

SOME PROPERTIES OF CERTAIN MULTIVALENT FUNCTIONS

Dedicated to Professor Yukihiro Kodama on his 60th birthday

By

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1. Introduction

Let A_p be the class of functions of the form

$$(1.1) \quad f(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n \quad (p \in N = \{1, 2, 3, \dots\})$$

which are analytic in the unit disk $U = \{z : |z| < 1\}$.

A function $f(z) \in A_p$ is said to be in the class $R_p(\alpha)$ if it satisfies

$$(1.2) \quad \operatorname{Re}\{f^{(p)}(z)\} > \alpha$$

for some $\alpha (0 \leq \alpha < p!)$ and for all $z \in U$. Let the functions $F(z)$ and $G(z)$ be analytic in the unit disk U . Then the function $F(z)$ is said to be subordinate to $G(z)$ if there exists a function $w(z)$ analytic in U , with $w(0) = 0$ and $|w(z)| < 1 (z \in U)$, such that $F(z) = G(w(z)) (z \in U)$. We denote this subordination by $F(z) \prec G(z)$.

For $f(z) \in R_p(\alpha)$, Saitoh has proved that

$$(1.3) \quad f(z) \in R_p(\alpha) \implies \frac{f^{(p-1)}(z)}{z} \prec 2\alpha - p! - \frac{2(p! - \alpha)}{z} \ln(1-z).$$

This is a generalization of the result for $p=1$ by Owa, Ma and Liu [5].

Let $S_p(\alpha)$ be the subclass of A_p consisting of functions which satisfy

$$(1.4) \quad f^{(p)}(z) \prec p! + (p! - \alpha)z \quad (z \in U)$$

for some $\alpha (0 \leq \alpha < p!)$. Then, it is easy to see that

$$S_p(\alpha) \subset R_p(\alpha) \quad (0 \leq \alpha < p!)$$

and that $f(z) \in A_p$ is in the class $S_p(\alpha)$ if and only if

$$(1.5) \quad |f^{(p)}(z) - p!| < p! - \alpha \quad (z \in U)$$

for some $\alpha (0 \leq \alpha < p!)$.

2. Some properties of the class $S_p(\alpha)$

We begin with the statement of the following lemma due to Jack [1] (also, due to Miller and Mocanu [4]).

LEMMA 1. *Let $w(z)$ be regular in U with $w(0)=0$. If $|w(z)|$ attains its maximum value in the circle $|z|=r$ at a point $z_0 \in U$, then we can write*

$$z_0 w'(z_0) = k w(z_0),$$

where k is real and $k \geq 1$.

An application of the above lemma leads to

THEOREM 1. *If $f(z) \in A_p$ satisfies*

$$(2.1) \quad |\beta f^{(p)}(z) + (1-\beta)z f^{(p+1)}(z) - p! \beta| < p! - \alpha \quad (z \in U)$$

for some $\alpha (0 \leq \alpha < p!)$ and $\beta (0 \leq \beta \leq 1)$, then $f(z) \in S_p(\alpha)$.

PROOF. We define the function $w(z)$ by

$$(2.2) \quad w(z) = \frac{f^{(p)}(z) - p!}{p! - \alpha}.$$

Then $w(z)$ is regular in U and $w(0)=0$. Since

$$(2.3) \quad z f^{(p+1)}(z) = (p! - \alpha) z w'(z),$$

we have

$$(2.4) \quad \begin{aligned} & \beta f^{(p)}(z) + (1-\beta)z f^{(p+1)}(z) - p! \beta \\ &= (p! - \alpha) \{ \beta w(z) + (1-\beta)z w'(z) \}. \end{aligned}$$

If there exists a point $z_0 \in U$ such that

$$\max_{|z| \leq |z_0|} |w(z)| = |w(z_0)| = 1,$$

then Lemma 1 gives

$$z_0 w'(z_0) = k w(z_0) \quad (k \geq 1).$$

Therefore, noting that $w(z_0) = e^{i\theta}$, we obtain that

$$(2.5) \quad \begin{aligned} & |\beta f^{(p)}(z_0) + (1-\beta)z_0 f^{(p+1)}(z_0) - p! \beta| \\ &= (p! - \alpha) (\beta + (1-\beta)k) \\ &\geq p! - \alpha, \end{aligned}$$

which contradicts our condition (2.1). Thus $|w(z)| < 1$ for all $z \in U$, that is,

$$(2.6) \quad |w(z)| = \left| \frac{f^{(p)}(z) - p!}{p! - \alpha} \right| < 1 \quad (z \in U).$$

This completes the proof of Theorem 1.

Letting $\beta=0$ in Theorem 1, we have

COROLLARY 1. *If $f(z) \in A_p$ satisfies*

$$(2.7) \quad |zf^{(p+1)}(z)| < p! - \alpha \quad (z \in U)$$

for some $\alpha(0 \leq \alpha < p!)$, then $f(z) \in S_p(\alpha)$.

Further making $\beta=1/2$, Theorem 1 leads to

COROLLARY 2. *If $f(z) \in A_p$ satisfies*

$$(2.8) \quad |f^{(p)}(z) + zf^{(p+1)}(z) - p!| < 2(p! - \alpha) \quad (z \in U)$$

for some $\alpha(0 \leq \alpha < p!)$, then $f(z) \in S_p(\alpha)$.

Next, we prove

THEOREM 2. *If $f(z)$ is in the class $S_p(\alpha)$, then*

$$(2.9) \quad \operatorname{Re} \left\{ e^{i\beta} \frac{f^{(p-1)}(z)}{z} \right\} > 0 \quad (z \in U),$$

where

$$(2.10) \quad |\beta| \leq \frac{\pi}{2} - \operatorname{Sin}^{-1} \left(\frac{p! - \alpha}{p!} \right).$$

PROOF. It follows from the definition of the class $S_p(\alpha)$ that

$$\begin{aligned} f(z) \in S_p(\alpha) &\iff |f^{(p)}(z) - p!| < p! - \alpha \quad (z \in U) \\ &\implies \operatorname{Re} \{ e^{i\beta} f^{(p)}(z) \} > 0 \quad (z \in U) \\ &\implies \operatorname{Re} \{ e^{i\beta} f^{(p)}(z) - ip! \sin \beta \} > 0 \quad (z \in U). \end{aligned}$$

Defining the function $w(z)$ by

$$(2.11) \quad e^{i\beta} \frac{f^{(p-1)}(z)}{z} - ip! \sin \beta = p! \cos \beta \cdot \frac{1+w(z)}{1-w(z)} \quad (w(z) \neq 1),$$

we see that $w(z)$ is regular in U with $w(0)=0$. Since $\cos \beta > 0$,

$$(2.12) \quad e^{i\beta} f^{(p-1)}(z) - ip! \sin \beta \cdot z = p! \cos \beta \cdot \frac{1+w(z)}{1-w(z)},$$

and

$$(2.13) \quad e^{i\beta} f^{(p)}(z) - ip! \sin \beta = p! \cos \beta \left\{ \frac{1+w(z)}{1-w(z)} + \frac{2zw'(z)}{(1-w(z))^2} \right\},$$

we have

$$(2.14) \quad \operatorname{Re}\{e^{i\beta} f^{(p)}(z) - ip! \sin\beta\} = p! \cos\beta \cdot \operatorname{Re}\left\{\frac{1+w(z)}{1-w(z)} + \frac{2zw'(z)}{(1-w(z))^2}\right\} > 0.$$

Suppose that there exists a point $z_0 \in U$ such that

$$\max_{|z| \leq |z_0|} |w(z)| = |w(z_0)| = 1 \quad (w(z_0) \neq 1).$$

Then, applying Lemma 1, and letting $w(z_0) = e^{i\theta}$, we obtain

$$(2.15) \quad \begin{aligned} & \operatorname{Re}\{e^{i\beta} f^{(p)}(z_0) - ip! \sin\beta\} \\ &= p! \cos\beta \cdot \operatorname{Re}\left\{\frac{1+e^{i\theta}}{1-e^{i\theta}} + \frac{2ke^{i\theta}}{(1-e^{i\theta})^2}\right\} \\ &= p! \cos\beta \cdot \frac{k}{\cos\theta - 1} \\ &< 0, \end{aligned}$$

which contradicts (2.14). Therefore, $|w(z)| < 1$ for all $z \in U$, which implies that

$$(2.16) \quad \begin{aligned} & \operatorname{Re}\left\{e^{i\beta} \frac{f^{(p-1)}(z)}{z} - ip! \sin\beta\right\} \\ &= \operatorname{Re}\left\{e^{i\beta} \frac{f^{(p-1)}(z)}{z}\right\} \\ &> 0. \end{aligned}$$

Taking $\alpha = 0$ in Theorem 2, we have

COROLLARY 3. *If $f(z)$ is in the class $S_p(\alpha)$, then*

$$(2.17) \quad \operatorname{Re}\left\{\frac{f^{(p-1)}(z)}{z}\right\} > 0 \quad (z \in U).$$

3. A Subclass $F_{p,b}(\alpha)$

Let $G(\alpha)$ be the class of functions $g(z)$ of the form

$$(3.1) \quad g(z) = 1 + \sum_{n=1}^{\infty} g_n z^n$$

which are analytic in U and satisfy

$$(3.2) \quad \operatorname{Re}\{g(z)\} > \alpha \quad (z \in U)$$

for some $\alpha (0 \leq \alpha < 1)$. Further, let $G_b(\alpha)$ be the subclass of $G(\alpha)$ consisting of functions $g(z)$ of the form (3.1) satisfying

$$(3.3) \quad g_1 = 2b(1-\alpha) \equiv g'(0) \quad (0 \leq b \leq 1).$$

For the above class $G_b(\alpha)$, McCarty ([2], [3]) has shown that

LEMMA 2. ([2]). *If $g(z) \in G_b(\alpha)$, then*

$$(3.4) \quad \left| \frac{g'(z)}{g(z)} \right| \leq \frac{2(1-\alpha)}{1-r^2} \left\{ \frac{b+2r+br^2}{1+2b(1-\alpha)r+(1-2\alpha)r^2} \right\} \quad (r=|z|<1).$$

LEMMA 3 ([3]). *If $g(z) \in G_b(\alpha)$, then*

$$(3.5) \quad \operatorname{Re} \left\{ \frac{zg'(z)}{g(z)} \right\} \geq \begin{cases} \frac{-2(1-\alpha)r(b+2r+br^2)}{(1+2abr+(2\alpha-1)r^2)(1+2br+r^2)} & (R' \leq R_b) \\ \frac{2\sqrt{\alpha A_1} - A_1 - \alpha}{1-\alpha} & (R' \geq R_b), \end{cases}$$

where

$$(3.6) \quad R_b = A_b - D_b,$$

$$(3.7) \quad A_b = \frac{(1+br)^2 - (2\alpha-1)(b+r)^2 r^2}{(1+2br+r^2)(1-r^2)},$$

$$(3.8) \quad D_b = \frac{2(1-\alpha)(b+r)(1+br)r}{(1+2br+r^2)(1-r^2)},$$

and

$$(3.9) \quad R' = \sqrt{\alpha A_1}.$$

Let $F_{p,b}(\alpha)$ be the subclass of $R_p(\alpha)$ consisting of functions $f(z) \in R_p(\alpha)$ satisfying

$$(3.10) \quad a_{p+1} = \frac{2b(p!-\alpha)}{(p+1)!} \equiv \frac{f^{(p+1)}(0)}{(p+1)!},$$

where $0 \leq \alpha < p!$ and $0 \leq b \leq 1$.

Now, we have

THEOREM 3. *If $f(z) \in F_{p,b}(\alpha)$, then*

$$(3.11) \quad \left| \frac{zf^{(p+1)}(z)}{f^{(p)}(z)} \right| \leq \frac{2(p!-\alpha)r}{1-r^2} \left\{ \frac{b+2r+br^2}{p!+2b(p!-\alpha)r+(p!-2\alpha)r^2} \right\} \\ (r=|z|<1).$$

PROOF. Note that $f(z) \in F_{p,b}(\alpha)$ implies

$$(3.12) \quad \frac{f^{(p)}(z)}{p!} = 1 + 2b \left(1 - \frac{\alpha}{p!} \right) z + \dots$$

and $0 \leq b \leq 1, 0 \leq \alpha < p!$. It follows from (3.12) that $f(z) \in F_{p,b}(\alpha)$ if and only if $f^{(p)}(z)/p! \in G_b(\alpha/p!)$. Therefore, (3.11) follows Lemma 2.

Also, using Lemma 3, we have

THEOREM 4. If $f(z) \in \mathbf{F}_{p,b}(\alpha)$, then

$$(3.13) \quad \operatorname{Re} \left\{ \frac{z f^{(p+1)}(z)}{f^{(p)}(z)} \right\} \geq \begin{cases} \frac{-2(p!-\alpha)r(b+2r+br^2)}{(p!+2\alpha br+(2\alpha-p!)r^2)(1+2br+r^2)} & (T' \leq T_b) \\ \frac{2\sqrt{p! \alpha B_1} - p! B_1 - \alpha}{p! - \alpha} & (T' \geq T_b), \end{cases}$$

where

$$(3.14) \quad T_b = B_b - C_b,$$

$$(3.15) \quad B_b = \frac{p!(1+br)^2 - (2\alpha-p!)(b+r)^2 r^2}{p!(1+2br+r^2)(1-r^2)},$$

$$(3.16) \quad C_b = \frac{2(p!-\alpha)(b+r)(1+br)r}{p!(1+2br+r^2)(1-r^2)},$$

and

$$(3.17) \quad T' = \sqrt{\frac{\alpha B_1}{p!}}.$$

4. Generalization of Saitoh's result

Finally, we give the generalization theorem of (1.3) which was recently proved by Saitoh [6].

THEOREM 5. If $f(z) \in \mathbf{A}_p$ satisfies

$$(4.1) \quad \operatorname{Re} \left\{ \frac{f^{(j)}(z)}{z^{p-j}} \right\} > \alpha \quad (z \in U)$$

for some $\alpha (0 \leq \alpha < p!/(p-j)!)$, then

$$(4.2) \quad \frac{\int_0^z \frac{f^{(j)}(t)}{t^{p-j}} dt}{z} < 2\alpha - q - \frac{2(q-\alpha)}{z} \ln(1-z),$$

where $1 \leq j \leq p$ and $q = p!/(p-j)!$.

PROOF. Let define $F(z)$ by

$$(4.3) \quad F'(z) = \frac{f^{(j)}(z)}{qz^{p-j}} = 1 + c_1 z + c_2 z^2 + \dots$$

It is easy to see that

$$(4.4) \quad \operatorname{Re}(F'(z)) > \beta \quad \left(\beta = \frac{\alpha}{q}, 0 \leq \beta < 1 \right)$$

and

$$(4.5) \quad F(z) = \frac{1}{q} \int_0^z \frac{f^{(j)}(t)}{t^{p-j}} dt.$$

Therefore, applying the result by Owa, Ma and Liu [5], we have

$$(4.6) \quad \frac{\int_0^z \frac{f^{(j)}(t)}{t^{p-j}} dt}{qz} < 2\beta - 1 - \frac{2(1-\beta)}{z} \ln(1-z),$$

or

$$(4.7) \quad \frac{\int_0^z \frac{f^{(j)}(t)}{t^{p-j}} dt}{z} < 2\alpha - q - \frac{2(q-\alpha)}{z} \ln(1-z).$$

REMARK.

(i) Letting $j=p$ in Theorem 5, we have the result (1.3) by Saitoh [6].

(ii) Letting $j=p=1$ in Theorem 5, we have the result by Owa, Ma and Liu [5].

References

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