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HOMOGENEITY OF $\mathscr{K}(\boldsymbol{Q})$

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Abstract. We prove that $\mathscr{K}(\boldsymbol{Q})$ is a topological group and characterize $\mathscr{K}(\boldsymbol{Q})$ as a first-category, zero-dimensional, separable, metrizable space of which every non-empty clopen subset is $\boldsymbol{\Pi}_1^1$-complete. In particular we answer a question of Fujita and Taniyama ([5]). With the additional assumption of Analytic Determinacy it was proved in [5] that $\mathscr{K}(\boldsymbol{Q})$ is a homogeneous space.

All spaces under consideration are separable and metrizable. Basic topological notions can be found in [4] and basic notions from descriptive set theory can be found in [6].

The rationals $\boldsymbol{Q}$ are identified with the set $\{x \in 2^\omega : \exists n \in \omega\ \forall k \in \omega,\ k \geq n\ x_k = 0\}$. The spaces $\mathscr{K}(\boldsymbol{Q})$ and $\mathscr{K}(2^\omega)$ are defined as the spaces of all nonempty compact subsets of $\boldsymbol{Q}$ (resp. 2^ω) with the Vietoris topology. A space X is of the *first category in itself* if there exists a family $\{X_n\}_{n \in N}$ of closed, nowhere dense subsets of X, such that $X = \bigcup_{n \in N} X_n$.

Our fundamental auxiliary result is the following theorem of van Engelen:

THEOREM 1 [2, LEMMA 3.1] *Let spaces X, Y be zero-dimensional and first category in itself, such that every nonempty clopen subset of the space X (resp. of Y) contains a closed copy of the space Y (resp. X). Then X and Y are homeomorphic.*

Our first result is the following theorem:

THEOREM 2 *$\mathscr{K}(\boldsymbol{Q})$ is a topological group.*

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Before the proof we should introduce some notions from descriptive set theory. In a remark below is given an alternative proof of homogeneity of $\mathcal{K}(\boldsymbol{Q})$, which does not refer to descriptive set theory. However, the other proof does not give information that $\mathcal{K}(\boldsymbol{Q})$ is a topological group.

We define the family $\boldsymbol{\Pi}_1^1(2^\omega)$ as the family of all coanalytic subsets of 2^ω. A subset $C \subset 2^\omega$ is $\boldsymbol{\Pi}_1^1$-*complete* if $C \in \boldsymbol{\Pi}_1^1(2^\omega)$ and for every $A \in \boldsymbol{\Pi}_1^1(2^\omega)$ there exists a continuous function $f : 2^\omega \to 2^\omega$, such that $x \in A$ iff $f(x) \in C$. The function f will be called a *reduction* of A to C. For every $\boldsymbol{\Pi}_1^1$-complete $C \subset 2^\omega$ and $A \in \boldsymbol{\Pi}_1^1(2^\omega)$ we can find a reduction, which is injective. The idea how to construct the injection starting from an ordinary reduction can be extracted from the proof of Lemma 3 in [8], and Steel refers to it as to a trick of Henderson. This result is also stated in [7] as Theorem 4. We can summarize these remarks in the following:

PROPOSITION. For every $\boldsymbol{\Pi}_1^1$-complete $C \subset 2^\omega$ and $A \in \boldsymbol{\Pi}_1^1(2^\omega)$ there exists an injective reduction of A to C. In particular, this reduction is a closed embedding of A into C.

The following theorem of Hurewicz give us an example of a $\boldsymbol{\Pi}_1^1$-complete set:

THEOREM 3 [6, ex. 33.5]. *The space* $\mathcal{K}(\boldsymbol{Q}) \subset \mathcal{K}(2^\omega)$ *is* $\boldsymbol{\Pi}_1^1$-*complete.* ∎

The PROOF OF THEOREM 2 consists of five parts:

1. Let $\omega^{<\omega}$ be the set of finite sequences of non-negative integers, endowed with the structure of Kleene-Brouwer linear order ([6, 2.G]) (one can find an order isomorphism of $\omega^{<\omega}$ and $\{x \in \boldsymbol{R} : x \text{ rational}, \ x \leq 0\}$). Let us observe that $\mathrm{WO}(\omega^{<\omega}) = \{A \subset \omega^{<\omega} : A \text{ is well-ordered}\}$ with topology inherited from the space $2^{\omega^{<\omega}}$, is a topological group with respect to the operation of symmetric difference. In particular it is a homogeneous space.

2. $\mathrm{WO}(\omega^{<\omega}) \subset 2^{\omega^{<\omega}}$ belongs to $\boldsymbol{\Pi}_1^1(2^{\omega^{<\omega}})$ and moreover it is a $\boldsymbol{\Pi}_1^1$-complete set according to Theorem 27.1 and Proposition 2.12 from [6].

3. CLAIM 1. Every nonempty clopen subset of $\mathrm{WO}(\omega^{<\omega})$ contains a closed copy of $\mathcal{K}(\boldsymbol{Q})$.

PROOF. Let A_1 and A_2 be finite disjoint subsets of $\omega^{<\omega}$ and let $W = \{A \subset \omega^{<\omega} : A_1 \subset A, A_2 \cap A = \varnothing\}$. It is enough to show, that $W \cap \mathrm{WO}(\omega^{<\omega})$ contains a closed copy of $\mathcal{K}(\boldsymbol{Q})$.

First we will prove, that $\mathrm{WO}(\omega^{<\omega}) \cap W$ is $\mathbf{\Pi}_1^1$-complete. Let us define $i : 2^{\omega^{<\omega}} \to W$, $i(A) = (A \cup A_1) \setminus A_2$. The function i is continuous and has the property that for every $A \in 2^{\omega^{<\omega}}$

$$A \in \mathrm{WO}(\omega^{<\omega}) \quad \text{iff} \quad i(A) \in \mathrm{WO}(\omega^{<\omega})$$

Consequently $\mathrm{WO}(\omega^{<\omega}) \cap W$ is $\mathbf{\Pi}_1^1$-complete.

Since $\mathscr{K}(\mathbf{Q})$ belongs to $\mathbf{\Pi}_1^1$, the Proposition implies that exists a closed copy of $\mathscr{K}(\mathbf{Q})$ in $\mathrm{WO}(\omega^{<\omega}) \cap W$.

4. Claim 2. Every nonempty clopen subset of $\mathscr{K}(\mathbf{Q})$ contains a closed copy of $\mathrm{WO}(\omega^{<\omega})$.

Proof. Let $W_0, \dots W_n$ be subsets of 2^ω. We define $\langle W_0, \dots W_n \rangle = \{K \in \mathscr{K}(2^\omega) : K \subset \bigcup_{k=0}^n W_k$ and for every $k = 0, \dots n, K \cap W_k \neq \emptyset\}$.

The family $\mathscr{B} = \{\langle W_0, \dots W_n \rangle : W_k \subset 2^\omega$ clopen, $k = 0, \dots n\}$ is a base of $\mathscr{K}(2^\omega)$ consisting of clopen sets. We fix $W = \langle W_0, \dots W_n \rangle \in \mathscr{B}$, where $W_0, \dots W_n$ are nonempty clopen sets in 2^ω.

It is enough to check that $W \cap \mathscr{K}(\mathbf{Q})$ contains a closed copy of $\mathrm{WO}(\omega^{<\omega})$. We fix points $x_1 \in W_1 \cap \mathbf{Q}, \dots, x_n \in W_n \cap \mathbf{Q}$ and $s \in 2^{<\omega}$, $n_0 = \mathrm{length}(s)$ such that $[s] = \{x \in 2^\omega : x|n_0 = s\}$ is contained in W_0 and $x_1 \notin [s], \dots x_n \notin [s]$.

We define for $x \in 2^\omega$, $(s^\frown x) \in 2^\omega$ as follows: for $n \in \omega, n < n_0$, $(s^\frown x)(n) = s(n)$ and for $n \geq n_0$, $(s^\frown x)(n) = x(n - n_0)$. Subsequently we define for $K \subset 2^\omega$ a new set $s^\frown K = \{s^\frown x : x \in K\} \subset 2^\omega$.

Now, let us define a continuous injection $i : \mathscr{K}(2^\omega) \to W$ by the formula $i(K) = (s^\frown K) \cup \{x_1, \dots x_n\}$. This function has the property, that for every $K \in \mathscr{K}(2^\omega)$ holds $i(K) \in \mathscr{K}(\mathbf{Q})$ iff $K \in \mathscr{K}(\mathbf{Q})$.

Function i is a reduction of $\mathscr{K}(\mathbf{Q})$ to $\mathscr{K}(\mathbf{Q}) \cap W$. Existence of the reduction together with Theorem 3 imply that $W \cap \mathscr{K}(\mathbf{Q})$ is $\mathbf{\Pi}_1^1$-complete.

Since $\mathrm{WO}(\omega^{<\omega})$ belongs to $\mathbf{\Pi}_1^1$, we can apply the Proposition.

5. Claim 3. $\mathrm{WO}(\omega^{<\omega})$ is of the first category in itself.

Proof. Since $\mathrm{WO}(\omega^{<\omega})$ belongs to $\mathbf{\Pi}_1^1(2^{\omega^{<\omega}})$, it has the Baire property ([6, Corollary 29.14]). On the other hand it is an ideal on $\omega^{<\omega}$. Hence $\mathrm{WO}(\omega^{<\omega})$ is of the first category in $2^{\omega^{<\omega}}$, because every ideal with the Baire property, which contains all finite subsets of $2^{\omega^{<\omega}}$ is of the first category in $2^{\omega^{<\omega}}$. Since $\mathrm{WO}(\omega^{<\omega})$ is dense in $2^{\omega^{<\omega}}$, it is first category in itself.

Now we can apply Theorem 1 and obtain a homeomorphism between the topological group $\mathrm{WO}(\omega^{<\omega})$ and $\mathscr{K}(\mathbf{Q})$. ■

Let us point out that a part of the above reasoning can be generalized in the following:

THEOREM 4. *Let X be of the first category in 2^ω and such that intersection of every nonempty clopen subset of 2^ω with X is Π^1_1-complete. Then X and $\mathscr{K}(\mathbf{Q})$ are homemorphic.* ■

REMARK 1. We sketch an alternative proof of homogeneity of $\mathscr{K}(\mathbf{Q})$. A space X is *strongly homogeneous* if every nonempty clopen set in X is homeomorphic to X itself. The alternative proof is based on the following result of van Engelen:

THEOREM 5 [2, THEOREM 4.1]. *Let a space X be zero-dimensional, first category in itself and such that every nonempty clopen subset of the space X contains a closed copy of X. Then X is strongly homogeneous.* ■

We will need one more theorem, which motivates the notion of strong homogeneity:

THEOREM 6 [3, THEOREM 1.9.1]. *If a space X is zero-dimensional and strongly homogeneous, then X is homogeneous.* ■

THEOREM 7. *The space $\mathscr{K}(\mathbf{Q})$ is strongly homogeneous.*

Theorems 6 and 7 give us the promised homogeneity of $\mathscr{K}(\mathbf{Q})$.

PROOF OF THEOREM 7. Let us observe that $\mathscr{K}(\mathbf{Q})$ is first category in itself. Indeed, $\mathscr{K}(\mathbf{Q}) \subset \bigcup_{q \in \mathbf{Q}} F_q$ where $F_q = \{K \in \mathscr{K}(2^\omega) : q \in K\}$ are closed and nowhere dense in $\mathscr{K}(2^\omega)$. Since $\mathscr{K}(\mathbf{Q})$ is dense in 2^ω, the intersections $F_q \cap \mathscr{K}(\mathbf{Q})$ are closed and nowhere dense in $\mathscr{K}(\mathbf{Q})$.

Let W be a clopen subset of $\mathscr{K}(2^\omega)$ defined as in the proof of Theorem 2, Claim 2. According to Theorem 5, to prove strong homogeneity of $\mathscr{K}(\mathbf{Q})$, it is sufficient to check that $W \cap \mathscr{K}(\mathbf{Q})$ contains a closed copy of $\mathscr{K}(\mathbf{Q})$.

Appropriate embedding is given by the function i from the proof of Theorem 2, Claim 2. The space $\mathscr{K}(2^\omega)$ is compact, hence i is a closed map and $i[\mathscr{K}(\mathbf{Q})]$ is a closed copy of $\mathscr{K}(\mathbf{Q})$ in $W \cap \mathscr{K}(\mathbf{Q})$. This finishes the proof. ■

REMARK 2. Let $\mathcal{K}_{\leq\aleph_0}(2^\omega) = \{K \in \mathcal{K}(2^\omega) : K \text{ is at most countable}\}$. Theorem 27.5 from [6] shows that $\mathcal{K}_{\leq\aleph_0}(2^\omega)$ is a $\mathbf{\Pi}_1^1$-complete subset of $\mathcal{K}(2^\omega)$.

We are going to show that every nonempty clopen set of $\mathcal{K}_{\leq\aleph_0}(2^\omega)$ is $\mathbf{\Pi}_1^1$-complete.

Let W be a clopen subset of $\mathcal{K}(2^\omega)$ defined as in the proof of Theorem 2, Claim 2. It is enough to verify, that there exists a reduction of $\mathcal{K}_{\leq\aleph_0}(2^\omega)$ to $W \cap \mathcal{K}_{\leq\aleph_0}(2^\omega)$. It is easy to check, that function i defined in the proof of Theorem 2, Claim 2 is suitable.

Lemma 1.2 of [5] shows that the set G of all nonempty compact subsets of 2^ω without isolated points is dense G_δ in $\mathcal{K}(2^\omega)$. $\mathcal{K}_{\leq\aleph_0}(2^\omega)$ is disjoint with G, hence it is first-category subset of $\mathcal{K}(2^\omega)$. Moreover, $\mathcal{K}_{\leq\aleph_0}(2^\omega)$ is dense in $\mathcal{K}(2^\omega)$, thus of the first category in itself.

Finally, Theorem 4 imply that the spaces $\mathcal{K}(\mathbf{Q})$, $\mathcal{K}_{\leq\aleph_0}(2^\omega)$ and $\mathrm{WO}(\omega^{<\omega})$ are homeomorphic and in particular, that each of them is homogenous.

The main aim of this remark is to point out an analogy between the last observation and the following result of R. Cauty:

THEOREM 8 [1, THEOREM 1.4]. *The space $\mathcal{K}_{\leq\aleph_0}(I)$ of countable compact subsets of the interval I with the Vietoris topology is homeomorphic with the space $\mathcal{D} = \{f \in \mathcal{C}(I) : f \text{ is everywhere differentiable}\}$ endowed with the topology of uniform convergence.*

The space $\mathcal{D}$ is a vector space and in particular a topological group. It implies that $\mathcal{K}_{\leq\aleph_0}(I)$ is homogeneous.

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