

RAYS AND THE FIXED POINT PROPERTY IN NONCOMPACT SPACES

By

Tadeusz DOBROWOLSKI and Witold MARCISZEWSKI*

Abstract. We are concerned with the question of whether a noncompact space with a nice local structure contains a ray, i.e., a closed homeomorph of $[0,1)$. We construct rays in incomplete locally path connected spaces, and also, in noncompact metrizable convex sets; as a consequence these spaces lack the fixed point property. On the other hand, we give an example of a noncompact (nonmetrizable) convex subset C of a locally convex topological vector space E which has the fixed point property.

1. Introduction

The classical Schauder-Tichonoff theorem states that, for a convex subset C of a locally convex topological vector space, the compactness of C implies the fixed point property of C . In [K], V. Klee observed that this implication can be reversed for a large class of topological vector spaces (including Banach spaces). His approach was very elementary; namely, he showed that every convex noncompact subset of a respective topological vector space contains a closed homeomorphic copy of $[0,1)$, called a (topological) ray. By (a little addition to) the Tietze theorem, a ray R in a normal topological space X is a retract of X ; and since $[0,1)$ lacks the fixed point property, so does the space X . (Let us point out that, in a nonmetric case, even if one constructs a ray in a convex set C , then C may not be normal and one cannot conclude that C lacks the fixed point property.) Klee has asked a question of whether an arbitrary noncompact convex subset C of a topological vector space E contains a ray (or lacks the fixed point property). He specified that the case of metrizable E is of some

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interest. In such a case, we obviously can assume that E is a complete metric linear space. Then, either C is not closed in E or C is completely metrizable. If C is not closed in E then, it is easy to construct a ray “through” a sequence $\{x_n\} \subset C$ that converges to a point $x_\infty \in E \setminus C$. If C is completely metrizable, then either C is locally compact or nonlocally compact; the locally compact case has been solved by Klee (see 2.1). The case of infinite-dimensional, completely metrizable, nonlocally compact C is treated in Proposition 3.1; we invoke therein a certain general statement concerning approximation of maps into C , a particular case of which yields the existence of a ray in C .

We also provide an answer to Klee’s question in case of a nonmetric space E . In Example 4.1 we construct a convex, noncompact subset W (of a compact convex set) in a locally convex topological vector space E such that W has the fixed point property (and does not contain a ray). On the other hand, we observe that every convex subset C (in an arbitrary topological vector space) which is not totally bounded must both contain a ray and lack the fixed point property.

It is reasonable to ask a more general question of whether a noncompact metrizable space with a nice local structure contains a ray. In particular, of whether a noncompact absolute retract contains a ray. In general, this is not the case, as classical examples of the “broken comb” and the “hedgehog” spaces show, see [C]. A special case of our Theorem 2.5 states that every absolute retract space X which is either locally compact or not completely metrizable contains a ray. For X which is not completely metrizable, the absolute retract property can be relaxed to the LC^0 -property. It is reasonable to ask the following

1.1. QUESTION. Let X be a completely metrizable absolute retract without the fixed point property. Does X contain a ray?

As far as we know, this question was tackled previously by S. Reich and Y. Sternfeld in [RS] who obtained an affirmative answer for some “hedgehog”-like spaces. More recently, V. Okhezin [O] has obtained some partial answer to this question as well.

Our approach to construct a ray in a noncompact X is very elementary. We simply find a completion \hat{X} of (X, d) , where d is some incomplete admissible metric on X (observe that every noncompact space X admits an incomplete metric d , see [E, 4.3.E(d)]). Having done this, we then construct a Peano continuum $Y \cup \{x_\infty\}$, where Y is a nonempty subset of X and $x_\infty \in \hat{X} \setminus X$. Next, there exists an arc $a : [0, 1] \rightarrow Y \cup \{x_\infty\}$ joining an element $y \in Y$ with x_∞ . The

restriction of a to $[0,1)$ gives us a required closed embedding of $[0,1)$ into X . Observe that such an embedding will be uniformly continuous with respect to the natural metrics on $[0,1)$ and d . This however does not bear any restriction because given any closed embedding $p : [0,1) \rightarrow X$, we can apply the Hausdorff metric extension theorem (see [E. p. 369]) to find a metric d on X , so that p will be an isometric embedding. It is clear now that p will extend to an embedding of $[0,1]$ into \hat{X} . Summarizing, a noncompact metrizable space X contains a ray if and only if X admits an admissible incomplete metric d and a path $p : [0,1] \rightarrow \hat{X}$ such that $p(t) \in X$ for every $0 \leq t < 1$ and $p(1) \in \hat{X} \setminus X$.

Here is how one can obtain such a path p for an LC^0 -space X which is not completely metrizable. We find a completely metrizable enlargement \tilde{X} of X so that every path in \tilde{X} can be instantly homotoped to a path in X . Such an enlargement \tilde{X} can be found for every LC^n (or, locally contractible) space X which is not completely metrizable; we additionally can require that \tilde{X} is LC^n (locally contractible) and that $\tilde{X} \setminus X$ is locally n -negligible in \tilde{X} (see Proposition 2.8). A corresponding result for absolute neighborhood retract spaces X was previously obtained by Toruńczyk in [Tor2].

2. LC^0 -spaces containing a ray

Let us start with the following observation due to Klee [K] which can be also found in [C].

2.1. PROPOSITION. *Every noncompact, connected, locally connected, locally compact metrizable space contains a ray (and consequently, lacks the fixed point property). \square*

The above fact admits the following reformulation.

2.2. PROPOSITION. *Let X be a noncompact metrizable space. Then X contains a ray if and only if X admits a completion \hat{X} such that for some $x_\infty \in \hat{X} \setminus X$, and connected, locally connected, completely metrizable subspace Y of X , the space $Y \cup \{x_\infty\}$ is locally connected.*

PROOF. Assume that X contains a ray R . Let d be a metric on the one-point compactification of R . By the Hausdorff theorem on extending metrics [E, p. 369], there exists an admissible metric on X whose restriction to R is d . Now, clearly our condition is satisfied with $Y = R$.

To show the converse statement observe that $Y \cup \{x_\infty\}$ is completely metrizable, locally connected and connected. This yields that $Y \cup \{x_\infty\}$ is path connected; hence an argument from the Introduction works. \square

The following abstraction of [Tor1, Proposition 2.1] has been suggested to us by H. Toruńczyk.

2.3. PROPOSITION. *Let X be a metrizable (resp., completely metrizable) space and \mathcal{T} be a set of pairs (U, V) of open subsets of X satisfying the following properties:*

- (a) $V \subseteq U$ for every $(U, V) \in \mathcal{T}$
- (b) for every $x \in X$ and every open neighborhood U of x there exists an open neighborhood $V \subseteq U$ of x such that $(U, V) \in \mathcal{T}$,
- (c) for every $(U, V) \in \mathcal{T}$ and every open sets $U', V' \subseteq X$ if $U \subseteq U'$ and $V' \subseteq V$ then $(U', V') \in \mathcal{T}$.

Then there exists an admissible metric (resp., complete metric) d on X such that for every $x \in X$ and $r \in (0, 1)$ the pair of open balls $(B_d(x, r), B_d(x, r/8))$ belongs to \mathcal{T} .

PROOF. By induction we will construct a sequence of admissible metrics d_n on X such that for every $n \in \omega$ and open set V such that $\text{diam}_{d_{n+1}} V < 1$ there exists an open U such that $(U, V) \in \mathcal{T}$ and $\text{diam}_{d_i} U < (n+1)^{-1} 2^{-(n+1)}$ for $i = 0, \dots, n$.

Let d_0 be any admissible metric (resp., complete metric) on X and suppose that metrics d_0, \dots, d_n have been defined. Let

$$\mathcal{U} = \left\{ U_x = \bigcap_{i=0}^n B_{d_i}(x, (n+1)^{-1} 2^{-(n+1)}) \mid x \in X \right\}.$$

By (b), for every $x \in X$ we can find an open neighborhood V_x of x such that $(U_x, V_x) \in \mathcal{T}$. From a result of Michael [Mi, p. 165] it follows that there exists an admissible metric d_{n+1} on X such that every set of d_{n+1} -diameter less than 1 is contained in some set V_x . Obviously, conditions (c) guarantees that d_{n+1} has the required property.

We define the metric d by the formula

$$d(x, y) = \sum_{n=0}^{\infty} \min(d_n(x, y), 2^{-(n+1)}) \quad \text{for } x, y \in X.$$

If our initial metric d_0 is complete then from the inequality $d(x, y) \geq \min(d_0(x, y), 2^{-1})$ it follows that d is also complete. Fix $x \in X$ and $r \in (0, 1)$.

Let $n \in \omega$ be such that $2^{-(n+1)} \leq r < 2^{-n}$. Since $r/8 < 2^{-(n+3)}$ we have $B_d(x, r/8) \subseteq B_{d_{n+2}}(x, r/8)$ and therefore $\text{diam}_{d_{n+2}} B_d(x, r/8) < 2^{-(n+2)} < 1$. By the property of d_{n+2} we can find an open set U such that $(U, B_d(x, r/8)) \in \mathcal{F}$ and $\text{diam}_d U < (n+2)^{-1} 2^{-(n+2)}$ for $i = 0, \dots, n+1$. Hence

$$\text{diam}_d U < (n+2)(n+2)^{-1} 2^{-(n+2)} + \sum_{i=n+2}^{\infty} 2^{-(i+1)} = 2^{-(n+1)} \leq r.$$

The above inequality shows that $U \subseteq B_d(x, r)$ and by (c) we conclude that the pair $(B_d(x, r), B_d(x, r/8))$ belongs to \mathcal{F} . \square

2.4. COROLLARY. *Let X be a metrizable LC^n -space, $n \in \omega$. Then there exists an admissible metric d on X such that for every $x \in X$ and $r \in (0, 1)$ each continuous map $f : \partial I^{k+1} \rightarrow B_d(x, r/8)$ can be extended to a continuous map $F : I^{k+1} \rightarrow B_d(x, r)$, $0 \leq k \leq n$.*

PROOF. Apply 2.3 for the set $\mathcal{F} = \{(U, V) \mid U \text{ and } V \text{ are open subsets of } X \text{ such that every continuous map } f : \partial I^{k+1} \rightarrow V \text{ can be extended to a continuous map } F : I^{k+1} \rightarrow U, 0 \leq k \leq n\}$. \square

2.5. THEOREM. *Let X be a metrizable LC^0 -space. If either X is noncompact, connected and locally compact or X is not completely metrizable then X contains a ray.*

PROOF. The locally compact case was settled in Proposition 2.1.

Assume that X is not completely metrizable. Then the metric d given by Cor. 2.4 is not complete. Let $(x_n)_{n \in \omega}$ be a Cauchy sequence in (X, d) which is not convergent. For every $n \in \omega$ we can find a path $p_n : [0, 1] \rightarrow X$ such that $p_n(0) = x_n, p_n(1) = x_{n+1}$ and $\text{diam}_d p_n([0, 1]) < 20d(x_n, x_{n+1})$. One can easily check that the set $Y = \bigcup_{n \in \omega} p_n([0, 1])$ is closed in X , noncompact, connected, locally connected and locally compact. By 2.1, Y contains a ray which is also a closed subset of X . \square

We see that a key ingredient in proving Theorem 2.5 was Proposition 2.3 which itself is an abstraction of [Tor1, Proposition 2.1]. The latter fact has been used by Toruńczyk (see [Tor1, Proposition 2.2]) to show that every absolute retract space X admits a so-called **regular** metric d . The last means that whenever (X, d) is isometrically (or more generally, uniformly) embedded onto a closed subset of a metric space (Y, ρ) , then there exists a retraction

$r : (Y, \rho) \rightarrow (X, d)$ which is regular, i.e., for every $\varepsilon > 0$ there exists $\delta > 0$ such that whenever $\text{dist}_\rho(x, X) < \delta$, then $d(r(x), x) < \varepsilon$. This fact provides an alternative proof of Theorem 2.5 for the case of noncompletely metrizable, absolute retract space X as follows. Embed isometrically (X, d) as a closed subset of a normed space $(E, \|\cdot\|)$ (see [BP, p. 49]), and let $r : (E, \|\cdot\|) \rightarrow (X, d)$ be a regular retraction. Since X is not completely metrizable, we can find a piecewise linear map $\phi : [0, 1] \rightarrow E$ with the nodes (i.e., the points of the set $\{\phi(t) \mid \phi \text{ is not affine at } t\} \cup \{\phi(0)\}$) lying in X and with $\lim_{t \rightarrow 1} \phi(t) \in \bar{X} \setminus X$, where \bar{X} is the closure of X in the completion \hat{E} of E . Using the regularity of r , we can construct ϕ in such a way that $\lim_{t \rightarrow 1} r(\phi(t)) = \lim_{t \rightarrow 1} \phi(t)$. This easily yields the existence of a ray in $r(\phi([0, 1])) \subseteq X$ (see the Introduction).

Observe that in the above argument we rather used the fact that (X, d) was not complete, than X is not completely metrizable. Consequently, if X has the fixed point property then any regular metric on X must be complete (otherwise, X would contain a ray, contradicting the fixed point property). Since, for completely metrizable X , the construction of [Tor1] yields a complete regular metric on X , it suggests that every regular metric on X must be complete. Here is a simple counterexample.

2.6. EXAMPLE. The standard Euclidean metric on $(0, 1]$ is regular. (Let $(0, 1]$ be a closed subset of a metric space (X, ρ) , let $U_n = \bigcup \{B(t, (1/n) - t) \mid 0 < t < (1/n)\}$ (here $B(x, \varepsilon)$ denotes the open ball at x and radius ε), and define inductively a retraction $r : X \rightarrow (0, 1]$ as follows: $r(X \setminus U_1) = \{1\}$, r transforms the boundaries of U_1 and of U_2 onto $\{1\}$ and $\{1/2\}$, respectively, and the set $U_1 \setminus U_2$ onto $[1/2, 1]$, and so on.) \square

Having in mind Question 1.1, it is reasonable to ask

2.7. QUESTION. Let X be a completely metrizable absolute retract space without the fixed point property. Does X admit an incomplete regular metric?

We now discuss a way of obtaining LC^0 -spaces which are not completely metrizable by employing the following notion of **local n -homotopy negligibility** due to Toruńczyk [Tor2]. A subset A of a metrizable space Y is locally n -negligible, $0 \leq n \leq \infty$, if for every $y \in Y$ and for every neighborhood U of y , there exists a neighborhood V of y , $V \subseteq U$, such that each map $f(I^k, \partial I^k) \rightarrow (V, V \setminus A)$ can be homotoped, via a homotopy $(h_t) : (I^k, \partial I^k) \rightarrow (U, U \setminus A)$, $0 \leq t \leq 1$, to h_1 so that $h_1(I^k) \subset U \setminus A$, $0 \leq k < n + 1$, see [Tor2]. Here by I^0 we mean a fixed one-point set, and $\partial I^0 = \emptyset$; $k < n + 1$ means “ $0 \leq k \leq n$ if $n \neq \infty$ ”.

and $k \in \omega$ if $n = \infty$. Locally ∞ -negligible sets are simply called locally negligible sets. It can easily be shown (see [Tor2, Remark 2.5]) that whenever Y is an LC^0 -space and A a locally 1-negligible subset of Y , then $X = Y \setminus A$ is an LC^0 -space. If additionally Y is completely metrizable and A is not an F_σ -subset of Y , then X is an LC^0 -space which additionally is not completely metrizable.

Our Proposition 2.3 enables us to derive the following result on enlarging incomplete LC^n -spaces to complete ones.

2.8. PROPOSITION. *Let Y be a metrizable space and let X be an LC^n -space (resp., X is an absolute neighborhood retract space) such that $X \subset Y$. Then X can be enlarged to $\tilde{X} \subseteq Y$ such that*

- (i) \tilde{X} is a G_δ -subset of Y ,
- (ii) \tilde{X} is an LC^n -space (resp., \tilde{X} is an absolute neighborhood retract space),
- (iii) $\tilde{X} \setminus X$ is locally n -negligible (resp., locally homotopy negligible) in \tilde{X} .

In addition, given a finite-dimensional polyhedron K , $\dim(K) < n + 1$, and a map $f : K \rightarrow X$, there exists a homotopy $(h_t) : K \rightarrow \tilde{X}$ such that $h_0 = f$ and $h_t(K) \subset X$ for $t > 0$.

The proof of the absolute neighborhood retract case of 2.8 has been provided in [Tor2, Proposition 4.1]. Since then the fact has become very useful. We hope that the cases of finite n or locally contractible X (treated in Proposition 2.9 below) will also find their applications.

Here is how the case of $n = 0$ can be applied to obtain a proof of 2.5 for an LC^0 -space which is not completely metrizable. Embed X in a completely metrizable space Y , and let \tilde{X} be an enlargement from 2.8. Then, by 2.8(i), \tilde{X} is also completely metrizable and therefore $\tilde{X} \setminus X \neq \emptyset$. By the ‘addition’ part of 2.8, one can find a path $h : [0, 1] \rightarrow \tilde{X}$ such that $p((0, 1]) \subset X$ and $p(0) \in \tilde{X} \setminus X$. Hence, X contains a ray.

PROOF OF 2.8. Let us first provide an argument for the case where X is an absolute neighborhood retract space (more direct than that of [Tor2, Proposition 4.1]). Embed Y as a closed linearly independent subset of a normed space E (see [BP, p. 49]). Then X is a closed subset of $E_0 = \text{span}(X)$. Write r for a retraction of an open subset U_0 in E_0 onto X . Let U be an open set in E such that $U \cap E_0 = U_0$. By the theorem of Lavrentiev, r can be extended to a map $\tilde{r} : \tilde{U}_0 \rightarrow \tilde{E}_0$, where $U_0 \subset \tilde{U}_0 \subset U \cap \tilde{E}_0$, \tilde{E}_0 is the closure of E_0 in the completion of E and \tilde{U}_0 is a G_δ -subset of $U \cap \tilde{E}_0$. Let $\tilde{X} = \{y \in Y \cap \tilde{U}_0 \mid \tilde{r}(y) = y\}$. Let $U'_0 = \tilde{r}^{-1}(\tilde{X})$. We see that $U_0 \subset E'_0 \subset U \cap \tilde{E}_0$. Since E_0 is a linear subspace of

$\bar{E}_0, \bar{E}_0 \setminus E_0$ is locally homotopy negligible in \bar{E}_0 . It follows that $U \cap \bar{E}_0 \setminus U$ (and hence $U \cap \bar{E}_0 \setminus \tilde{U}$) is locally homotopy negligible in $U \cap \bar{E}_0$. Since $U \cap \bar{E}_0$ is an absolute neighborhood retract, by [Tor2, Theorem 3.1], \tilde{X} is also an absolute neighborhood retract. It is easy to see that (iii) holds.

Assume that X is an LC^n -space. Denote by \hat{Y} a completion of Y . Let d be a metric on X satisfying the assertion of 2.4. By the Lavrientiev theorem, the metric d can be extended to a G_δ -subset \tilde{Y} of the closure of X in \hat{Y} . Now, it is easy to see, that given $\tilde{y} \in \tilde{Y}$, every map $f : \partial I^k \rightarrow B(\tilde{y}, \varepsilon) \cap X$ extends to a map $\tilde{f} : I^k \rightarrow B(\tilde{y}, 16\varepsilon) \cap X$ for $0 \leq k < n + 2$. According to Eilenberg—Wilder terminology (and used in [Tor2]), the set X is LC^n rel. \tilde{Y} at each point $\tilde{y} \in \tilde{Y}$. We can now apply [Tor2, Theorem 2.8] to conclude that $\tilde{Y} \setminus X$ is locally n -homotopy negligible, and that the ‘addition’ part of 2.8 holds with \tilde{X} being replaced by \tilde{Y} .

Set $\tilde{X} = \tilde{Y} \cap Y$. We easily check that (i), (iii) and the ‘addition’ part of 2.8 hold for such \tilde{X} . It remains to show that \tilde{X} is an LC^n -space. Pick $\tilde{x} \in \tilde{X}$, a map $f : \partial I^k \rightarrow B(\tilde{x}, \varepsilon)$, $0 \leq k < n + 2$. By the ‘addition’ part, f can be homotoped within $B(\tilde{x}, \varepsilon)$ to a map $f_1 : \partial I^k \rightarrow B(\tilde{x}, \varepsilon) \cap X$. By the above property of d , f_1 can be extended to a map $I^k \rightarrow B(\tilde{x}, 16\varepsilon) \cap X$. This shows that f can be extended to a map $I^k \rightarrow B(\tilde{x}, 16\varepsilon)$, hence \tilde{x} is an LC^n -space. \square

2.9. PROPOSITION. *Let X be a metrizable locally contractible space. Then X can be enlarged to a completely metrizable space \tilde{X} such that*

- (i) \tilde{X} is locally contractible,
- (ii) $\tilde{X} \setminus X$ is locally homotopy negligible in \tilde{X} .

In addition, given a finite-dimensional polyhedron K and a map $f : K \rightarrow \tilde{X}$, there exists a homotopy $(h_t) : K \rightarrow \tilde{X}$ such that $h_0 = f$ and $h_t(K) \subset X$ for $t > 0$.

PROOF. We use a similar construction as in the proof of 2.3. We will construct an admissible metric d in X such for every $x \in X$ and $r \in (0, 1)$ the ball $B_d(x, r/8)$ can be contracted within the ball $B_d(x, r)$ by a homotopy which can be extended to a homotopy contracting the corresponding ball in the completion \tilde{X} of X with respect to d .

By induction we will construct sequences of admissible metrics d_n on X , locally finite open covers \mathcal{U}_n of X and families of homotopies $\mathcal{H}_n = \{h_U : U \times [0, 1] \rightarrow X \mid U \in \mathcal{U}_n\}$ satisfying the following conditions for every $n \in \omega$:

- (a) $\forall (U \in \mathcal{U}_n) h_U \in \mathcal{H}_n$ is a contraction of U in X ,
- (b) $\forall (U \in \mathcal{U}_n) \forall (i \leq n) \text{diam}_{d_i} h_U(U \times [0, 1]) < (n + 1)^{-1} 2^{-(n+1)}$,

- (c) $\forall(U \in \mathcal{U}_n)\forall(j < n)\forall(V \in \mathcal{U}_j)\forall(x, y \in U \cap V)\forall(t \in [0, 1])\forall(i \leq n)$
 $d_i(h_V(x, t), h_V(y, t)) < (n + 1)^{-1}2^{-(n+1)},$
- (d) every set of d_{n+1} -diameter less than 1 is contained in some set $U \in \mathcal{U}_n$.

We start the construction with any admissible metric d_0 on X . Using the local contractibility of X we find \mathcal{U}_0 and \mathcal{H}_0 in order to satisfy (a) and (b). Suppose that metrics d_0, \dots, d_n , covers $\mathcal{U}_0, \dots, \mathcal{U}_{n-1}$ and families $\mathcal{H}_0, \dots, \mathcal{H}_{n-1}$ have been constructed for $n \geq 1$. Again using the local contractibility of X one can easily find a locally finite open cover \mathcal{U}_n and a family of homotopies \mathcal{H}_n satisfying (a) and (b). Since the covers $\mathcal{U}_0, \dots, \mathcal{U}_{n-1}$ are locally finite, we can additionally assure (c). Finally, a metric d_{n+1} can be obtained from the result of Michael [Mi, p. 165].

We define a metric d by the formula

$$d(x, y) = \sum_{n=0}^{\infty} \min(d_n(x, y), 2^{-(n+1)}) \quad \text{for } x, y \in X.$$

Let \tilde{X} be the completion of X with respect to the metric d . Fix $x \in \tilde{X}$ and $r \in (0, 1)$. We will show that the ball $B_d(x, r/8)$ can be contracted within the ball $B_d(x, r)$. Let $n \in \omega$ be such that $2^{-(n+1)} \leq r < 2^{-n}$. We take $x' \in X$ and $s < 2^{-(n+3)}$ such that $B_d(x, r/8) \subseteq B_d(x', s)$. Since $s < 2^{-(n+3)}$ we have $B_d(x', s) \cap X \subseteq B_{d_{n+2}}(x', s) \cap X$ and therefore $\text{diam}_{d_{n+2}} B_d(x', s) \cap X < 2^{-(n+2)} < 1$. By the property (d) of d_{n+2} , we can find an open set $U \in \mathcal{U}_{n+1}$ such that $B_d(x', s) \cap X \subseteq U$. Let $C([0, 1], \tilde{X})$ denotes the space of continuous maps from $[0, 1]$ into \tilde{X} equipped with the standard (complete) supremum metric. We will check that the map $g : U \rightarrow C([0, 1], \tilde{X})$ defined by

$$g(u)(t) = h_U(u, t) \quad \text{for } u \in U \text{ and } t \in [0, 1]$$

is uniformly continuous (in fact, Lipschitz). Let $y, z \in U$ be such that $d(y, z) < 2^{-(k+2)}$, for $k > n + 1$. Then $d_{k+1}(y, z) < 1$ and (d) implies that $y, z \in V$ for some $V \in \mathcal{U}_k$. From (c) it follows that $d_i(h_U(y, t), h_U(z, t)) < (k + 1)^{-1}2^{-(k+1)}$, for every $t \in [0, 1]$ and $i \leq k$. Therefore

$$d(h_U(y, t), h_U(z, t)) < (k + 1)(k + 1)^{-1}2^{-(k+1)} + \sum_{i=k+1}^{\infty} 2^{-(i+1)} = 2^{-k}.$$

This means that the distance between $g(y)$ and $g(z)$ is less than or equal 2^{-k} . Consequently, the map g can be extended to a continuous map $G : \text{Cl}_{\tilde{X}} U \rightarrow C([0, 1], \tilde{X})$. Obviously, $B_d(x, r/8) \subseteq \text{Cl}_{\tilde{X}} U$ and the homotopy

$H : B_d(x, r/8) \times [0, 1] \rightarrow \tilde{X}$ defined by

$$H(y, t) = G(y)(t) \quad \text{for } y \in B_d(x, r/8) \text{ and } t \in [0, 1]$$

is a contraction of $B_d(x, r/8)$ in \tilde{X} . Using the condition (b) one can easily calculate that $\text{diam}_d H(B_d(x, r/8) \times [0, 1]) < 2^{-(n+1)} \leq r$ hence $H(B_d(x, r/8) \times [0, 1]) \subseteq B_d(x, r)$. Property (ii) and the ‘addition’ part of 2.9 can be verified in the same way as in the proof of 2.8. \square

We do not know whether or not Proposition 2.9 holds in a version of 2.8. Such a version can be obtained if the answer to the following question is affirmative.

2.10. QUESTION. Let A be a locally homotopy negligible subset of a locally contractible space X . Is $X \setminus A$ locally contractible?

Note that the complement $X \setminus A$ enjoys the following strong version of the LC^∞ -property: For every $x \in X \setminus A$ and neighborhood U of x there exists a neighborhood V of x , $V \subseteq U$, such that spheres of all dimensions in V are contractible in U . Aiming at a negative answer to 2.10, it follows from [Tor2, Theorem 3.1] that X cannot be an absolute neighborhood retract. Let us consider the example of Borsuk of a compactum $X = X_0 \cup \bigcup_{k=1}^\infty X_k^\circ$ which is locally contractible but not an absolute neighborhood retract; we employ the notation of [Bor, p. 125]. It is easy to see that the identity map on X can be arbitrarily closely approximated by maps (even retractions) into $\bigcup_{k=1}^\infty X_k^\circ$. Hence, X_0 is locally homotopy negligible in X . (Since $\bigcup_{k=1}^\infty X_k^\circ$ is locally finite dimensional and locally contractible, it is an absolute neighborhood retract; this provides another counterexample to the converse implication of [Tor2, Theorem 3.1].) For an arbitrary subset A of X_0 , form the space $Y_A = A \cup \bigcup_{k=1}^\infty X_k^\circ$. Clearly, A is locally homotopy negligible in Y_A . Yet, it can be shown that Y_A is locally contractible as well, and hence Y_A cannot serve as a suitable solution to 2.10.

3. Convex sets which lack the fixed point property

First we show that every noncompact convex subset C of a metric linear space lacks the fixed point property. As said in the Introduction, it is enough to settle the case of an infinite-dimensional, nonlocally compact closed subset C in a complete metric linear space E .

3.1. PROPOSITION. *Let C be a nonlocally compact closed convex subset of a separable completely metrizable linear space E . Then, every map of a separable, completely metrizable, finite-dimensional space M into C can be strongly approximated by a closed embedding.*

PROOF. By [DT] and [D], the space C enjoys the so-called strong approximation property, i.e., given an open cover \mathcal{U} of C , every map $\bigoplus_{n \in \mathbb{N}} I^n \rightarrow C$ can be \mathcal{U} -approximated by a map g so that $\{g(I^n)\}_{n=1}^\infty$ forms a discrete family in C . Next, since C is locally contractible, and hence, is LC^n for every n , our statement can be obtained by inspecting reasonings of [Tor3, p. 255], see also [Bo1, p. 127] and [Bo2, p.10]. \square

3.2. REMARK. In 3.1, M can be replaced by an arbitrary separable, completely metrizable absolute neighborhood space. (Apply the approach of [Tor3, p. 255].) \square

A discussion from the Introduction and the statement of 3.1 yield the following answer to a question of Klee [K].

3.3. COROLLARY. *Every noncompact convex subset of a metric linear space contains a ray, and therefore fails the fixed point property.* \square

Let us recall that a subset A of a topological vector space E is said to be **totally bounded**, if for every neighborhood U of 0 in E there are $x_1, x_2, \dots, x_n \in E$ such that $A \subset \bigcup_{i=1}^n x_i + U$.

3.4. THEOREM. *Let C be a convex subset of a topological vector space E . If C is not totally bounded, then C contains a ray and does not have the fixed point property.*

PROOF. We can assume (see [KN, p. 50]) that $E = \prod_\alpha E_\alpha$, where each E_α is a complete metric linear space. Let $\pi_\alpha : E \rightarrow E_\alpha$ be the projection. If each $\pi_\alpha(C)$ is compact, then C , as a subset of the compact set $\prod_\alpha \pi_\alpha(C)$, would be totally bounded. Consequently, there exists α such that $C_\alpha = \pi_\alpha(C)$ is a noncompact, convex subset of the metric linear space $(E_\alpha, |\cdot|_\alpha)$. By 3.3, there exists a closed embedding $p : [0, 1) \rightarrow C_\alpha$. We may assume that p is piecewise linear.

To see this first we approximate p by a piecewise linear map $p' : [0, 1) \rightarrow C_\alpha$ in a following way. We construct piecewise linear maps $p'_n : [1 - 2^{-n},$

$1 - 2^{-n-1}] \rightarrow C_\alpha$, $n = 0, 1, \dots$, so that $|p(t) - p'_n(t)|_\alpha < 2^{-n}$ for every $1 - 2^{-n} \leq t \leq 1 - 2^{-n-1}$. The map p' is defined as the union of p'_0, p'_1, \dots . Using the fact that p is closed and the approximation property of p' one can easily see that the image $p'([0, 1])$, a closed subset of C_α , is a noncompact, connected, locally compact LC^0 -space. Therefore, by 2.5 we can find a closed embedding $p'' : [0, 1] \rightarrow p'([0, 1])$. Since $p'([0, 1])$ is a union of a locally finite family of segments we may assume that p'' is piecewise linear.

Now, we take the increasing sequence $\{t_n\}$, $0 < t_n < 1$, $\lim t_n = 1$ such that p is affine on each subinterval $[t_n, t_{n+1}]$. Put $x_n = p(t_n)$. We claim that p can be 'lifted' to a map $q : [0, 1] \rightarrow C$, that is, we have $\pi_\alpha q = p$. To arrange that, pick $y_n \in \pi_\alpha^{-1}\{x_n\} \cap C$. Let q be an affine map on each $[t_n, t_{n+1}]$ joining y_n with y_{n+1} . Clearly, q is continuous and $\pi_\alpha q = p$ since π_α is linear. It follows that q is a closed embedding.

Let us now show that C fails to have the fixed point property. We simply will exhibit a retraction $r : C \rightarrow R$, where $R = q([0, 1])$. Write $r' : C_\alpha \rightarrow p([0, 1])$ for a retraction (C_α is metrizable!). By our construction, $\pi_\alpha|_R$ is a homeomorphism of R onto $p([0, 1])$. Let s be the inverse of $\pi_\alpha|_R$. It is clear that $r = sr'\pi_\alpha$ is a required retraction. \square

4. A noncompact convex set with the fixed point property

In Example 4.1 below we will show that Theorem 3.4 cannot be extended to all totally bounded noncompact convex sets C in topological vector spaces E . In view of 3.3, C must be nonmetric. Surprisingly, a suitable C can be found in a locally convex space E . This provides a negative answer to a question of Klee [K, p. 32].

4.1. EXAMPLE. There exists a locally convex topological vector space E and a noncompact convex subset W of E which has the fixed point property.

Let $K = \omega_1 + 1$ be the compact space of all ordinals $\leq \omega_1$ with the order topology. Consider the Banach space $C(K)$. Let $E = C(K)^*$ be the dual of $C(K)$ equipped with the weak* topology. The space E can be identified with the space of all regular Borel measures on K . Since K is scattered (i.e. does not contain any dense-in-itself subset) every measure $\mu \in E$ is purely atomic (has countable support $\text{supp}(\mu)$), see [R].

Let C be the subspace of E consisting of probability measures and let $W = \{\mu \in C \mid \text{supp}(\mu) \subseteq \omega_1\}$. (Clearly, W is dense in C , and since C is compact, W is totally bounded.)

4.2. LEMMA. *The space W has the fixed point property.*

PROOF. Let $f : W \rightarrow W$ be a continuous map. For every $\alpha < \omega_1$, let $W_\alpha = \{\mu \in W \mid \text{supp}(\mu) \subseteq [0, \alpha]\}$. The convex set W_α , regarded as a subspace of all probability measures in the dual space $C([0, \alpha])^*$ with the weak* topology, is closed, and therefore, W_α is compact. Since the compact space $[0, \alpha]$ is metrizable, $C([0, \alpha])$ is separable and therefore the weak* topology in $C([0, \alpha])^*$ and W_α is metrizable. For infinite α the space W_α is infinite-dimensional and by Keller's Theorem (see [BP]) is homeomorphic to the Hilbert cube I^ω .

We will show that for some infinite $\alpha < \omega_1$ we have $f(W_\alpha) \subseteq W_\alpha$. Therefore $f|W_\alpha$ (and f) has a fixed point.

By induction we construct a strictly increasing sequence of countable ordinals $(\alpha_n)_{n \in \omega}$ such that $f(W_{\alpha_n}) \subseteq W_{\alpha_{n+1}}$. Let $\alpha_0 = 0$ and suppose that $\alpha_0, \dots, \alpha_n$ have been defined. Since W_{α_n} is separable and every countable subset of W is contained in W_α , for some $\alpha < \omega_1$ we can find $\alpha_{n+1} < \omega_1$ such that $W_{\alpha_{n+1}}$ contains $f(W_{\alpha_n})$. We may assume that $\alpha_{n+1} > \alpha_n$. Now, we take $\alpha = \sup\{\alpha_n \mid n \in \omega\}$. The set $\bigcup_{n \in \omega} W_{\alpha_n}$ is dense in W_α and $f(\bigcup_{n \in \omega} W_{\alpha_n}) \subseteq \bigcup_{n \in \omega} W_{\alpha_n}$ therefore $f(W_\alpha) \subseteq W_\alpha$. \square

4.3. REMARK. Since every closed separable subset of W is compact, W does not contain a ray. This also follows from Lemma 4.3 and the following property of W .

4.4. LEMMA. *The space W is normal.*

PROOF. By normality of the compact space C it is enough to prove that disjoint closed subsets A, B of W have disjoint closures in C . Assume, to the contrary, that there exists $\mu \in \text{Cl}_C A \cap \text{Cl}_C B$. Let $\alpha < \omega_1$ be such $\text{supp}(\mu) \subseteq \alpha + 1 \cup \{\omega_1\}$. Let $\{f_n \mid n \in \omega\}$ be the norm dense subset of $\{f \in C(\omega_1 + 1) \mid f(\beta) = 0 \text{ for any } \beta > \alpha\}$. Define open sets

$$U_n = \left\{ \nu \in C \mid \forall (i \leq n) \left(\left| \int f_i d\nu - \int f_i d\mu \right| < \frac{1}{n+1} \right) \right\}.$$

Then for every $\nu \in \bigcap_{n \in \omega} U_n$ and every $\beta \leq \alpha$ we have $\nu(\beta) = \mu(\beta)$. For $\beta < \omega_1$ let g_β be the characteristic function of the interval $(\beta, \omega_1] = [\beta + 1, \omega_1]$. The function g_β is continuous on $\omega_1 + 1$. Now, define by induction the increasing sequence β_n and measures ν_n in the following way:

We set $\beta_0 = \alpha + 1$. Suppose that β_n and ν_k for $k < n$ have been defined. We take the following neighborhood of μ :

$$V_n = \left\{ \nu \in C \mid \nu \in U_n \text{ and } \left| \int g_{\beta_n} d\nu - \int g_{\beta_n} d\mu \right| < \frac{1}{n+1} \right\}.$$

If n is even then take $\nu_n \in A \cap V_n$; if n is odd then we choose $\nu_n \in B \cap V_n$. Finally, we define $\beta_{n+1} = \sup(\text{supp}(\nu_n))$.

Let $\beta = \sup \beta_n$ and let the measure $\nu \in W$ be defined by:

$$\nu(\gamma) = \begin{cases} \mu(\gamma) & \text{for } \gamma \leq \alpha \\ \mu(\omega_1) & \text{for } \gamma = \beta \\ 0 & \text{otherwise} \end{cases}$$

Let $f \in C(\omega_1 + 1)$. Since $\nu_n \in U_n$ it follows that $\int_{[0, \alpha]} f d\nu_n \rightarrow \int_{[0, \alpha]} f d\nu$. Also $\nu_n([0, \alpha]) \rightarrow \mu([0, \alpha]) = \nu([0, \alpha])$. We have

$$\nu_n((\beta_n, \beta]) = \nu_n((\beta_n, \omega_1]) = \int g_{\beta_n} d\nu_n \rightarrow \int g_{\beta_n} d\mu = \mu(\omega_1) = \nu(\beta).$$

Therefore $\nu_n((\alpha, \beta_n]), \nu_n((\beta, \omega_1]) \rightarrow 0$. By continuity of f at β it also follows that $\int_{(\beta_n, \beta]} f d\nu_n \rightarrow \int_{(\beta, \beta]} f d\nu = f(\beta)\nu(\beta)$. Now we can evaluate that

$$\begin{aligned} \int f d\nu_n &= \int_{[0, \alpha]} f d\nu_n + \int_{(\alpha, \beta_n]} f d\nu_n + \int_{(\beta_n, \beta]} f d\nu_n + \int_{(\beta, \omega_1]} f d\nu_n \\ &\rightarrow \int_{[0, \alpha]} f d\nu + 0 + f(\beta)\nu(\beta) + 0 = \int f d\nu. \end{aligned}$$

This shows that $\lim_n \nu_n = \nu$ in W . Therefore $\nu \in A \cap B$ contradicting our assumption on sets A, B . \square

Let us present another example of a noncompact convex set with the fixed point property; this example has been suggested to us by Roman Pol.

4.5. EXAMPLE. Let S be the Σ -product of intervals $[0, 1]$ in \mathbb{R}^{ω_1} , i.e.

$$S = \{(x_\alpha) \in [0, 1]^{\omega_1} \mid \{\alpha < \omega_1 \mid x_\alpha \neq 0\} \text{ is countable}\} \subseteq \mathbb{R}^{\omega_1}.$$

The set S is a noncompact convex subset of the locally convex space \mathbb{R}^{ω_1} and S has the fixed point property.

The fixed point property of S can be proved in a similar way as for the set W of Example 4.1. Namely, let $f : S \rightarrow S$ be a continuous map and let

$S_\alpha = \{(x_\alpha) \in S \mid x_\beta = 0 \text{ for } \beta > \alpha\}$. Using the same argument as in the proof of Lemma 4.2 one can find an infinite α such that $f(S_\alpha) \subseteq S_\alpha$ (here again we employ the fact that every countable subset of S is contained in S_β for some $\beta < \omega_1$). Since S_α can be identified with the Hilbert cube $[0, 1]^\alpha$, f has a fixed point in S_α .

By [E, 2.7.14] the space S is also normal.

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Department of Mathematics,
The University of Oklahoma,
Norman, OK 73019-0315
Current address: Pittsburg State University,
Department of Mathematics,
Pittsburg, Kansas 66762
E-mail address: tdobrowo@mail.pittstate.edu

Institute of Mathematics,
Warsaw University, ul.
Banacha 2, 02-097 Warszawa,
and

Institute of Mathematics,
Polish Academy of Sciences,
Warszawa, Poland

E-mail address: wmarcisz@mimuw.edu.pl