k,t,\mu}(\boldsymbol{R}^n)$ the space of all functions f such that:

$$\|f\|_{k,t,\mu} := \left\| \exp\left(\frac{k}{\delta}(T^\delta - t^\delta)\langle D_x\rangle^{1/s}\right) f \right\|_\mu < \infty,$$

$\|g\|_\mu$ the norm of g in the usual Sobolev space $H^\mu(\boldsymbol{R}^n)$.

From Paley-Wiener theorem it follows that

$$\|f\|_{k,t,\mu} \le C\|f\|_{s,A}, \quad f \in \gamma_A^{(s)}(\boldsymbol{R}^n) \cap C_0^\infty(\boldsymbol{R}^n), \quad 0 \le kT^\delta/\delta \le T_0$$

with T_0 and C positive constants depending on A. Conversely, for every $T_1 < T$ and $k > 0$ there is $A_1 > 0$ such that

$$H^{k,t,\mu}(\boldsymbol{R}^n) \subset \gamma_A^{(s)}(\boldsymbol{R}^n), \quad t \in [0,T_1], A > A_1, \mu > n/2.$$

For functions $u(t,x)$ we define the space

$$C_T^j(H^{k,t,\mu}) := \left\{ u; t \to \exp\left(\frac{k}{\delta}(T^\delta - t^\delta)\langle D_x\rangle^{1/s}\right)\partial_t^h u(t,\cdot) \text{ is continuous from } [0,T] \text{ to } H^{\mu-h}(\boldsymbol{R}^n), h = 0,\dots,j \right\}.$$

Theorem 4. *Under the hypotheses of Theorem* 3 *there are positive constants* k_0, T_0, C *such that for every* $u \in C_T^1(H^{k,t,\mu+1})$, $kT^\delta/\delta \le T_0$, $k \ge k_0$, *which satisfies* $Pu = 0$ *we have*

$$(1.8)\quad \|u(t)\|_{k,t,\mu} + \|\partial_t u(t)\|_{k,t,\mu-1} \le C(\|u(0)\|_{k,0,\mu} + \|\partial_t u(0)\|_{k,0,\mu-1}), \quad 0 \le t \le T.$$

We shall prove Theorem 4 in Section 3. From estimate (1.8) we can solve problem (1.1), (1.2) in $[0,T_1]$, $T_1 = (\delta T_0/k_0)^{1/\delta}$. This is sufficient to prove Theorem 3 since we have $a_{j,k} \in C^1([T_1,T];\gamma_A^{(s)}(\boldsymbol{R}^n))$ that ensures $\gamma^{(s)}$ well posedness in $[T_1,T]$.

2. C^∞ Well Posedness

In this section we prove Theorem 2 which implies Theorem 1. Writing

$$P = \partial_t^2 + a(t,x,D_x) + b(t,x,D_x), \quad D_x = \frac{1}{i}\partial_x \quad (i = \sqrt{-1}),$$

$$a(t,x,\xi) = \sum_{j,k=1}^{n} a_{j,k}(t,x)\xi_j\xi_k, \quad b(t,x,\xi) = i\sum_{j=1}^{n} b_j(t,x)\xi_j + b_{n+1}(t,x),$$

the assumptions on P are the following:

$$a(t,x,\xi) \geq c_0|\xi|^2, \quad c_0 > 0, \tag{2.1}$$

$$|\partial_x^\beta \partial_\xi^\alpha a(t,x,\xi)| \leq C_{\alpha,\beta} t^{-p} \langle\xi\rangle^{2-|\alpha|}, \quad |\alpha| \geq 0, |\beta| > 0, \tag{2.2}$$

$$|\partial_x^\beta \partial_\xi^\alpha \partial_t a(t,x,\xi)| \leq C_{\alpha,\beta} t^{-1-r|\beta|} \langle\xi\rangle^{2-|\alpha|}, \quad |\alpha| \geq 0, |\beta| \geq 0, \tag{2.3}$$

$$|\partial_x^\beta \partial_\xi^\alpha b(t,x,\xi)| \leq C_{\alpha,\beta} \langle\xi\rangle^{1-|\alpha|}, \quad |\alpha| \geq 0, |\beta| \geq 0, \tag{2.4}$$

$p, r \in [0,1[$, $(t,x,\xi) \in]0,T] \times \boldsymbol{R}^n \times \boldsymbol{R}^n$, $\langle\xi\rangle = (1+|\xi|^2)^{1/2}$.

In particular (2.3) gives also

$$|\partial_\xi^\alpha a(t,x,\xi)| \leq C_\alpha \log(1+1/t) \langle\xi\rangle^{2-|\alpha|}, \quad |\alpha| \geq 0. \tag{2.5}$$

We modify the symbol a for $\langle\xi\rangle \leq 2/t$ defining

$$a_0(t,x,\xi) = \varphi(t\langle\xi\rangle)\langle\xi\rangle^2 + (1-\varphi(t\langle\xi\rangle))a(t,x,\xi),$$

$$\varphi \in C^\infty(\boldsymbol{R}), \quad 0 \leq \varphi \leq 1, \quad \varphi = 1 \text{ in } [0,1], \quad \varphi = 0 \text{ in } [2,+\infty[.$$

Then $\lambda(t) = \sqrt{a_0(t)}$, $0 \leq t \leq T$, is a family of symbols of pseudodifferential operators in $\boldsymbol{R}^n$ which satisfies

$$\lambda(t,x,\xi) \geq c\langle\xi\rangle, \quad c > 0 \tag{2.6}$$

$$|\partial_x^\beta \partial_\xi^\alpha \lambda(t,x,\xi)| \leq C_{\alpha,\beta} \langle\xi\rangle^{1-|\alpha|}[1 + H(t\langle\xi\rangle)t^{-p|\beta|}(\log(1+1/t))^{1+|\alpha|}], \tag{2.7}$$

$$H(y) = 0 \text{ for } y < 1, \quad H(y) = 1 \text{ for } y \geq 1.$$

In particular, if we denote as usual by $S^m_{\rho,\delta}$ the class of all symbols $q(x,\xi)$ such that $|\partial_x^\beta \partial_\xi^\alpha q(x,\xi)| \leq C_{\alpha,\beta}\langle\xi\rangle^{m-\rho|\alpha|+\delta|\beta|}$, $0 \leq \delta < \rho \leq 1$, we have that $\{\lambda(t); 0 \leq t \leq T\}$ is bounded in $S^{1+\varepsilon}_{\rho,p}$ and $\{\lambda^{-1}(t); 0 \leq t \leq T\}$ is bounded in $S^{-1}_{\rho,p}$ for every $\varepsilon > 0$ and every $\rho \in]p,1[$.

Another consequence is that the symbol $r(t,x,\xi)$ of the operator $a_0 - \lambda^2$ verifies $t^{1-\varepsilon}r \in C([0,T]; S^1_{1,p})$ for every $\varepsilon \in]0, 1-p[$.

From (2.2), (2.3) and (2.5) we get:

$$|\partial_x^\beta \partial_\xi^\alpha \partial_t \lambda(t,x,\xi)| \le C_{\alpha,\beta}\langle\xi\rangle^{1-|\alpha|}H(t\langle\xi\rangle)t^{-1-\delta|\beta|}(\log(1+1/t))^{1+|\alpha|}, \tag{2.8}$$

$$\delta = \max\{p, r\},$$

which implies the boundness in $S^{1+\varepsilon'}_{1,\delta}$ of $\{t^{1-\varepsilon}\partial_t\lambda(t); 0 \le t \le T\}$ for any given $\varepsilon \in]0,1[$, $\varepsilon' \in]\varepsilon, 1[$, by using $t^{-\varepsilon}(\log(1+1/t))^{1+|\alpha|} \le C_\alpha t^{-\varepsilon'} \le C_\alpha\langle\xi\rangle^{\varepsilon'}$ on the support of $H(t\langle\xi\rangle)$.

Now we factorize the principal part of the operator $P = \partial_t^2 + a + b$:

$$P = (\partial_t - i\lambda)(\partial_t + i\lambda) + a - a_0 + a_1, \tag{2.9}$$

$$a_1 = -i[\partial_t, \lambda] + a_0 - \lambda^2 + b.$$

Obviously $t^{p+m}(a(t) - a_0(t))$, $0 \le t \le T$, is a bounded and continuous family in $S^{2-m}_{1,0}$ for any $m \ge 0$ while $t^{1-\varepsilon}a_1(t)$, $0 \le t \le T$ is bounded and continuous in $S^{1+\varepsilon'}_{1,\delta}$ for every $\varepsilon \in]0, 1-\delta[$, $\varepsilon' \in]\varepsilon, 1[$. Hereafter we fix $0 < \varepsilon < \varepsilon' < 1-\delta$.

We have not $a - a_0 + a_1 \in L^1([0,T]; S^1_{\rho,\delta})$ that by Gronwall's method would give the classical energy inequality

$$\|u(t)\|_s + \|\partial_t u(t)\|_{s-1} \le C(\|u(0)\|_s + \|\partial_t u(0)\|_{s-1}), \quad C > 0, 0 \le t \le T$$

for every $u \in C([0,T]; H^{s+1}(\boldsymbol{R}^n)) \cap C^1([0,T]; H^s(\boldsymbol{R}^n))$ such that $Pu = 0$.

Anyway a weaker condition in this direction holds true: $a - a_0 = \varphi(t\langle\xi\rangle)(a - \langle\xi\rangle^2)$ is bounded by $C\langle\xi\rangle^2\log(1+1/t)$ and vanishes for $t\langle\xi\rangle > 2$ so we can find a smooth function $\psi_0(t,\xi)$ such that

$$|a - a_0|\langle\xi\rangle^{-1} \le \psi_0, \quad (\log(1+1/t))^{-1}\psi_0 \in C([0,T]; S^1_{1,0})$$

and

$$\int_0^T |\partial_\xi^\alpha \psi_0(t,\xi)|\,dt \le C_\alpha\langle\xi\rangle^{1-|\alpha|}\int_0^{2/\langle\xi\rangle}\log(1+1/t)\,dt \le h_\alpha\langle\xi\rangle^{-|\alpha|}\log(1+\langle\xi\rangle).$$

Concerning a_1 we have that

$$\partial_t\lambda = (2\lambda)^{-1}[\langle\xi\rangle\varphi'(t\langle\xi\rangle)(\langle\xi\rangle^2 - a) + (1 - \varphi(t\langle\xi\rangle))\partial_t a]$$

is bounded by $C\langle\xi\rangle^2\log(1+1/t)$ for $t\langle\xi\rangle \le 2$ and by $t^{-1}\langle\xi\rangle$ for $t\langle\xi\rangle > 2$ while the symbol of $a_0 - \lambda^2 + b$ is bounded by $Ct^{-1+\varepsilon}\langle\xi\rangle$. So we can find $\psi_1(t,\xi)$ such that

$$|a_1|\langle\xi\rangle^{-1} \le \psi_1, \quad t^{1-\varepsilon}\psi_1 \in C([0,T]; S^{\varepsilon'}_{1,0})$$

and

$$\int_0^T |\partial_\xi^\alpha \psi_1(t,\xi)|\,dt \le C_\alpha \langle\xi\rangle^{-|\alpha|}\left[1+\langle\xi\rangle\int_0^{2/\langle\xi\rangle}\log(1+1/t)\,dt+\int_{2/\langle\xi\rangle}^T \frac{1}{t}\,dt\right]$$
$$\le h_\alpha \langle\xi\rangle^{-|\alpha|}\log(1+\langle\xi\rangle).$$

Now we use the factorization (2.9) to reduce the equation $Pu=0$ to a first order system. For $u \in C([0,T];H^{s+1}(\boldsymbol{R}^n)) \cap C^1([0,T];H^s(\boldsymbol{R}^n))$ let us define

$$U = {}^t(u_1,u_2), \quad u_1 = (\partial_t + i\lambda)u, \quad u_2 = \langle D_x\rangle u - mu_1,$$

m the operator with symbol $m(t,x,\xi) = \dfrac{(1-\varphi(t\langle\xi\rangle/3))\langle\xi\rangle}{2i\lambda(t,x,\xi)}$ to have $\langle\xi\rangle = 2i\lambda m$ for $t\langle\xi\rangle > 6$ and $(\operatorname{supp} m) \cap (\operatorname{supp} a - a_0) = \varnothing$.

Then it is easy to see that the equation $Pu=0$ is equivalent to a first order 2×2 system $LU=0$,

$$L = \partial_t + K(t,x,D_x), \quad K = D + A, \quad A = A_0 + A_1, \tag{2.10}$$

where

$$D = \begin{pmatrix} -i\lambda & 0 \\ 0 & i\lambda \end{pmatrix}, \quad t^p A_0 \in C([0,T];S^1_{1,0}), \tag{2.11}$$
$$A_0(t,x,\xi) = 0 \text{ for } t\langle\xi\rangle > 6, \quad t^{1-\varepsilon}A_1 \in C([0,T];S^{\varepsilon'}_{1,\delta})$$

and there are two positive functions $\psi_0(t,\xi), \psi_1(t,\xi)$ such that:

$$|A_0| \le \psi_0, \quad (\log(1+1/t))^{-1}\psi_0 \in C([0,T];S^1_{1,0}), \tag{2.12}$$
$$|A_1| \le \psi_1, \quad t^{1-\varepsilon}\psi_1 \in C([0,T];S^{\varepsilon'}_{1,0}),$$
$$\int_0^T |\partial_\xi^\alpha \psi(t,\xi)|\,dt \le h_\alpha\langle\xi\rangle^{-|\alpha|}\log(1+\langle\xi\rangle), \quad \psi = \psi_0 + \psi_1.$$

Since it is

$$C^{-1}(\|u(t)\|_{s+1} + \|\partial_t u(t)\|_{s-\varepsilon}) \le \|U(t)\|_s \le C(\|u(t)\|_{s+1+\varepsilon} + \|\partial_t u(t)\|_s), \quad 0 \le t \le T,$$

we prove Theorem 2 by the following result:

THEOREM 2.1. *There are positive constants C, h such that for every $U \in C([0,T];H^{s+1}(\boldsymbol{R}^n)) \cap C^1([0,T];H^s(\boldsymbol{R}^n))$ which satisfies $LU=0$ we have*

$$\|U(t)\|_{s-h} \le C\|U(0)\|_s, \quad 0 \le t \le T. \tag{2.13}$$

Proof. It is sufficient to prove (2.13) for $s = 0$ since $\langle D_x \rangle^s L \langle D_x \rangle^{-s}$ satisfies the same hypotheses as L.

We look for lower bounds of the operator $K = D + A$ in (2.10). As it concerns the diagonal part D, from (2.7) we have that the symbol $d(t,x,\xi)$ of the operator $D(t) + D^*(t)$ satisfies $t^{1-\varepsilon} d \in C([0,T]; S^0_{1,\delta})$ so it follows

$$2\,\mathrm{Re}\langle DU(t), U(t)\rangle \geq -Ct^{-1+\varepsilon}\langle U(t), U(t)\rangle, \quad C > 0 \tag{2.14}$$

for every $U \in C([0,T]; H^1(\boldsymbol{R}^n))$.

Next we make the change of variable

$$V = w(t, D_x)U, \quad w(t,\xi) = \exp\left(-\int_0^t \psi(s,\xi)\,ds\right),$$

$\psi = \psi_0 + \psi_1$ the function in (2.12). We have

$$\|U(t)\|_{-h_0} \leq 2\|V(t)\|_0, \quad U(0) = V(0), h_0 > 0, 0 < t \leq T \tag{2.15}$$

and $LU = 0$ if and only if $L_1 V = 0$ with

$$\begin{aligned} L_1 &= wLw^{-1} = \partial_t + K_1(t,x,D_x), \\ K_1 &= D + (\psi I + A) + R_1, \\ &t^{1-\varepsilon}(\log(1+\langle\xi\rangle))^{-1} R_1 \in C([0,T]; S^0_{1,\delta}) \end{aligned} \tag{2.16}$$

Now the symbol of $\psi I + A$ satisfies

$$\begin{aligned} &t^{1-\varepsilon}(\psi_0 I + A_0) \in C([0,T]; S^1_{1,0}), \quad \psi_0 I + (A_0 + A_0^*)/2 \geq 0 \quad \text{for large } |\xi|, \\ &t^{1-\varepsilon}(\psi_1 I + A_1) \in C([0,T]; S^{\varepsilon'}_{1,\delta}), \quad \psi_1 I + (A_1 + A_1^*)/2 \geq 0 \quad \text{for large } |\xi|, \\ &\varepsilon < \varepsilon' < 1 - \delta, \quad \delta = \max\{p, r\}, \end{aligned}$$

so the sharp Garding inequality gives

$$2\,\mathrm{Re}\langle(\psi I + A)V(t), V(t)\rangle \geq -Ct^{-1+\varepsilon}\langle V(t), V(t)\rangle, \quad C > 0 \tag{2.17}$$

for every $V \in C([0,T]; H^1(\boldsymbol{R}^n))$.

For the operator R_1 we have

$$2\,\mathrm{Re}\langle R_1 V(t), V(t)\rangle \geq -h_1 t^{-1+\varepsilon}\langle \log(1+\langle D_x\rangle)V(t), V(t)\rangle, \quad h_1 > 0 \tag{2.18}$$

that leads us to make the further change of variable (cf. [2]):

$$W = (1+\langle D_x\rangle)^{-\alpha(t)} V = (1+\langle D_x\rangle)^{-\alpha(t)} w(t,D_x)U, \quad \alpha(t) = h_1 t^\varepsilon/\varepsilon,$$

h_1 the constant in (2.18). It is

(2.19)
$$\|U(t)\|_{-h} \le 2^{\alpha(T)+1}\|W(t)\|_0, \quad U(0) = W(0), \quad h = h_0 + h_1 T^{\varepsilon}/\varepsilon, \quad 0 \le t \le T,$$

h_0 the constant in (2.15), and $LU = 0$ if and only if $L_2 W = 0$ with

(2.20)
$$\begin{aligned} L_2 &= (1+\langle D_x\rangle)^{-\alpha(t)} L_1 (1+\langle D_x\rangle)^{\alpha(t)} = \partial_t + K_2(t,x,D_x),\\ K_2 &= D + (\psi I + A) + (h_1 t^{-1+\varepsilon}\log(1+\langle D_x\rangle) + R_1) + R_2,\\ t^{1-\varepsilon}&R_2 \in C([0,T]; S^0_{1,\delta}). \end{aligned}$$

Now $h_1 t^{-1+\varepsilon}\log(1+\langle D_x\rangle) + R_1$ is a positive operator by (2.18) while $t^{1-\varepsilon}R_2(t)$ is uniformly bounded in $L^2(\boldsymbol{R}^n)$ for $0 < t \le T$. From this, (2.14) and (2.17) we get

$$2\,\mathrm{Re}\langle K_2 W(t), W(t)\rangle \ge -Ct^{-1+\varepsilon}\langle W(t), W(t)\rangle, \quad C > 0$$

for every $W \in C([0,T]; H^1(\boldsymbol{R}^n))$, hence

$$\frac{d}{dt}\|W(t)\|_0^2 \le Ct^{-1+\varepsilon}\|W(t)\|_0^2$$

for every $W \in C([0,T]; H^1(\boldsymbol{R}^n)) \cap C^1([0,T]; H^0(\boldsymbol{R}^n))$ such that $L_2 W = 0$. This gives

$$\|W(t)\|_0^2 \le \exp(Ct^{\varepsilon}/\varepsilon)\|W(0)\|_0^2$$

that is (2.13) with $s = 0$ by (2.19).

3. $\gamma^{(s)}$ Well Posedness

In this section we prove Theorem 4 which implies Theorem 3.

We need to introduce a class pseudodifferential operators in Gevrey spaces:

Definition 3.1. For $m \in \boldsymbol{R}$, $s > 1$, $A > 0$ we denote by $\Gamma^m_{s,A}$ the space of all symbols $a(x,\xi)$ such that

(3.1)
$$|a|_{\Gamma^m_{s,A,l}} := \sup_{(x,\xi)\in\boldsymbol{R}^{2n},\,|\alpha+\beta|\le l,\,\gamma\in\boldsymbol{Z}^n_+} |\partial_\xi^\alpha \partial_x^{\beta+\gamma} a(x,\xi)| A^{-|\gamma|}(\gamma!)^{-s}\langle\xi\rangle^{-m+|\alpha|}$$

is finite for every $l \in \boldsymbol{Z}_+$.

Set $a_\Lambda(x,D_x) = e^{\Lambda}a(x,D_x)e^{-\Lambda}$, $\Lambda = k\langle D_x\rangle^{1/s}$, and denote by

$$|a|_{S^m_l} := \sup_{(x,\xi)\in\boldsymbol{R}^{2n},\,|\alpha+\beta|\le l} |\partial_\xi^\alpha\partial_x^\beta a(x,\xi)|\langle\xi\rangle^{-m+|\alpha|}$$

the usual norms in $S^m_{1,0}$. In [4] Kajitani proved the following result:

Proposition 3.2. *For every $A > 0$ there is $T_0 > 0$ such that*

$$|k| < T_0, \quad a \in \Gamma^m_{s,A} \Rightarrow a_\Lambda \in S^m_{1,0}, \quad a_\Lambda = a + r, \quad r \in S^{m-1+1/s}_{1,0} \tag{3.2}$$

and for every $l \in \boldsymbol{Z}^n_+$ there are $C_l > 0$ and $l' \in \boldsymbol{Z}^n_+$ such that

$$|r|_{S^{m-1+1/s}_l} \le C_l |a|_{\Gamma^m_{s,A,l'}}. \tag{3.3}$$

In particular we have that $a(x, D_x)$, $a \in \Gamma^m_{s,A}$, is a continuous operator from $H^{k,t,\mu}(\boldsymbol{R}^n)$ to $H^{k,t,\mu-m}(\boldsymbol{R}^n)$, $H^{k,t,\mu}(\boldsymbol{R}^n) = \exp\left(-\frac{k}{\delta}(T^\delta - t^\delta)\langle D_x\rangle^{1/s}\right)H^\mu(\boldsymbol{R}^n)$, for $0 < kT^\delta/\delta \le T_0$, $0 \le t \le T$.

Now we can begin the proof of Theorem 4. In this section the assumptions on the operator

$$P = \partial_t^2 + a(t, x, D_x) + b(t, x, D_x)$$

are the following:

$$a(t, x, \xi) \ge c_0|\xi|^2, \quad c_0 > 0 \tag{3.4}$$

$$a \in C([0, T]; \Gamma^2_{s,A}) \tag{3.5}$$

(3.6) $t^q \partial_t a, 0 < t \le T$, is a continuous and bounded family in $\Gamma^2_{s,A}$,

$q > 1, s < q/(q-1)$

$$b \in C([0, T]; \Gamma^1_{s,A}). \tag{3.7}$$

Here we define

$$a_0(t, x, \xi) = \varphi(t^q\langle\xi\rangle)\langle\xi\rangle^2 + (1 - \varphi(t^q\langle\xi\rangle))a(t, x, \xi), \tag{3.8}$$

$$\varphi \in C^\infty(\boldsymbol{R}), \quad 0 \le \varphi \le 1, \quad \varphi = 1 \text{ in } [0, 1], \quad \varphi = 0 \text{ in } [2, +\infty[$$

and take $\delta > 0$ so that $1/s = (q - 1 + \delta)/q$ to have

$$t^{1-\delta}(a - a_0) \in C([0, T]; \Gamma^{1+1/s}_{s,A}) \tag{3.9}$$

using $\langle\xi\rangle \le 2t^{-q}$ in the support of $\varphi(t^q\langle\xi\rangle)$.

We have also

$$\lambda = \sqrt{a_0} \in C([0, T]; \Gamma^1_{s,A}), \quad \lambda^{-1} \in C([0, T]; \Gamma^{-1}_{s,A})$$

and from (3.6) we get

$$t^{1-\delta}\partial_t\lambda \in C([0, T]; \Gamma^{1+1/s}_{s,A}) \tag{3.10}$$

by $\langle\xi\rangle \le 2t^{-q}$ in $\operatorname{supp}\varphi'(t^q\langle\xi\rangle)$ and $t^{-q} \le \langle\xi\rangle$ in $\operatorname{supp}(1 - \varphi(t^q\langle\xi\rangle))$.

So we can write

$$P = (\partial_t - i\lambda)(\partial_t + i\lambda) + r, \quad t^{1-\delta} r \in C([0, T]; \Gamma_{s,A}^{1+1/s})$$

and define

$$U = {}^t(u_1, u_2), \quad u_1 = (\partial_t + i\lambda)u, \quad u_2 = \langle D_x \rangle u - m u_1,$$

$m(t, x, \xi) = \langle \xi \rangle / 2i\lambda(t, x, \xi)$ to have that the equation $Pu = 0$ is equivalent to a first order 2×2 system $LU = 0$,

$$L = \partial_t + K(t, x, D_x), \quad K = D + R, \tag{3.11}$$

$$D = \begin{pmatrix} -i\lambda & 0 \\ 0 & i\lambda \end{pmatrix}, \quad t^{1-\delta} R \in C([0, T]; \Gamma_{s,A}^{1/s}).$$

Denoting by $\|u\|_{k,t,\mu}$ the norm of u in $H^{k,t,\mu}(\boldsymbol{R}^n)$, it is

$$C^{-1}(\|u(t)\|_{k,t,\mu+1} + \|\partial_t u(t)\|_{k,t,\mu}) \le \|U(t)\|_{k,t,\mu} \le C(\|u(t)\|_{k,t,\mu+1} + \|\partial_t u(t)\|_{k,t,\mu}),$$

$0 \le t \le T$, $0 < kT^\delta/\delta \le T_0$, T_0 the constant in Proposition 3.2, thus we prove Theorem 4 by the following result:

Theorem 3.3. *There are positive constants k_0, C such that for every $U \in C_T^1(H^{k,t,\mu+1})$, $kT^\delta/\delta \le T_0$, $k \ge k_0$, which satisfies $LU = 0$ we have*

$$\|U(t)\|_{k,t,\mu} \le C\|U(0)\|_{k,0,\mu}, \quad 0 \le t \le T. \tag{3.12}$$

Proof. It is sufficient to prove (3.12) for $\mu = 0$ since $\langle D_x \rangle^\mu L \langle D_x \rangle^{-\mu}$ satisfies the same hypotheses as L and this is equivalent to prove

$$\|V(t)\|_0 \le C\|V(0)\|_0, \quad 0 \le t \le T \tag{3.13}$$

for every $V \in C([0, T]; H^1(\boldsymbol{R}^n)) \cap C^1([0, T]; H^0(\boldsymbol{R}^n))$ such that $L_\Lambda V = 0$, $L_\Lambda = e^\Lambda L e^{-\Lambda}$, $\Lambda = \dfrac{k}{\delta}(T^\delta - t^\delta)\langle D_x \rangle^{1/s}$.

From Proposition 3.2 and (3.11) we have

$$L_\Lambda = \partial_t + kt^{-1+\delta}\langle D_x \rangle^{1/s} + D + R_1, \quad t^{1-\delta} R_1 \in C([0, T]; S_{1,0}^{1/s}), \quad kT^\delta/\delta \le T_0,$$

so we can take k large enough, say $k \ge k_0$, to make $kt^{-1+\delta}\langle D_x \rangle^{1/s} + R_1(t)$ a positive operator while $D(t) + D^*(t)$ is uniformly bounded in $L^2(\boldsymbol{R}^n)$ for $0 \le t \le T$. This gives

$$\frac{d}{dt}\|V(t)\|_0^2 \le C\|V(t)\|_0^2, \quad 0 \le t \le T \le (\delta T_0/k)^{1/\delta}$$

for every $V \in C([0,T]; H^1(\boldsymbol{R}^n)) \cap C^1([0,T]; H^0(\boldsymbol{R}^n))$ such that $L_\Lambda V = 0$ which proves (3.13).

Remark. It is possible to prove Theorem 4 also for the critical index $s = q/(q-1)$. This needs the use of the Sharp Garding inequality as in the proof of Theorem 2 after an *ad hoc* version of Proposition 3.2 for more general functions Λ.

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