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TOTALLY GEODESIC SUBMANIFOLDS OF SYMMETRIC-LIKE RIEMANNIAN MANIFOLDS

By

J. BERNDT, F. PRÜFER and L. VANHECKE

Abstract. We study for various kinds of geometric structures on Riemannian manifolds whether these structures induce on totally geodesic submanifolds structures of the same kind.

1. Introduction

A submanifold M of a Riemannian manifold $\overline{M}$ is said to be totally geodesic if every geodesic in M is also a geodesic in $\overline{M}$. This is equivalent to saying that the second fundamental form of M vanishes. Suppose $\overline{M}$ is equipped with some special geometric structure. A natural question is whether this structure induces a structure of the same kind on its totally geodesic submanifolds. For instance, when $\overline{M}$ is a Riemannian locally symmetric space, then any totally geodesic submanifold of it is also locally symmetric for the induced Riemannian structure. In this paper we study this question for various classes of Riemannian manifolds which are natural generalizations of locally symmetric spaces. In Section 2 we give a summary of their definitions. In Section 3 we show that for most of these classes the totally geodesic submanifolds inherit a similar structure, but in Section 4 we use generalized Heisenberg groups to prove that this is not the case for naturally reductive homogeneous spaces.

2. Symmetric-like Riemannian manifolds

We start by summarizing the definitions of the various Riemannian manifolds which we call *symmetric-like*. A more thorough account and many further references can be found in [1] and [6].

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Let M be an n-dimensional, connected, smooth manifold of dimension $n \geq 2$ and equipped with some Riemannian metric g. We denote its Levi Civita connection by ∇ and its Riemannian curvature tensor by R, where we use the convention $R_{XY} = \nabla_{[X,Y]} - [\nabla_X, \nabla_Y]$. Further, $I(M)$ denotes the isometry group of M and $I^0(M)$ its identity component. The tangent bundle of M is denoted by TM, the tangent space of M at some point $m \in M$ by T_mM. If γ is a geodesic in M, the associated (Riemannian) Jacobi operator R_γ is the self-adjoint tensor field along γ defined by $R_\gamma := R(\dot{\gamma}, \cdot)\dot{\gamma}$. For any $v \in TM$ the (Riemannian) Jacobi operator R_v with respect to v is the self-adjoint endomorphism $R_v := R(v, \cdot)v$ on T_mM. Let $\exp_m : T_mM \to M$ be the exponential map of M at m. At least locally we have a well-defined smooth map given for every unit vector $\xi \in T_mM$ by

$$s_m : p = \exp_m(t\xi) \mapsto s_m(p) = \exp_m(-t\xi)$$

which is called a local geodesic symmetry of M at m. We shall also work with normal coordinates $p \mapsto (x^1(p), \ldots, x^n(p))$ centered at $m \in M$. The Riemannian metric g in such coordinates is given by the matrix-valued map $p \mapsto (g_{ij}(p))$, and we have the normal volume density function

$$\omega_m(p) = (\det(g_{ij}))^{1/2}(p)$$

defined on a normal coordinate neighborhood. We denote by $\mu_k(m,p)$ the k-th elementary symmetric function of the characteristic polynomial of the symmetric matrix $(g_{ij}(p))^{-1}$. Each μ_k is a symmetric two-point function [11].

(1) A Riemannian manifold M is said to be a *k-D'Atri space* if for any $m \in M$ the function $\mu_k(m,p)$ is left-centrally symmetric, that is,

$$\mu_k(m, \exp_m(t\xi)) = \mu_k(m, \exp_m(-t\xi)).$$

Note that a Riemannian manifold is a 1-D'Atri space if and only if it is a D'Atri space [9]. A Riemannian manifold is said to be a *D'Atri space* if its local geodesic symmetries are volume-preserving up to sign, that is, $\mu_n(m,p)$ is left-centrally symmetric.

(2) A Riemannian manifold M is said to be a *k-harmonic space* if for any $m \in M$ the symmetric two-point function $\mu_k(m,p)$ is radial in its first variable (and hence also in its second variable). M is *harmonic* in the usual sense if $\mu_n(m,p)$ is radial. Note that a Riemannian manifold is 1-harmonic if and only if it is harmonic [10]. A necessary and sufficient condition for a Riemannian manifold to be harmonic is that each small geodesic sphere in the manifold has constant mean curvature.

(3) A Riemannian homogeneous space M is said to be *naturally reductive* if there exists a connected Lie subgroup G of $I(M)$ acting transitively on M and a reductive decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ of the Lie algebra $\mathfrak{g}$ of G, where $\mathfrak{h}$ is the Lie algebra of the isotropy group H under the action of G at some point in M, such that every geodesic in M is the orbit of a one-parameter subgroup of $I^0(M)$ generated by some $X \in \mathfrak{m}$.

(4) A Riemannian manifold M is said to be a *g.o. space* if every geodesic in M is the orbit of a one-parameter group of isometries.

(5) A Riemannian homogeneous space M is said to be a *commutative space* if the algebra of all $I(M)$-invariant differential operators on M is commutative.

(6) A Riemannian manifold M is said to be a $\mathfrak{C}$-*space* if for every geodesic γ in M the eigenvalues of the associated Jacobi operator R_γ are constant. This is equivalent to saying that for any geodesic γ in M there exists a skew-symmetric tensor field T_γ along γ such that $R'_\gamma = [R_\gamma, T_\gamma]$, where the prime denotes covariant differentiation of R_γ with respect to $\dot{\gamma}$. If for any geodesic γ in M there exists a parallel T_γ with that property, then M is said to be a $\mathfrak{C}_0$-*space*.

(7) A Riemannian manifold M is called a $\mathfrak{P}$-*space* if for any geodesic γ in M the associated Jacobi operator R_γ is diagonalizable by a parallel orthonormal frame field along γ.

(8) A Riemannian manifold is called an $\mathfrak{SC}$-*space* if for each small geodesic sphere in it the principal curvatures at antipodal points coincide.

(9) A Riemannian manifold M is called a $\mathfrak{TC}$-*space* if for any two small geodesic spheres in M with the same radii and touching each other at some point $m \in M$ the principal curvatures of these two spheres coincide at m.

(10) A Riemannian manifold M with the property that the eigenvalues of the Jacobi operator R_v do not depend on the choice of the unit vector $v \in TM$ is called an *Osserman space*. If these eigenvalues do not depend on the choice of the unit vector $v \in T_mM$ for any $m \in M$, but may vary with the point m, then M is called a *pointwise Osserman space*.

3. Totally geodesic submanifolds

Next, we investigate whether totally geodesic submanifolds of the symmetric-like Riemannian manifolds as defined in Section 2 admit corresponding properties. In the following, M is a totally geodesic submanifold of a Riemannian manifold $(\tilde{M}, \tilde{g})$. The induced Riemannian metric on M will be denoted by g. Let $(x^1, \ldots, x^{\tilde{n}})$ be normal coordinates centered at some point $m \in \tilde{M}$. We call these normal coordinates *adapted* if $\partial/\partial x^i(m)$ $(i = 1, \ldots, n)$ is tangent to M at m and

$\partial/\partial x^a(m)$ $(a = n+1, \ldots, \tilde{n})$ is perpendicular to M at m. We will denote by $\alpha, \beta \in \{1, \ldots, \tilde{n}\}$ tangential indices with respect to $\tilde{M}$, by $i, j \in \{1, \ldots, n\}$ tangential indices with respect to M, and by $a, b \in \{n+1, \ldots, \tilde{n}\}$ normal indices with respect to M. We start with the following useful lemma.

LEMMA 1. *With respect to an adapted normal coordinate system centered at $m \in M$ we have for any $p \in M$ lying in the coordinate neighborhood the following representation of the metric tensor of $\tilde{M}$:*

$$(\tilde{g}_{\alpha\beta}(p)) = \begin{pmatrix} \tilde{g}_{ij}(p) & 0 \\ 0 & \tilde{g}_{ab}(p) \end{pmatrix} = \begin{pmatrix} g_{ij}(p) & 0 \\ 0 & g_{ab}(p) \end{pmatrix}.$$

PROOF. Let γ be a geodesic in M parametrized by arc length and with $\gamma(0) = m$. We consider the Jacobi vector fields $\tilde{Y}_i : t \mapsto t\partial/\partial x^i(\gamma(t))$ and $\tilde{Y}_a : t \mapsto t\partial/\partial x^a(\gamma(t))$ along γ in $\tilde{M}$. Since M is totally geodesic in $\tilde{M}$ and the initial values of $\tilde{Y}_i$ at 0 are tangent to M, this Jacobi vector field is the variational vector field of a geodesic variation in M. Thus $\tilde{Y}_i(t)$ is tangent to M for each t. Further, since M is totally geodesic in $\tilde{M}$, the Gauss equation implies that the tangential component $\tilde{Y}_a^T$ of $\tilde{Y}_a$ is a Jacobi vector field in M. But $\tilde{Y}_a^T$ has initial values $\tilde{Y}_a^T(0) = 0$ and $(\tilde{Y}_a^T)'(0) = 0$. This implies that $\tilde{Y}_a^T$ vanishes and hence $\tilde{Y}_a(t)$ is normal to M for each t. This gives $g_{ia}(\gamma(t)) = (1/t^2)g(\tilde{Y}_i(t), \tilde{Y}_a(t)) = 0$ for each t, and the lemma follows.

We now come to the main result of this section.

THEOREM 1. *Let M be a connected totally geodesic submanifold of $\tilde{M}$.*

(i) *If $\tilde{M}$ is a k-D'Atri space (resp. a k-harmonic space) for all $k = 1, \ldots, \tilde{n}$, then M is a k-D'Atri space (resp. a k-harmonic space) for all $k = 1, \ldots, n$.*

(ii) *If $\tilde{M}$ is a $\mathfrak{C}$-space, a $\mathfrak{C}_0$-space, a $\mathfrak{P}$-space, an $\mathfrak{SC}$-space, a $\mathfrak{IC}$-space, an Osserman space, or a pointwise Osserman space, respectively, then M belongs to the same class as $\tilde{M}$.*

(iii) *If $\tilde{M}$ is a g.o. space or a commutative space, then M is a k-D'Atri space for each $k = 1, \ldots, n$.*

(iv) *If $\tilde{M}$ is a g.o. space and M is complete, then M is also a g.o. space.*

PROOF. (i) Let $\tilde{M}$ be a k-D'Atri space (resp. a k-harmonic space) for all $k = 1, \ldots, \tilde{n}$. Then Lemma 1 shows that the same holds for M.

(ii) First, suppose $\tilde{M}$ is a $\mathfrak{C}$-space and let γ be a geodesic in M. Then there exists a skew-symmetric tensor field $\tilde{T}_\gamma$ along γ in $\tilde{M}$ such that $\tilde{R}_\gamma' = [\tilde{R}_\gamma, \tilde{T}_\gamma]$.

Since M is totally geodesic in $\tilde{M}$, restriction and orthogonal projection of $\tilde{T}_\gamma$ to the tangent spaces of M along γ gives a skew-symmetric tensor field T_γ along γ in M with $R'_\gamma = [R_\gamma, T_\gamma]$. This shows that M is also a $\mathfrak{C}$-space. When $\tilde{T}_\gamma$ is $\tilde{\nabla}$-parallel, then T_γ is ∇-parallel, which gives the corresponding statement for $\mathfrak{C}_0$-spaces. Next, suppose $\tilde{M}$ is a $\mathfrak{P}$-space and γ a geodesic in M. As M is totally geodesic in $\tilde{M}$, the associated Jacobi operator $\tilde{R}_\gamma$ leaves the tangent space of M at any point on γ invariant. Since the tangent bundle of M restricted to γ is $\tilde{\nabla}$-parallel, it follows that $\tilde{R}_\gamma$ can be diagonalized by a parallel orthonormal frame field along γ whose first n elements are tangent to M everywhere. These n parallel vector fields diagonalize the Jacobi operator R_γ in M, and it follows that M is also a $\mathfrak{P}$-space. Since M is totally geodesic in $\tilde{M}$, the Weingarten equation implies that any principal curvature of a small geodesic sphere in M is also a principal curvature of the corresponding geodesic sphere in $\tilde{M}$. This implies the statement for $\mathfrak{SC}$- and $\mathfrak{IC}$-spaces. Finally, the statement for Osserman spaces and pointwise Osserman spaces follows from the fact that each eigenvalue of $R_v, v \in TM$, is also an eigenvalue of $\tilde{R}_v$, because of the Gauss equation and since M is totally geodesic in $\tilde{M}$.

(iii) This follows from (i) and the fact that every g.o. space [8] and every commutative space [7] is a k-D'Atri space for all k.

(iv) Let γ be a maximal geodesic in M. As $\tilde{M}$ is a g.o. space, there exists a one-parameter group of isometries of $\tilde{M}$ having γ as an orbit. This one-parameter group generates a Killing vector field $\tilde{X}$ on $\tilde{M}$. The restriction and orthogonal projection of $\tilde{X}$ to M gives a Killing vector field X on M, and by construction γ is an orbit of the one-parameter group of isometries of M determined by X.

REMARK. Below we will provide an example of a complete totally geodesic submanifold of a naturally reductive space which is not naturally reductive. Nevertheless, the following questions remain open:

1. Is any complete totally geodesic submanifold of a commutative space also commutative?

2. Is, for a fixed k, any totally geodesic submanifold of a k-D'Atri space (resp. a k-harmonic space) also a k-D'Atri space (resp. a k-harmonic space)?

4. Totally geodesic subgroups of generalized Heisenberg groups

In this section we investigate totally geodesic submanifolds of generalized Heisenberg groups. As a consequence we get the example mentioned in the previous remark. A thorough treatment of generalized Heisenberg groups can be

found in [1], from which we also take several facts without providing the proofs here.

Let $\mathfrak{v}$ and $\mathfrak{z}$ be real vector spaces with finite dimensions n and m, respectively, and $\beta : \mathfrak{v} \times \mathfrak{v} \to \mathfrak{z}$ a skew-symmetric bilinear map. Then the direct sum $\mathfrak{n} = \mathfrak{v} \oplus \mathfrak{z}$ becomes a 2-step nilpotent Lie algebra with m-dimensional center $\mathfrak{z}$ by means of

$$[U + X, V + Y] = \beta(U, V) \quad (U, V \in \mathfrak{v}, X, Y \in \mathfrak{z}).$$

We choose some inner product $\langle \cdot, \cdot \rangle$ on $\mathfrak{n}$ such that $\mathfrak{v}$ and $\mathfrak{z}$ are perpendicular and define a homomorphism

$$J : \mathfrak{z} \to \text{End}(\mathfrak{v}), Z \mapsto J_Z$$

by

$$\langle J_Z U, V \rangle = \langle [U, V], Z \rangle \quad (U, V \in \mathfrak{v}, Z \in \mathfrak{z}).$$

Then $\mathfrak{n}$ is said to be a *generalized Heisenberg algebra* if

$$J_Z^2 = -|Z|^2 \,\text{id} \quad (Z \in \mathfrak{z}).$$

The associated connected, simply connected 2-step nilpotent Lie group N equipped with the induced left-invariant Riemannian metric g is called a *generalized Heisenberg group*. For $m = 1$ these are precisely the classical Heisenberg algebras and groups.

The classification of generalized Heisenberg algebras and groups can be obtained by means of the classification of finite-dimensional real representations of Clifford algebras over negative definite real quadratic spaces. Denote by $Cl(\mathfrak{z}, q)$ the real Clifford algebra of the real quadratic space $(\mathfrak{z}, q)$, where q is the negative of the quadratic form associated to the inner product on $\mathfrak{z}$. If $m \not\equiv 3 \pmod 4$, then there exists (up to equivalence) precisely one irreducible real Clifford module ϑ. The $\mathfrak{v}$ is isomorphic to $\oplus^k \vartheta$ for some positive integer k. If $m \equiv 3 \pmod 4$, then there exist (up to equivalence) precisely two non-equivalent irreducible real Clifford modules ϑ_1, ϑ_2 over the Clifford algebra $Cl(\mathfrak{z}, q)$. The modules ϑ_1, ϑ_2 have the same dimension and $\mathfrak{v}$ is isomorphic to $(\oplus^{k_1} \vartheta_1) \oplus (\oplus^{k_2} \vartheta_2)$ for some non-negative integers k_1, k_2. Any two pairs (k_1, k_2) and $(\tilde{k}_1, \tilde{k}_2)$ of non-negative integers with $k_1 + k_2 = \tilde{k}_1 + \tilde{k}_2$ yield generalized Heisenberg algebras $\mathfrak{n}(k_1, k_2)$ and $\mathfrak{n}(\tilde{k}_1, \tilde{k}_2)$ of the same dimension. These are isomorphic if and only if $(\tilde{k}_1, \tilde{k}_2) \in \{(k_1, k_2), (k_2, k_1)\}$, that is, the corresponding generalized Heisenberg groups, $N(k_1, k_2)$ and $N(\tilde{k}_1 . \tilde{k}_2)$ are isometric if and only if $(\tilde{k}_1, \tilde{k}_2) \in \{(k_1, k_2), (k_2, k_1)\}$.

Viewing elements in the Lie algebra $\mathfrak{n}$ as left-invariant vector fields on the Lie groups N, the Levi Civita connection ∇ is determined by

$$\nabla_{V+Y}(U+X) = -\frac{1}{2}J_X V - \frac{1}{2}J_Y U - \frac{1}{2}[U,V]$$

for all $U, V \in \mathfrak{v}$ and $X, Y \in \mathfrak{z}$.

We will now study totally geodesic submanifolds of generalized Heisenberg groups. For general investigations about totally geodesic submanifolds of 2-step nilpotent Lie groups we refer to [4].

PROPOSITION 1. *Let N, $\tilde{N}$ be generalized Heisenberg groups and suppose N is a totally geodesic submanifold of $\tilde{N}$. Let $\mathfrak{n} = \mathfrak{v} \oplus \mathfrak{z}$ and $\tilde{\mathfrak{n}} = \tilde{\mathfrak{v}} \oplus \tilde{\mathfrak{z}}$ be the associated decomposition of the Lie algebra of N and $\tilde{N}$, respectively. Then we have $\mathfrak{v} \subset \tilde{\mathfrak{v}}$ and $\mathfrak{z} \subset \tilde{\mathfrak{z}}$.*

PROOF. The basic idea for the proof is as follows. Since N is totally geodesic in $\tilde{N}$, the spectrum and the eigenspaces of the Jacobi operator R_ξ must be contained in the spectrum and the eigenspaces of the Jacobi operator $\tilde{R}_\xi$ for each unit vector $\xi \in TN$. The spectra of these Jacobi operators and the corresponding eigenspaces have been computed explicitly in [1, p. 36–38]. Comparing these data, we get the result.

Let $V \in \mathfrak{v}$ be a unit vector. We have to show that $V \in \tilde{\mathfrak{v}}$. First we note that R_V has three distinct eigenvalues 0, $-3/4$ and $1/4$. We decompose V into $V = \tilde{V} + \tilde{Y}$ with $\tilde{V} \in \tilde{\mathfrak{v}}$ and $\tilde{Y} \in \tilde{\mathfrak{z}}$. If $\tilde{Y} = 0$ we are done. Thus we assume $\tilde{Y} \neq 0$. If $\tilde{V} = 0$, then $\tilde{R}_{\tilde{Y}}$ has just the two eigenvalues 0 and $1/4$, but not $-3/4$. Therefore we must have $\tilde{V} \neq 0$ and $\tilde{Y} \neq 0$. We decompose $\tilde{\mathfrak{n}}$ into

$$\tilde{\mathfrak{n}} = \tilde{\mathfrak{n}}_3 + \tilde{\mathfrak{p}} + \tilde{\mathfrak{q}}$$

with

$$\tilde{\mathfrak{n}}_3 := \operatorname{span}\{\tilde{V}, J_{\tilde{Y}}\tilde{V}, \tilde{Y}\},$$

$$\tilde{\mathfrak{p}} := \ker \operatorname{ad}(\tilde{V}) \cap \ker \operatorname{ad}(J_{\tilde{Y}}\tilde{V}),$$

$$\tilde{\mathfrak{q}} := \operatorname{span}\{\tilde{Y}^\perp, J_{\tilde{Y}^\perp}\tilde{V}, J_{\tilde{Y}^\perp}J_{\tilde{Y}}\tilde{V}\}.$$

Each of these three subspaces is invariant under the action of $\tilde{R}_{\tilde{V}+\tilde{Y}}$. The endomorphism $\tilde{R}_{\tilde{V}+\tilde{Y}}|\tilde{\mathfrak{n}}_3$ has the eigenvalues 0 and $1/4$ (if $|\tilde{V}|^2 = 1/4$) or 0, $1/4$ and $1/4 - |\tilde{V}|^2$ (if $|\tilde{V}|^2 \neq 1/4$); the endomorphism $\tilde{R}_{\tilde{V}+\tilde{Y}}|\tilde{\mathfrak{p}}$ (if $\tilde{\mathfrak{p}} \neq \{0\}$) has only one eigenvalue $(1 - |\tilde{V}|^2)/4 = |\tilde{Y}|^2/4$. The situation is more complicated for

$\tilde{R}_{\tilde{V}+\tilde{Y}}|\tilde{\mathfrak{q}}$ (if $\tilde{\mathfrak{q}} \neq \{0\}$). We put $\hat{V} = \tilde{V}/|\tilde{V}|$, $\hat{Y} = \tilde{Y}/|\tilde{Y}|$, and consider the skew-symmetric endomorphism

$$\tilde{K}_{\tilde{V},\tilde{Y}} : \tilde{Y}^{\perp} \to \tilde{Y}^{\perp}, \tilde{X} \mapsto [\hat{V}, \tilde{J}_{\tilde{X}}\tilde{J}_{\tilde{Y}}\hat{V}].$$

Using the abbreviation $\tilde{K} := \tilde{K}_{\tilde{V},\tilde{Y}}$, we have an orthogonal decomposition of $\tilde{Y}^{\perp}$ into

$$\tilde{Y}^{\perp} = \tilde{L}_0 \oplus \cdots \oplus \tilde{L}_k$$

where $\tilde{L}_j := \ker(\tilde{K}^2 - \tilde{\mu}_j \operatorname{id}_{\tilde{Y}^{\perp}})$ $(j = 0, 1, \ldots, k)$ and $0 \geq \tilde{\mu}_0 > \tilde{\mu}_1 > \cdots > \tilde{\mu}_k \geq -1$ are the distinct eigenvalues of $\tilde{K}^2$. We define

$$\tilde{\mathfrak{q}}_j : \operatorname{span}\{\tilde{L}_j, \tilde{J}_{\tilde{L}_j}\tilde{V}, \tilde{J}_{\tilde{L}_j}\tilde{J}_{\tilde{Y}}\tilde{V}\}, \quad j = 0, \ldots, k, \quad \tilde{\mu}_k \neq -1,$$

$$\tilde{\mathfrak{q}}_k := \operatorname{span}\{\tilde{L}_k, \tilde{J}_{\tilde{L}_k}\tilde{V}\}, \text{ if } \tilde{\mu}_k = -1.$$

Then $\tilde{\mathfrak{q}} = \tilde{\mathfrak{q}}_0 \oplus \cdots \oplus \tilde{\mathfrak{q}}_k$ and each space $\tilde{\mathfrak{q}}_j$ is invariant under the action of $\tilde{R}_{\tilde{V}+\tilde{Y}}$ with

$$\dim \tilde{\mathfrak{q}}_j \equiv \begin{cases} 0(\mathrm{mod}3) & \text{if } \tilde{\mu}_j = 0 \\ 0(\mathrm{mod}4) & \text{if } \tilde{\mu}_j = -1 \\ 0(\mathrm{mod}6) & \text{otherwise.} \end{cases}$$

Finally, we put

$$\tilde{\rho}_1 = \frac{1}{4} - |\tilde{V}|^2,$$

$$\tilde{\rho}_2 = \frac{1}{8}(1 + \sqrt{1 + 32|\tilde{V}|^2|\tilde{Y}|^2}),$$

$$\tilde{\rho}_3 = \frac{1}{8}(1 - \sqrt{1 + 32|\tilde{V}|^2|\tilde{Y}|^2}).$$

Now, if $j = k$ and $\tilde{\mu}_k = -1$, then $\tilde{R}_{\tilde{V}+\tilde{Y}}|\tilde{\mathfrak{q}}_k$ has two different eigenvalues $\tilde{\kappa}_{k1}$ and $\tilde{\kappa}_{k2}$ which are thee solutions of the quadratic equation

$$\left(\tilde{\rho} - \frac{1}{4}|\tilde{V}|^2\right)(\tilde{\rho} - \tilde{\rho}_1) = \frac{9}{16}|\tilde{V}|^2|\tilde{Y}|^2.$$

Otherwise $\tilde{R}_{\tilde{V}+\tilde{Y}}|\tilde{\mathfrak{q}}_j$ has three distinct eigenvalues $\tilde{\kappa}_{j1}$, $\tilde{\kappa}_{j2}$, $\tilde{\kappa}_{j3}$ which are the solutions of the third order equation

$$(\tilde{\rho} - \tilde{\rho}_1)(\tilde{\rho} - \tilde{\rho}_2)(\tilde{\rho} - \tilde{\rho}_3) = \frac{27}{64}|\tilde{V}|^4|\tilde{Y}|^2\tilde{\mu}_j.$$

The substitution of $\tilde{\rho} = -3/4$ and the explicit expressions for $\tilde{\rho}_1$, $\tilde{\rho}_2$, $\tilde{\rho}_3$ into the previous third order equation gives

$$(t-1)\left(\frac{3}{4} - \frac{1}{2}t + \frac{1}{2}t^2\right) = \frac{27}{64}t^2(1-t)s,$$

where $t := |\tilde{V}|^2$ and $s := \tilde{\mu}_j$. Since $t \neq 0$ and $t \neq 1$, this implies

$$s = \frac{32t^2 - 32t + 48}{-27t^2}.$$

Regarding s as a function in the variable t, we get for the derivative $s'(t) = 32(3-t)/27t^3 > 0$ for $0 < t < 1$, whence the function $s(t)$ is strictly monotone increasing for $0 < t \leq 1$. Since $s(1) = -16/9 < -1$, this shows $s(t) < -1$ for $0 < t < 1$, in contrast to $-1 \leq s = \tilde{\mu}_j \leq 0$. This shows that $-3/4$ is not an eigenvalue of $\tilde{R}_{\tilde{V}+\tilde{Y}}|\tilde{\mathfrak{q}}_j$. Now we consider the case $\tilde{\mu}_k = -1$. Then the above quadratic equation gives

$$\left(\frac{1}{4}t + \frac{3}{4}\right)(1-t) = \frac{9}{16}t(1-t)$$

with $t := |\tilde{V}|^2$ and $\tilde{\rho} := -3/4$. However, this is possible only for $t = 1$ or $t = 12/5$, in contradiction to $0 < t = |\tilde{V}|^2 < 1$. Thus $-3/4$ is not an eigenvalue of $\tilde{R}_{\tilde{V}+\tilde{Y}}|\tilde{\mathfrak{q}}_k$. The eigenvalue of $\tilde{R}_{\tilde{V}+\tilde{Y}}|\tilde{\mathfrak{p}}$ is positive, so $-3/4$ is not an eigenvalue of $\tilde{R}_{\tilde{V}+\tilde{Y}}|\tilde{\mathfrak{p}}$. Eventually, we know that $\tilde{R}_{\tilde{V}+\tilde{Y}}|\tilde{\mathfrak{n}}_3$ has only the eigenvalues 0, 1/4 and $1/4 - |\tilde{V}|^2$. Because of $|\tilde{V}|^2 < 1$, $-3/4$ cannot be an eigenvalue of $\tilde{R}_{\tilde{V}+\tilde{Y}}|\tilde{\mathfrak{n}}_3$. Altogether we now see that $-3/4$ is not an eigenvalue of $\tilde{R}_{\tilde{V}+\tilde{Y}}$. On the other hand, R_V has the eigenvalue $-3/4$, which gives a contradiction. Thus $\tilde{Y} \neq 0$ is not possible and so $V = \tilde{V} \in \tilde{\mathfrak{v}}$. Thus we have now proved that $\mathfrak{v} \subset \tilde{\mathfrak{v}}$.

Now let $Y \in \mathfrak{z}$ be a unit vector. We have to show that $Y \in \tilde{\mathfrak{z}}$. The Jacobi operator R_Y has two different eigenvalues 0 and 1/4 with corresponding eigenspaces $\mathfrak{z}$ and $\mathfrak{v}$, respectively. We decompose Y into $Y = \tilde{V} + \tilde{Y}$ with $\tilde{V} \in \tilde{\mathfrak{v}}$ and $\tilde{Y} \in \tilde{\mathfrak{z}}$. If $\tilde{V} = 0$ we are done. Thus we assume $\tilde{V} \neq 0$. If $\tilde{Y} = 0$, then $\tilde{R}_{\tilde{V}}$ has the eigenvalues 0, $-3/4$ and 1/4, and the eigenspace corresponding to 1/4 is $\tilde{\mathfrak{z}}$. But this implies $\mathfrak{v} \subset \tilde{\mathfrak{z}}$, in contradiction to $\mathfrak{v} \subset \tilde{\mathfrak{v}}$ which was established above. Therefore we must have $\tilde{Y} \neq 0$. Since 1/4 is not an eigenvalue of $\tilde{R}_{\tilde{V}+\tilde{Y}}|\tilde{\mathfrak{p}}$ and of $\tilde{R}_{\tilde{V}+\tilde{Y}}|\tilde{\mathfrak{q}}$, we have to consider only $\tilde{R}_{\tilde{V}+\tilde{Y}}|\tilde{\mathfrak{n}}_3$. The eigenspace of 1/4 is spanned by $-|\tilde{Y}|^2\tilde{V} + |\tilde{V}|^2\tilde{Y}$, which gives a contradiction to $\mathfrak{v} \subset \tilde{\mathfrak{v}}$. Therefore, 1/4 is not an eigenvalue of $\tilde{R}_{\tilde{V}+\tilde{Y}}$, which is another contradiction. Consequently, we must have $\tilde{V} = 0$ and it follows that $\mathfrak{z} \subset \tilde{\mathfrak{z}}$. This finishes the proof of Proposition 1.

REMARK. Proposition 1 says that any generalized Heisenberg group N which is totally geodesically embedded in some other generalized Heisenberg group $\tilde{N}$ is *well-positioned* in the sense of [4]. This means that $\mathfrak{n} = (\mathfrak{n} \cap \tilde{\mathfrak{v}}) \oplus (\mathfrak{n} \cap \tilde{\mathfrak{z}})$, $\mathfrak{v} = \mathfrak{n} \cap \tilde{\mathfrak{v}}$ and $\mathfrak{z} = \mathfrak{n} \cap \tilde{\mathfrak{z}}$.

A totally geodesic Lie subgroup N of a Lie group $\tilde{N}$ is a Lie subgroup N which is embedded totally geodesically in $\tilde{N}$. A Lie subalgebra $\mathfrak{n}$ of the Lie algebra $\tilde{\mathfrak{n}}$ of $\tilde{N}$ is said to be totally geodesic if $\tilde{\nabla}_X Y \in \mathfrak{n}$ for all $X, Y \in \mathfrak{n}$. There is an obvious one-to-one correspondence between totally geodesic Lie subalgebra of $\tilde{\mathfrak{n}}$ and connected totally geodesic Lie subgroups of $\tilde{N}$.

The lowest dimension for a generalized Heisenberg group with 2-dimensional center is 6. We will now show that this 6-dimensional generalized Heisenberg group can be embedded totally geodesically into any generalized Heisenberg group not satisfying the J^2-condition. For this we must first recall the meaning of the J^2-condition, which has been formulated first in [2]. A generalized Heisenberg group satisfies the J^2-condition if any only if for all $X, Y \in \mathfrak{z}$ with $\langle X, Y \rangle = 0$ and all non-zero vectors $U \in \mathfrak{v}$ there exists some vector $Z \in \mathfrak{z}$ so that $J_X J_Y U = J_Z U$, that is, so that $J_X J_Y U \in \ker \operatorname{ad}(U)^{\perp}$. It was shown in [2] that a generalized Heisenberg group satisfies the J^2-condition if and only if it is isomorphic to the nilpotent part in the Iwasawa decomposition of the identity component of the isometry group of a non-compact rank-one symmetric space. More precisely, a generalized Heisenberg group satisfies the J^2-condition if and only if $m = 1$ (this corresponds to complex hyperbolic space), $m = 3$ and $\mathfrak{v}$ is an isotypic module (this corresponds to quaternionic hyperbolic space), or $m = 7$ and $\mathfrak{v}$ is an irreducible module (this corresponds to Cayley hyperbolic plane).

THEOREM 2. *Let $\tilde{N}$ be a generalized Heisenberg group which does not satisfy the J^2-condition. Then the 6-dimensional generalized Heisenberg group N with 2-dimensional center can be embedded totally geodesically into $\tilde{N}$.*

PROOF. As $\tilde{N}$ does not satisfy the J^2-condition, the solvable extension of $\tilde{N}$ known as a Damek-Ricci space has zero sectional curvature for some suitable 2-plane [3]. Such a 2-plane exists if and only if there exists a unit vector $V + Y \in \tilde{\mathfrak{n}}$ with $|V|^2 = 2/3$ and a non-zero vector $x \in Y^{\perp}$ so that $J_X J_Y V$ is orthogonal to $J_{\tilde{\mathfrak{z}} V}$ [1, p. 104]. A straightforward calculation shows that V, $J_X V$, $J_Y V$, $J_X J_Y V$, X, Y are orthogonal and span a 6-dimensional totally geodesic Lie subalgebra $\mathfrak{n}$ of $\tilde{\mathfrak{n}}$ isomorphic to a 6-dimensional generalized Heisenberg algebra with 2-dimensional center. The corresponding totally geodesic Lie subgroup N of $\tilde{N}$ is a 6-dimensional generalized Heisenberg group with 2-dimensional center.

COROLLARY 1. *A totally geodesic submanifold of a naturally reductive space is not naturally reductive in general.*

PROOF. A generalized Heisenberg group is naturally reductive if and only if $\dim \mathfrak{z} \in \{1,3\}$ [5]. Any generalized Heisenberg group $\tilde{N}$ with 3-dimensional center and arising from a non-isotypic module $\mathfrak{v}$ is therefore naturally reductive and does not satisfy the J^2-condition. The previous theorem implies that one can find a generalized Heisenberg group N with 2-dimensional center embedded totally geodesically in $\tilde{N}$. Since the center of N is 2-dimensional, N is not naturally reductive, and the corollary is proved.

We finish this paper with another existence theorem concerning totally geodesic embeddings of generalized Heisenberg groups with 2-dimensional center into generalized Heisenberg groups with 3-dimensional center.

THEOREM 3. *Let $N = N(k_1,k_2)$ be a generalized Heisenberg group with 3-dimensional center and suppose that $k := \min\{k_1,k_2\} \geq 1$. Then there exists for each $l \in \{1,\dots,k\}$ a $(4l+2)$-dimensional generalized Heisenberg group with 2-dimensional center embedded totally geodesically in N.*

PROOF. We begin with an explicit description of the generalized Heisenberg algebra $\mathfrak{n} = \mathfrak{n}(k_1,k_2)$. Denote by $\boldsymbol{H}$ the algebra of quaternions. We define a linear map

$$\Phi : \boldsymbol{R}^3 \to \boldsymbol{H}, (s,t,u) \mapsto si + tj + uk.$$

For $Z \in \boldsymbol{R}^3$ we consider the automorphism

$$J_Z : \boldsymbol{H}^{k_1+k_2} \to \boldsymbol{H}^{k_1+k_2}$$

given by

$$(W_1,\dots,W_{k_1+k_2}) \mapsto (W_1\Phi(Z),\dots,W_{k_1}\Phi(Z),\Phi(Z)W_{k_1+1},\dots,\Phi(Z)W_{k_1+k_2}).$$

Then we have $J_Z^2 = -|Z|^2\mathrm{id}$ and $\mathfrak{n} = \mathfrak{n}(k_1,k_2) = \boldsymbol{H}^{k_1+k_2} \oplus \boldsymbol{R}^3$. Let X, Y, Z be the standard basis of $\boldsymbol{R}^3$ which corresponds via Φ to i,j,k. Let π_1 (respectively π_2) be the projection of $\mathfrak{v} = \mathfrak{v}_1 \oplus \mathfrak{v}_2 \cong \boldsymbol{H}^{k_1} \oplus \boldsymbol{H}^{k_2}$ onto the first (respectively second) factor and $V_1 \in \mathfrak{v}$ a unit vector with $|\pi_1(V_1)| = |\pi_2(V_1)|$. Then

$$\mathfrak{L}_1 = \mathrm{span}\{X, Y, V_1, J_X V_1, J_Y V_1, J_X J_Y V_1\}$$

is a totally geodesic Lie subalgebra isomorphic to the 6-dimensional generalized Heisenberg algebra with 2-dimensional center. If $k \geq 2$ we may find a unit vector

$V_2 \in \mathfrak{v}$ with $|\pi_1(V_2)| = |\pi_2(V_2)|$ and perpendicular to the 8-dimensional subspace

$$\mathfrak{Q}_1(Z) := (\mathfrak{Q}_1 \cap \mathfrak{v}) \oplus J_Z(\mathfrak{Q}_1 \cap \mathfrak{v})$$

of $\mathfrak{v}$. Then

$$\mathfrak{Q}_2 := \mathfrak{Q}_1 \oplus \operatorname{span}\{V_2, J_X V_2, J_Y V_2, J_X J_Y V_2\}$$

is a totally geodesic Lie subalgebra of $\mathfrak{n}$ isomorphic to the 10-dimensional generalized Heisenberg algebra with 2-dimensional center. In this manner we may construct successively for each $l \in \{1, \ldots, k\}$ a totally geodesic Lie subalgebra of $\mathfrak{n}$ isomorphic to the $(4l+2)$-dimensional generalized Heisenberg algebra with 2-dimensional center. This proves the theorem.

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Universität Köln
Mathematisches Institut
Weyertal 86–90
50931 Köln
Germany

Universität Leipzig
Mathematisches Institut
Augustusplatz 10/11
04109 Leipzig
Germany

Katholieke Universiteit Leuven
Department of Mathematics
Celestijnenlaan 200 B
3001 Leuven
Belgium