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ON HYPOELLIPTICITY FOR A CLASS OF PSEUDO-DIFFERENTIAL OPERATORS

By

Nobuo NAKAZAWA

1. Introduction and Results

We shall study hypoellipticity for a class of pseudo-differential operators which includes the operator $-a(x)\Delta+1$ with $a(x)\geq 0$ as a typical example. We shall use the Weyl symbols and the Weyl calculus in this paper. For the Weyl calculus we refer to Hörmander [2]. Let $p(x,\xi)\in S^m(\equiv S^m_{1,0}(\boldsymbol{R}^{2n}))$, i.e., $|p^{(\alpha)}_{(\beta)}(x,\xi)|\leq C_{\alpha,\beta}\langle\xi\rangle^{m-|\alpha|}$ for $(x,\xi)\in\boldsymbol{R}^{2n}$ and any multi-indices α and β, where $x=(x_1,\dots,x_n)\in\boldsymbol{R}^n$, $\xi=(\xi_1,\dots,\xi_n)\in\boldsymbol{R}^n$, $\langle\xi\rangle=\sqrt{1+|\xi|^2}$, $|\xi|=\sqrt{\sum_{j=1}^n|\xi_j|^2}$, $p^{(\alpha)}_{(\beta)}(x,\xi)=\partial^\alpha_\xi D^\beta_x p(x,\xi)$ and $D_x=(D_1,\dots,D_n)\equiv -i\partial_x=-i(\partial/\partial x_1,\dots,\partial/\partial x_n)$. We define for $u\in\mathscr{S}$

$$Pu\equiv p^w(x,D)u=(2\pi)^{-n}\int\left(\int e^{-i(x-y)\cdot\xi}p\left(\frac{x+y}{2},\xi\right)u(y)\,dy\right)d\xi,$$

where $x\cdot\xi=x_1\xi_1+\cdots+x_n\xi_n$ and $\mathscr{S}$ denotes the Schwartz space of rapidly decreasing functions on $\boldsymbol{R}^n$. We call the symbol $p(x,\xi)$ the Weyl symbol of P and write $\sigma_w(P)(x,\xi)=p(x,\xi)$. For pseudo-differential operators we also refer to Kumano-go [5] and Shubin [7].

For simplicity we denote $p^w(x,D)$ and $\sigma_w(P)(x,\xi)$ by $p(x,D)$ and $\sigma(P)(x,\xi)$ respectively, in this paper.

DEFINITION 1.1. *Let $x^0\in\boldsymbol{R}^n$. We say that P is hypoelliptic at x^0 if there exists a neighborhood U of x^0 such that*

$$U\cap\operatorname{sing\,supp}Pu=U\cap\operatorname{sing\,supp}u\quad \textit{for } u\in H_{-\infty},$$

where sing supp u *denotes the singular support of u, $H_{-\infty}=\bigcup_s H_s$ and H_s denotes the Sobolev space of order $s\in\boldsymbol{R}$.*

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We impose the following conditions on $p(x,\xi)$:

(A-0) The symbol $p(x,\xi)$ can be written in the form

$$p(x,\xi)=p_m(x,\xi)+p_{m-1}(x,\xi)+p_{m-2}(x,\xi)+p_{m-3}(x,\xi),$$

where $p_{m-j}(x,\xi)\in S^{m-j}$ $(0\le j\le 3)$ and $p_{m-j}(x,\xi)$ is homogeneous of degree $m-j$ in ξ for $|\xi|\ge 1$ $(0\le j\le 2)$.

(A-1) There exist a neighborhood U of 0 in $\boldsymbol{R}^n$ and $C>0$ such that

$$s(x,\xi)\equiv p_m(x,\xi)+\operatorname{Re}p_{m-1}(x,\xi)+\operatorname{Re}p_{m-2}(x,\xi)\ge -C\langle\xi\rangle^{m-3}$$

for $(x,\xi)\in U\times\boldsymbol{R}^n$.

(A-2) There exist a neighborhood U of 0, constants $c_0>0$ and $C_0\in\boldsymbol{R}$ such that

$$\operatorname{Re}(p(x,D)u,u)\ge c_0\|u\|^2_{m/2-1}-C_0\|u\|^2_{m/2-2}$$

for $u\in C_0^\infty(U)$, where $(u,v)=\int u(x)\overline{v(x)}\,dx$ and $\|u\|_s=(\langle D\rangle^s u,\langle D\rangle^s u)^{1/2}$.

(A-3) There exist a neighborhood U of 0 and $r\in\boldsymbol{Z}$ with $0\le r\le n$ such that

$$p_m(x,\xi)\ne 0\quad\text{if } x\in U,\quad |\xi|=1\quad\text{and } x'=(x_1,\dots,x_r)\ne 0,$$

where we consider $x'=0$ if $r=0$.

(A-4) There exists a neighborhood U of 0 such that for any $\nu>0$ there is a constant $C_\nu>0$ satisfying

$$\text{(i)}\qquad \sum_{\substack{|\alpha|+|\beta|=2\\ \alpha'=0}}^{n}(\log\langle\xi\rangle)^{|\alpha|}|p^{(\alpha)}_{m(\beta)}(x,\xi)|\langle\xi\rangle^{-|\beta|}\le \nu s(x,\xi)+C_\nu\langle\xi\rangle^{m-3},$$

$$\text{(ii)}\quad \log\langle\xi\rangle|\operatorname{Im}p_{m-1}(x,\xi)|\langle\xi\rangle^{-1}+\sum_{\substack{|\alpha|+|\beta|=1\\ \alpha'=0}}^{n}(\log\langle\xi\rangle)^{|\alpha|}|\operatorname{Im}p^{(\alpha)}_{m-1(\beta)}(x,\xi)|\langle\xi\rangle^{-|\beta|}$$

$$\le \nu s(x,\xi)+C_\nu\langle\xi\rangle^{m-3}$$

if $x\in U$ and $|\xi|\ge 1$, where $\alpha'=(\alpha_1,\dots,\alpha_r)$.

We note that (A-3) is always valid if $r=0$. Now we can state our main theorem.

THEOREM 1.2. *Under* (A-0)–(A-4), $p(x,D)$ *is hypoelliptic at* $x=0$.

Now we mention several known results relating to the above theorem.

RESULT 1. Hörmander [1] constructed a local parametrix at 0 of the operator

$$L_1 = a(x)(-\Delta)^m + (-\Delta)^{m'},$$

where $m, m' \in \boldsymbol{Z}_+ (= \boldsymbol{N} \cup \{0\})$ and $m > m'$, under the following conditions:

(B-1) $a(x) \in C^\infty$ and $a(x) \geq 0$.

(B-2) In a neighborhood of 0

$$|D_x^\beta a(x)| \leq M_\beta a(x)^{1-\tau|\beta|} \quad (1 - \tau|\beta| \geq 0, 0 < \tau < \{2(m - m')\}^{-1}).$$

Therefore, L_1 is hypoelliptic at 0 under the above conditions.

RESULT 2. Katsuta [4] showed that the existence of a local parametrix at 0 of the operator

$$L_2 = -a(x)\Delta + 1,$$

when L_2 satisfies (B-1) and the following condition:

(B-3) There exist a neighborhood U of 0, $\delta \in \boldsymbol{R}$ with $0 < \delta < 1/2$ and $M > 0$ such that

$$|\partial_{x_j} a(x)| \leq M a(x)^{1/2+\delta} \quad (x \in U, 1 \leq j \leq n).$$

Consequently L_2 is hypoelliptic at 0 under (B-1) and (B-3).

RESULT 3. We showed in [6] that L_2 is hypoelliptic at 0 under (B-1) and the condition

(B-4) there exists a neighborhood U of 0 such that $\partial_x^\alpha a(x) = 0$ if $x \in U$, $a(x) = 0$ and $|\alpha| = 2$.

Concerning the above results, it is easy to see that (B-2) implies (B-3) and that (B-3) does (B-4) under the assumption (B-1). Furthermore, if L_2 satisfies (B-1) and (B-4), then L_2 satisfies (A-0)–(A-4). This follows from Propositions 4.1 and 4.2 in Section 4 (see Section 4).

In addition, (A-1) and (A-2) are satisfied if the following conditions are satisfied (see Proposition 4.1 below):

(A-1)$'$ there exists a neighborhood U of 0 such that

$$p_m(x,\xi)\ge 0,\quad \mathrm{Re}\, p_{m-1}(x,\xi)\ge 0,\quad \mathrm{Re}\, p_{m-2}(x,\xi)>0$$

for $x\in U$ and $|\xi|=1$.

(A-2)$'$ $p_{m(\beta)}(0,\xi)=0$ for any $\xi\in \boldsymbol{R}^n$ with $|\xi|=1$ and $\beta\in \boldsymbol{Z}^n_+$ with $|\beta|\le 2$ if $p_m(0,\xi^0)=0$ for some $\xi^0\in \boldsymbol{R}^n$ with $|\xi^0|=1$.

The plan of this paper is as follows. In Section 2, we give a general criterion of hypoellipticity which is a simple variant of criteria given in Kajitani and Wakabayashi [3] and Wakabayashi and Suzuki [8]. We also reduce the operator $p(x,D)\in S^m$ to $\tilde{p}(x,D)\in S^2$. In Sectition 3, we complete the proof of Theorem 1.2. Finally in Sectition 4, we give some remarks and examples.

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2. Preliminaries

In this section, we shall give propositions for the proof of Theorem 1.2 and reduce the problem for $p(x,D)$ to the problem for $\tilde{p}(x,D)=\langle D\rangle^{-m/2+1}p(x,D)\langle D\rangle^{-m/2+1}$.

First we assume that $p(x,\xi)\in S^m$ and that $p(x,\xi)$ satisfies (A-3). Let $x^0=(0,x^{0\prime\prime})\in U$, and choose $\varphi(x'')\in C_0^\infty(\boldsymbol{R}^{n-r})$ so that

$$\varphi(x'')=\begin{cases}|x''-x^{0\prime\prime}|^2 & (|x''-x^{0\prime\prime}|\le 1),\\ 2 & (|x''-x^{0\prime\prime}|\ge 2),\end{cases}$$

where $x''=(x_{r+1},\dots,x_n)\in \boldsymbol{R}^{n-r}$. Here we consider $x^0=0$ and $\varphi(x'')\equiv 0$ if $r=n$.

Define

$$\begin{aligned}\Lambda(x'',\xi)&=\Lambda_\delta(x'',\xi;s,a,N)\\ &=(-s+a\varphi(x''))\log\langle\xi\rangle+N\log(1+\delta|\xi|^2),\\ p_\Lambda(x,D)&=(e^{-\Lambda})(x'',D)p(x,D)(e^{\Lambda})(x'',D).\end{aligned}$$

The following proposition is essentially due to Kajitani and Wakabayashi [3] and Wakabayashi and Suzuki [8].

PROPOSITION 2.1. *Assume that there exist a neighborhood U_0 of x^0, $l_1,l_2,l_3\in \boldsymbol{R}$, $a_0\ge 0$, N_0, $s_0\in \boldsymbol{R}$ and $\chi(x')\in C_0^\infty(\boldsymbol{R}^r)$ satisfying $\chi(x')=1$ near 0 so that for any $a\ge a_0$, $N\ge N_0$, $s\ge s_0$ there are constants $\delta_0>0$ and $C>0$ such that*

$$\|u\|_{l_1} \le C(\|p_\Lambda(x,D)u\|_{l_2} + \|u\|_{l_1-1} + \|(1-\chi)u\|_{l_3}), \tag{2.1}$$

for $u \in C_0^\infty(U_0)$ *if* $0 < \delta \le \delta_0$. *Here we consider* $\chi(x') \equiv 1$ *if* $r = 0$. *Then* $p(x,D)$ *is hypoelliptic at* x^0 *namely,* $x^0 \notin \operatorname{sing\,supp} u$ *if* $u \in H_{-\infty}$ *and* $x^0 \notin \operatorname{sing\,supp} p(x,D)u$.

PROOF. Let $u \in H_{-\infty}$. Then there exists a constant $s' \in \boldsymbol{R}$ such that $u \in H_{s'}$. Assume that $x^0 \notin \operatorname{sing\,supp} p(x,D)u$. For simplicity we assume that $r \le n-1$. Then there is a neighborhood $U_1 = U_1' \times U_1''$ of x^0 such that

$$U_1 \subset\subset U \cap U_0 \cap \{x = (x', x'') \in \boldsymbol{R}^n; |x'' - x^{0\prime\prime}| \le 1\},$$

$$\operatorname{sing\,supp} p(x,D)u \cap \bar{U}_1 = \emptyset.$$

where $A \subset\subset B$ means that $\bar{A}$ is compact and included in the interior of B. Choose a neighborhood $U_2 = U_2' \times U_2''$ of x^0, $\Psi_1(x') \in C_0^\infty(U_1')$ and $\Psi_2(x'') \in C_0^\infty(U_1'')$ so that

$$U_2 \subset\subset U,$$

$$\Psi_1(x') = 1 \quad \text{in } U_2',$$

$$\Psi_2(x'') = 1 \quad \text{in } U_2''.$$

Here we consider $\Psi_1(x') \equiv 1$ if $r = 0$. Then there is a positive constant ε such that

$$\varphi(x'') = |x'' - x^{0\prime\prime}| \ge 2\varepsilon \quad \text{for } x'' \in U_1'' \backslash U_2''.$$

Fix $\tau > s'$ and choose $a > 0$, N, $s \in \boldsymbol{R}$ so that $a \ge a_0$, $N \ge N_0$, $s \ge s_0$ and

$$\begin{cases} 2a\varepsilon - s \ge l_2 + m - 1 - s', \\ \tau \le l_1 + s - a\varepsilon, \\ 2N \ge s - s' + \max\{l_1 - 1, l_2 + m, l_3\}. \end{cases} \tag{2.2}$$

It follows from the symbol calculus that there exists a symbol $q(x'', \xi)$ $(\equiv q(x'', \xi; \delta)) \in C([0,1]; S^0)$ satisfying

$$(e^\Lambda)(x'', D)(e^{-\Lambda})(x'', D)q(x'', D) - I \in S^{-\infty} \quad \text{uniformly in } \delta \in [0,1]. \tag{2.3}$$

We have

$$\begin{aligned} &p(x,D)(\Psi_1(x')\Psi_2(x'')u(x)) \\ &\quad = \Psi_1(x')\Psi_2(x'')p(x,D)u(x) + [p(x,D), \Psi_1(x')\Psi_2(x'')]u(x), \end{aligned} \tag{2.4}$$

where $[A, B] = AB - BA$. Operating $(e^{-\Lambda})(x'', D)$ to the both sides of (2.4) we have

$$p_\Lambda(x,D)(e^{-\Lambda})(x'',D)q(x'',D)(\Psi_1(x')\Psi_2(x'')u(x))$$
$$= (e^{-\Lambda})(x'',D)(\Psi_1(x')\Psi_2(x'')p(x,D)u(x))$$
$$+ (e^{-\Lambda})(x'',D)([p(x,D),\Psi_1(x')\Psi_2(x'')]u(x))$$
$$+ (e^{-\Lambda})(x'',D)p(x,D)((e^{\Lambda})(x'',D)e^{-\Lambda}(x'',D)q(x'',D)-I)$$
$$\times (\Psi_1(x')\Psi_2(x'')u(x))$$
$$\equiv f_1+f_2+f_3.$$

Put $v_\delta = (e^{-\Lambda})(x'',D)q(x'',D)(\Psi_1(x')\Psi_2(x'')u(x))$. Then we have

$$p_\Lambda(x,D)v_\delta(x) = f_1+f_2+f_3.$$

Since $\Psi_1(x')\Psi_2(x'')p(x,D)u(x)\in H_\infty$, there is a constant C such that

$$\|f_1\|_{l_2}\le C \quad \text{for } 0\le\delta\le 1.$$

Here and after the constants do not depend on δ unless stated. By (2.3) we have

$$\|f_3\|_{l_2}\le C \quad \text{for } 0\le\delta\le 1.$$

As for f_2, we know that

$$[p(x,D),\Psi_1(x')\Psi_2(x'')]u(x)$$
$$= [p(x,D),\Psi_1(x')]\Psi_2(x'')u(x) + \Psi_1(x')[p(x,D),\Psi_2(x'')]u(x),$$

and

$$\operatorname{supp}\sigma([p(x,D),\Psi_1(x')]\Psi_2(x''))\subset\subset (U_1'\backslash U_2')\times U_1''\times \boldsymbol{R}^n \bmod S^{-\infty},$$
$$\operatorname{supp}\sigma(\Psi_1(x')[p(x,D),\Psi_2(x'')])\subset\subset U_1'\times(U_1''\backslash U_2'')\times \boldsymbol{R}^n \bmod S^{-\infty}.$$

In virtue of (A-3), we have

$$u\in C^\infty \quad \text{in } (U_1'\backslash U_2')\times U_1''.$$

Therefore there exists a constant C such that

$$\|(e^{-\Lambda})(x'',D)[p(x,D),\Psi_1(x')]\Psi_2(x'')u\|_{l_2}\le C \quad \text{for } 0\le\delta\le 1.$$

For $x''\in U_1''\backslash U_2''$

$$|e^{-\Lambda(x'',\xi)}|\le\langle\xi\rangle^{s-2a\varepsilon} \quad \text{for } 0\le\delta\le 1.$$

Then by (2.2) we obtain, with some $C > 0$,

$$\|(e^{-\Lambda})(x'', D)\Psi_1(x')[p(x, D), \Psi_2(x')]u\|_{l_2} \leq C \quad \text{for } 0 \leq \delta \leq 1.$$

Therefore there is a constant C such that

$$\|f_2\|_{l_2} \leq C \quad \text{for } 0 \leq \delta \leq 1.$$

Hence, we have

$$\|p_\Lambda(x, D)v_\delta\|_{l_2} \leq C \quad \text{for } 0 \leq \delta \leq 1.$$

Let $\Psi \in C_0^\infty(U_0)$ satisfy

$$\Psi(x) = 1 \quad \text{in } U_1.$$

Then

$$\|p_\Lambda(x, D)(\Psi(x)v_\delta(x))\|_{l_2} \leq C \quad \text{for } 0 \leq \delta \leq 1.$$

If $0 < \delta \leq 1$ then

$$\Psi(x)v_\delta(x) \in H_{s'-s+2N} \subset H_{\max\{l_1-1, l_2+m, l_3\}}.$$

Therefore by using an inequality (2.1) with $u = \Psi v_\delta$, we have

$$\|\Psi v_\delta\|_{l_1} \leq C(\|p_\Lambda(x, D)\Psi v_\delta\|_{l_2} + \|\Psi v_\delta\|_{l_1-1} + \|(1 - \chi(x'))\Psi v_\delta\|_{l_3}),$$

for $0 < \delta \leq \delta_0$. Since $\Psi(x')\Psi(x'')u(x)$ belongs to C^∞ in $\{x' \neq 0\}$, we have

$$\|(1 - \chi(x'))\Psi v_\delta\|_{l_3} \leq C' \quad \text{for } 0 \leq \delta \leq 1$$

with some $C' > 0$. We can find a constant C'' so that

$$C\|\Psi v_\delta\|_{l_1-1} \leq \frac{1}{2}\|\Psi v_\delta\|_{l_1} + C''\|u\|_{s'}.$$

Then we obtain, with another constant C,

$$\|\Psi v_\delta\|_{l_1} \leq C \quad \text{for } 0 < \delta \leq \delta_0.$$

Therefore, we have

$$\|v_\delta\|_{l_1} \leq C \quad \text{for } 0 < \delta \leq \delta_0,$$

modifying C if necessary. This means that $\{v_\delta\}$ is bounded in a Hilbert space H_{l_1}. So we can see that there exists a subsequence which converges weakly in H_{l_1}. Therefore we have

$$v_0 = (e^{-\Lambda_0})(x'', D)q(x'', D; 0)\Psi_1(x')\Psi_2(x'')u(x) \in H_{l_1}.$$

Let $U_3(\subset\subset U_2)$ be a neighborhood of 0 satisfying

$$\varphi(x'') < \varepsilon \quad \text{for } x \in U_3.$$

Then

$$e^{\Lambda_0(x'',\xi)} \leq \langle\xi\rangle^{-s+a\varepsilon} \leq \langle\xi\rangle^{l_1-\tau}$$

for $x \in U_3$. Since $(e^{\Lambda_0})(x'', D)v_0 - \Psi_1(x')\Psi_2(x'')u(x) \in H_\infty$, we have

$$u(x) \in H^\tau \quad \text{in } U_3,$$

which implies that

$$x^0 \notin \operatorname{sing\,supp} u.$$

This completes the proof of Proposition 2.1. □

Next, we shall give the reduction as mentioned before. In addition to (A-3), we assume that $p(x,\xi)$ satisfies (A-0)–(A-2) and (A-4). Put

$$\tilde{p}(x,D) = \langle D\rangle^{-m/2+1} p(x,D)\langle D\rangle^{-m/2+1},$$

$$a_2(x,\xi) = \langle\xi\rangle^{-m+2} p_m(x,\xi),$$

$$a_1(x,\xi) = \langle\xi\rangle^{-m+2} p_{m-1}(x,\xi),$$

$$a_0(x,\xi) = \langle\xi\rangle^{-m+2} p_{m-2}(x,\xi) - \frac{1}{4}\sum_{j,k=1}^{n} (\partial_{\xi_j}\partial_{\xi_k}\langle\xi\rangle^{-m/2+1})(\partial_{x_j}\partial_{x_k} p_m(x,\xi))\langle\xi\rangle^{-m/2+1}$$

$$+\frac{1}{4}\sum_{j,k=1}^{n} (\partial_{\xi_j}\langle\xi\rangle^{-m/2+1})(\partial_{x_j}\partial_{x_k} p_m(x,\xi))(\partial_{\xi_k}\langle\xi\rangle^{-m/2+1})$$

and

$$a(x,\xi) = a_2(x,\xi) + a_1(x,\xi) + a_0(x,\xi),$$

$$b(x,\xi) = \tilde{p}(x,\xi) - a(x,\xi).$$

Then we have $a(x,\xi) \in S^2$, $b(x,\xi) \in S^{-1}$ and

$$\tilde{p}(x,D) = a(x,D) + b(x,D).$$

Since $\langle D\rangle$ is elliptic, $p(x,D)$ is hypoelliptic at 0 if $\tilde{p}(x,D)$ is hypoelliptic at 0.

By (A-4) there is a constant C' such that

$$\operatorname{Re} a(x,\xi) \geq \langle\xi\rangle^{-m+2}\left(s(x,\xi) - C'\sum_{j,k=1}^{n}\{\nu s(x,\xi) + C_\nu\langle\xi\rangle^{m-3}\}\right),$$

$$\geq \langle\xi\rangle^{-m+2}\{(1-C'\nu)s(x,\xi) - C'C_\nu\langle\xi\rangle^{m-3}\}$$

if $\nu > 0$ and $x \in U$. We choose $\nu > 0$ so that $C'\nu \le 1/2$. Then we have

$$\operatorname{Re} a(x,\xi) \ge \tfrac{1}{2}\langle\xi\rangle^{-m+2} s(x,\xi) - C''\langle\xi\rangle^{-1}.$$

By virtue of (A-1), we see that $a(x,\xi)$ satisfies the following:

($\tilde{\text{A}}$-1) There exist a neighborhood U of 0 and a constant C such that

$$\operatorname{Re} a(x,\xi) \ge -C\langle\xi\rangle^{-1} \quad (x \in U).$$

By the definition of $a(x,\xi)$, we have for $u \in C_0^\infty(U)$

$$\begin{aligned}\operatorname{Re}(a(x,D)u,u) &= \operatorname{Re}(\langle D\rangle^{-m/2+1} p(x,D)\langle D\rangle^{-m/2+1}u,u) - \operatorname{Re}(b(x,D)u,u)\\ &= \operatorname{Re}(p(x,D)\langle D\rangle^{-m/2+1}u, \langle D\rangle^{-m/2+1}u) - \operatorname{Re}(b(x,D)u,u). \qquad (2.5)\end{aligned}$$

Let U_1 be a neighborhood of 0 satisfying $U_1 \subset\subset U$, and choose $\chi \in C_0^\infty(U)$ so that $\chi(x) = 1$ near $\overline{U_1}$. Then for each s there exists $C_s > 0$ such that

$$\|(1-\chi)\langle D\rangle^{-m/2+1}u\|_s \le C_s\|u\|_{-1} \quad \text{for } u \in C_0^\infty(U_1).$$

Assume that $u \in C_0^\infty(U_1)$, and put $v = \chi\langle D\rangle^{-m/2+1}u$. Then we have

$$\begin{aligned}\operatorname{Re}(a(x,D)u,u) &\ge \operatorname{Re}(p(x,D)u,u) - \operatorname{Re}(b(x,D)u,u) - C(\|u\|_{-1}^2 + \|v\|_{m/2-2}^2)\\ &\ge c_0\|v\|_{m/2-1}^2 - C'(\|u\|_{-1/2}^2 + \|v\|_{m/2-2}^2)\\ &\ge \frac{c_0}{2}\|u\|_0^2 - c_0\|(1-\chi)\langle D\rangle^{-m/2+1}u\|_{m/2-1}^2 - C''\|u\|_{-1/2}^2\\ &\ge \frac{c_0}{2}\|u\|_0^2 - C_0\|u\|_{-1/2}^2.\end{aligned}$$

Therefore the following condition is satisfied:

($\tilde{\text{A}}$-2) There exist a neighborhood U_1 of 0 and constants $c_0 > 0$ and C_0 such that

$$\operatorname{Re}(a(x,D)u,u) \ge \frac{c_0}{2}\|u\|_0^2 - C_0\|u\|_{-1/2}^2 \quad \text{for } u \in C_0^\infty(U_1).$$

By (A-3) we see that

($\tilde{\text{A}}$-3) there exists a neighborhood U of 0 such that

$$a_2^0(x,\xi) \ne 0 \quad \text{if } x = (x',x'') \in U, \quad |\xi| = 1 \text{ and } x' \ne 0,$$

where

$$a_2^0(x,\xi) = |\xi|^{-m+2} p_m(x,\xi) \quad \text{for } |\xi| \ge 1.$$

Next we consider (A-4). Let $|\alpha|+|\beta|=2$ and $\alpha'=0$. Then

$$
\begin{aligned}
&(\log\langle\xi\rangle)^{|\alpha|}|a^{(\alpha)}_{2(\beta)}(x,\xi)|\langle\xi\rangle^{-|\beta|}\\
&\quad=(\log\langle\xi\rangle)^{|\alpha|}|\partial_\xi^\alpha(\langle\xi\rangle^{-m+2}p_{m(\beta)}(x,\xi))|\langle\xi\rangle^{-|\beta|}\\
&\quad\le\langle\xi\rangle^{-m+2}(\log\langle\xi\rangle)^{|\alpha|}p^{(\alpha)}_{m(\beta)}(x,\xi)\langle\xi\rangle^{-|\beta|}\\
&\qquad+\sum_{\substack{\alpha^1+\alpha^2=\alpha\\ \alpha^1>0}}\frac{\alpha!}{\alpha^1!\alpha^2!}(\log\langle\xi\rangle)^{|\alpha|}|\partial_\xi^{\alpha^1}\langle\xi\rangle^{-m+2}|\,|p^{(\alpha^2)}_{m(\beta)}(x,\xi)|\langle\xi\rangle^{-|\beta|}\\
&\quad\le\nu\langle\xi\rangle^{-m+2}s(x,\xi)+C_\nu\langle\xi\rangle^{-1}+C_1(\log\langle\xi\rangle)^{|\alpha|}\sqrt{p_m(x,\xi)}\langle\xi\rangle^{-m/2}\\
&\qquad+C_2(\log\langle\xi\rangle)^2|p_m(x,\xi)|\langle\xi\rangle^{-m} \qquad (2.6)
\end{aligned}
$$

if $\nu>0$, where C_1 and C_2 are some positive constants. Note that

$$
\begin{aligned}
(\log\langle\xi\rangle)^{|\alpha|}\sqrt{p_m(x,\xi)}\langle\xi\rangle^{-m/2}&=\sqrt{p_m(x,\xi)\langle\xi\rangle^{-m+2\varepsilon}}\langle\xi\rangle^{-\varepsilon}(\log\langle\xi\rangle)^{|\alpha|}\\
&\le\frac{1}{2}(p_m(x,\xi)\langle\xi\rangle^{-m+2\varepsilon}+\langle\xi\rangle^{-2\varepsilon}(\log\langle\xi\rangle)^{2|\alpha|})\\
&\le\frac{1}{2}s(x,\xi)\langle\xi\rangle^{-m+2\varepsilon}+C'\langle\xi\rangle^{-1+2\varepsilon}\\
&\quad+\frac{1}{2}\langle\xi\rangle^{-2\varepsilon}(\log\langle\xi\rangle)^{2|\alpha|}
\end{aligned}
$$

for $\varepsilon>0$. Let $\varepsilon=1/3$. Then we have

$$(\log\langle\xi\rangle)^{|\alpha|}\sqrt{p_m(x,\xi)}\langle\xi\rangle^{-m/2}\le\frac{1}{2}s(x,\xi)\langle\xi\rangle^{-m+2}+C''\langle\xi\rangle^{-1/3}.$$

Therefore for any $\nu>0$ there are constants C_ν and C'_ν such that

$$
\begin{aligned}
(\log\langle\xi\rangle)^{|\alpha|}|a^{(\alpha)}_{2(\beta)}(x,\xi)|\langle\xi\rangle^{-|\beta|}&\le\nu s(x,\xi)\langle\xi\rangle^{-m+2}+C_\nu\langle\xi\rangle^{-1/3}\\
&\le2\nu\,\mathrm{Re}\,a(x,\xi)+C'_\nu\langle\xi\rangle^{-1/3}.
\end{aligned}
$$

Similarly we can deal with (ii) in (A-4). Then we have the following:

($\tilde{\text{A}}$-4) There is a constant $C_\nu>0$ such that

(i) $$\sum_{\substack{|\alpha|+|\beta|=2\\ \alpha'=0}}(\log\langle\xi\rangle)^{|\alpha|}|a^{(\alpha)}_{2(\beta)}(x,\xi)|\langle\xi\rangle^{-|\beta|}\le\nu\,\mathrm{Re}\,a(x,\xi)+C_\nu\langle\xi\rangle^{-1/3},$$

(ii) $$\sum_{\substack{|\alpha|+|\beta|=1 \\ \alpha'=0}} (\log\langle\xi\rangle)^{|\alpha|}|\mathrm{Im}\, a_{1(\beta)}^{(\alpha)}(x,\xi)|\langle\xi\rangle^{-|\beta|} \le \nu \mathrm{Re}\, a(x,\xi) + C_\nu\langle\xi\rangle^{-1/3}$$

if $x \in U$.

Therefore, in order to prove Theorem 1.2 it suffices to show that $\tilde{p}(x,D)$ is hypoelliptic at 0 under ($\tilde{\mathrm{A}}$-1)–($\tilde{\mathrm{A}}$-4).

We need a simple variant of the Fefferman-Phong inequality to prove Theorem 1.2.

PROPOSITION 2.2. *Let $q(x,\xi) \in S^2$ satisfy*

$$|q_{(\beta)}^{(\alpha)}(x,\xi)| \le C_{\alpha,\beta}\langle\xi\rangle^{2-|\alpha|} \quad \textit{for } (x,\xi) \in \mathbf{R}^n \times \mathbf{R}^n.$$

Let U and U_1 be open sets in $\mathbf{R}^n$ satisfying $U_1 \subset\subset U$. If $q(x,\xi) \ge 0$ for $x \in U$, then there exists a constant $C \equiv C(\{C_{\alpha,\beta}\}, U, U_1)$ such that

$$(q(x,D)u,u) \ge -C\|u\|_0^2 \quad \textit{for } u \in C_0^\infty(U_1). \tag{2.7}$$

PROOF. We choose a cut-off function $\chi \in C_0^\infty(\mathbf{R}^n)$ so that

$$\chi(x) = 1 \quad \text{in a neighborhood of } \overline{U_1}.$$

Then

$$(q(x,D)u,u) = (\chi(x)q(x,D)u,u) + ((1-\chi(x))q(x,D)u,u).$$

Since $\chi(x)q(x,\xi) \ge 0$, we can apply the Fefferman-Phong inequality. So there exists a constant C such that

$$(\chi(x)q(x,D)u,u) \ge -C\|u\|_0^2 \quad \text{for } u \in C_0^\infty(U_1).$$

On the other hand,

$$((1-\chi(x))q(x,D)u,u) = 0 \quad \text{for } u \in C_0^\infty(U_1),$$

since $\chi = 1$ in a neighborhood of $\overline{U_1}$ and $u \in C_0^\infty(U_1)$. Therefore we obtain the estimate (2.7). □

3. Proof of Theorem 1.2

In this section, we shall show that $\tilde{p}(x,D)$ is hypoelliptic at 0 applying Proposition 2.1. Put

$$\tilde{p}_\Lambda(x,D) = (e^{-\Lambda})(x'',D)\tilde{p}(x,D)(e^{\Lambda})(x'',D).$$

Then we can write

$$\begin{aligned}\tilde{p}_\Lambda(x,\xi) &= e^{-\Lambda(x'',\xi)}\sharp(a(x,\xi)+b(x,\xi))\sharp e^{\Lambda(x'',\xi)}\\ &= a(x,\xi)+b(x,\xi)+i\{\Lambda,a\}(x,\xi)-\frac{1}{2}(\mathrm{Hess}\,a)(-H_\Lambda)\\ &\quad+\frac{1}{2}\sum_{j,k=r+1}^{n}(\Lambda_{x_jx_k}\Lambda_{\xi_k}-\Lambda_{x_j\xi_k}\Lambda_{x_k})a_{\xi_j}(x,\xi)\\ &\quad+\frac{1}{2}\sum_{j=1}^{n}\sum_{k=r+1}^{n}(\Lambda_{\xi_j\xi_k}\Lambda_{x_k}-\Lambda_{\xi_jx_k}\Lambda_{\xi_k})a_{x_j}(x,\xi)\\ &\quad+\frac{1}{4}\sum_{j=r+1}^{n}\{\Lambda_{\xi_j},\Lambda_{x_j}\}a(x,\xi)+r_1(x,\xi),\end{aligned}$$

where $a(x,\xi)\sharp b(x,\xi)=\sigma(a(x,D)b(x,D))(x,\xi)$, $\{a,b\}(x,\xi)=\sum_{j=1}^{n}\{a_{\xi_j}(x,\xi)b_{x_j}(x,\xi) - a_{x_j}(x,\xi)b_{\xi_j}(x,\xi)\}$, $(\mathrm{Hess}\,a)(x,\xi)$ stands for the Hessian matrix of $a(x,\xi)$, $(\mathrm{Hess}\,a)(\delta z)={}^t\delta z(\mathrm{Hess}\,a)(x,\xi)\delta z$, H_Λ does the Hamilton vector field of $\Lambda(x,\xi)$, $\Lambda_{x_j}(x,\xi)=(\partial/\partial x_j)\Lambda(x,\xi)$, $\Lambda_{x_jx_k}(x,\xi)=\partial^2/(\partial x_j\partial x_k)\Lambda(x,\xi)$ and $r_1(x,\xi)\in\bigcap_{\varepsilon>0}S^{-1+\varepsilon}$.

Let

$$A=(\mathrm{Re}\,a)(x,D)+C_0\langle D\rangle^{-1},$$

where C_0 is the constant in ($\tilde{\mathrm{A}}$-2). Let U and U_1 be neighborhoods of 0 which appeared in ($\tilde{\mathrm{A}}$-1)–($\tilde{\mathrm{A}}$-4). We may assume that $U_1\subset\subset U$. Then by ($\tilde{\mathrm{A}}$-2) we have

$$(Au,u)\geq\frac{c_0}{2}\|u\|_0^2\quad\text{for } u\in C_0^\infty(U_1). \tag{3.1}$$

Further, we have

$$\begin{aligned}\mathrm{Re}(\tilde{p}_\Lambda(x,D)u,u) &\geq (Au,u)-C\|u\|_{-1/4}^2\\ &\quad-\frac{1}{4}\sum_{j=r+1}^{n}|(\mathrm{Op}(\{\Lambda_{\xi_j},\Lambda_{x_j}\}a_2)u,u)|\\ &\quad-\frac{1}{2}\sum_{j=1}^{n}\sum_{k=r+1}^{n}|\mathrm{Op}((\Lambda_{\xi_j\xi_k}\Lambda_{x_j}-\Lambda_{\xi_jx_k}\Lambda_{\xi_k})a_{x_j})u,u)|\\ &\quad-\frac{1}{2}\sum_{j,k=r+1}^{n}|\mathrm{Op}((\Lambda_{x_jx_k}\Lambda_{\xi_j}-\Lambda_{x_j\xi_k}\Lambda_{x_k})a_{\xi_j})u,u)|\\ &\quad-\left|\left(\mathrm{Op}\left(\{\Lambda,\mathrm{Im}\,a_1\}+\frac{1}{2}(\mathrm{Hess}\,a)(-H_\Lambda)\right)u,u\right)\right|\\ &\equiv (Au,u)-C\|u\|_{-1/4}^2-\frac{1}{4}I_1-\frac{1}{2}I_2-\frac{1}{2}I_3-I_4,\end{aligned} \tag{3.2}$$

where $\mathrm{Op}(q)$ denotes the pseudo-differential operator with the Weyl symbol $q(x,\xi)$.

As for I_1, Schwarz' inequality shows that

$$|(Au,v)| \le (Au,u)^{1/2}(Av,v)^{1/2}$$

for $u,v \in C_0^\infty(U_1)$. Let $u \in C_0^\infty(U_1)$. Since $\{\Lambda_{\xi_j},\Lambda_{x_j}\} \in S^{-2+\varepsilon}$ $(\varepsilon > 0)$, we obtain

$$\begin{aligned} I_1 &\le \sum_{j=r+1}^{n} |(Au, \mathrm{Op}(\{\Lambda_{\xi_j},\Lambda_{x_j}\})u| + C\|u\|_{-1/4}^2 \\ &\le C(Au,u)^{1/2}\|u\|_{-1/4} + C\|u\|_{-1/4}^2. \end{aligned} \tag{3.3}$$

Therefore for any $\nu > 0$ there is a constant C_ν such that

$$I_1 \le \nu(Au,u) + C_\nu\|u\|_{-1/4}^2.$$

Next we estimate I_2. We can see that

$$I_2 \le C\sum_{j=1}^{n}\sum_{k=r+1}^{n} \|\mathrm{Op}(\langle\xi\rangle^{-1}\operatorname{Re} a_{x_j})u\|_0\|u\|_{-1/2},$$

and

$$\|\mathrm{Op}(\langle\xi\rangle^{-1}\operatorname{Re} a_{x_j})u\|_0^2 = (\mathrm{Op}(\langle\xi\rangle^{-1}\operatorname{Re} a_{x_j})\,\mathrm{Op}(\langle\xi\rangle^{-1}\operatorname{Re} a_{x_j})u,u).$$

Set

$$c(x,\xi) = (\langle\xi\rangle^{-1}\operatorname{Re} a_{x_j}(x,\xi))\sharp(\langle\xi\rangle^{-1}\operatorname{Re} a_{x_j}(x,\xi)).$$

We have

$$c(x,\xi) = \langle\xi\rangle^{-2}(\operatorname{Re} a_{x_j}(x,\xi))^2 + r_1(x,\xi),$$

where $r_1(x,\xi) \in S^0$. We choose a constant C so that

$$\operatorname{Re} a(x,\xi) + C\langle\xi\rangle^{-1} \ge 0 \quad \text{for } x \in U,$$

that is, C is just the same appeared in ($\tilde{\mathrm{A}}$-1). We write

$$c(x,\xi) = \langle\xi\rangle^{-2}\{(\operatorname{Re} a(x,\xi) + C\langle\xi\rangle^{-1})_{x_j}\}^2 + r_1(x,\xi).$$

Therefore

$$|c(x,\xi)| \le C'(\operatorname{Re} a(x,\xi) + 1) \quad (x \in U).$$

So we obtain

$$C'(\mathrm{Re}\, a(x,\xi)+1) \pm c(x,\xi) \geq 0 \quad (x \in U).$$

Applying Proposition 2.2 we have

$$((C'\,\mathrm{Op}(\mathrm{Re}\, \dot{a}) \pm c(x,D))u,u) \geq -C\|u\|_0^2.$$

This shows that

$$|(c(x,D)u,u)| \leq C'(Au,u) + C\|u\|_0^2.$$

So we have

$$I_2 \leq \nu((Au,u) + \|u\|_0^2) + C_\nu \|u\|_{-1/4}^2. \tag{3.4}$$

As for I_3, we have

$$I_3 \leq C \sum_{j,k=r+1}^{n} \|\mathrm{Op}(\mathrm{Re}\, a_{\xi_j})u\|_0 \|u\|_{-1/2}.$$

Let

$$\tilde{c}(x,\xi) = (\mathrm{Re}\, a_{\xi_j}(x,\xi)) \sharp (\mathrm{Re}\, a_{\xi_j}(x,\xi)).$$

Then

$$\begin{aligned}\tilde{c}(x,\xi) &= (\mathrm{Re}\, a_{\xi_j}(x,\xi))^2 + r_2(x,\xi)\\ &= \{(\mathrm{Re}\, a(x,\xi) + C\langle\xi\rangle^{-1})_{\xi_j}\}^2 + r_2'(x,\xi),\end{aligned}$$

where $r_2(x,\xi)$, $r_2'(x,\xi) \in S^0$ and the constant C is as in ($\tilde{\mathrm{A}}$-1). Therefore we have

$$|\tilde{c}(x,\xi)| \leq C'(\mathrm{Re}\, a(x,\xi)+1) \quad (x \in U),$$

for some C'. This gives

$$I_3 \leq \nu((Au,u) + \|u\|_0^2) + C_\nu \|u\|_{-1/4}^2. \tag{3.5}$$

Choose $\Psi(\xi) \in C^\infty(\boldsymbol{R}^n)$ so that

$$\Psi(\xi) = \begin{cases} 1 & (|\xi| \geq 2)\,, \\ 0 & (|\xi| \leq 1). \end{cases}$$

For $0 < \nu \leq 1$ we put

$$q_\nu^\pm(x,\xi) = \left(\nu\, \mathrm{Re}\, a(x,\xi) + C_\nu\langle\xi\rangle^{-1} \pm \frac{1}{\nu}\{\Lambda, \mathrm{Im}\, a_1\} \pm \frac{1}{2\nu}(\mathrm{Hess}\, a)(-H_\Lambda)\right)\Psi(s_\nu\xi),$$

where the s_ν satisfy $0 < s_\nu \leq 1$ and are determined later. By virtue of ($\tilde{\mathrm{A}}$-4) we can choose C_ν so that

$$q_\nu^\pm(x,\xi) \geq 0 \quad (x \in U).$$

Therefore we have

$$|q_{\nu(\beta)}^{\pm(\alpha)}(x,\xi)| \leq \langle\xi\rangle^{2-|\alpha|}\left(C_{\alpha,\beta} + C_\nu s_\nu^3 C_{\alpha,\beta} + \frac{s_\nu^{3/2}}{\nu} C_{\alpha,\beta}\right).$$

Now we choose s_ν so that $C_\nu s_\nu^3 \leq 1$, $\frac{1}{\nu} s_\nu^{3/2} \leq 1$. Then

$$|q_{\nu(\beta)}^{\pm(\alpha)}(x,\xi)| \leq C_{\alpha,\beta}\langle\xi\rangle^{2-|\alpha|},$$

where $C_{\alpha,\beta}$ are independent of ν. Therefore by Proposition 2.2 we have

$$(q_\nu^\pm(x,D)u,u) \geq -C\|u\|_0^2,$$

where C does not depend on ν. Therefore

$$\begin{aligned}&|(\mathrm{Op}(\{\Lambda, \mathrm{Im}\, a_1\} + \tfrac{1}{2}(\mathrm{Hess}\, a)(-H_\Lambda))u,u)| \\ &\quad \leq \nu^2(\mathrm{Op}(\mathrm{Re}\, a)u,u) + \nu C_\nu\|u\|_{-1/2}^2 + \nu C\|u\|_0^2 + C\|u\|_{-1}^2.\end{aligned}$$

Thus,

$$I_4 \leq \nu((Au,u) + \|u\|_0^2) + C_\nu\|u\|_{-1/2}^2. \tag{3.6}$$

Consequenyly, by (3.1)–(3.6) we have

$$\mathrm{Re}(\tilde{p}_\Lambda(x,D)u,u) \geq \frac{c_0}{4}\|u\|_0^2 - C\|u\|_{-1/4}^2. \tag{3.7}$$

Schwarz' inequality gives

$$\mathrm{Re}(\tilde{p}_\Lambda(x,D)u,u) \leq C\|\tilde{p}_\Lambda(x,D)u\|_0^2 + \frac{c_0}{8}\|u\|_0^2. \tag{3.8}$$

Therefore in virtue of (3.7) and (3.8), there is a constant C such that

$$\|u\|_0 \leq C(\|\tilde{p}_\Lambda(x,D)u\|_0 + \|u\|_{-1}).$$

Applying Proposition 2.1 with $x^0 \in U_1$, we see that $\tilde{p}(x,D)$ is hypoelliptic at 0. This completes the proof of Theorem 1.2. □

4. Remarks and Examples

In this section we shall first study the conditions which we impose on $p(x,\xi)$. Finally we shall give several examples.

PROPOSITION 4.1. *If* (A-0), (A-1)$'$ *and* (A-2)$'$, *then* (A-1) *and* (A-2) *hold.*

PROOF. It is obvious that (A-1) holds. Without loss of generality, we may assume that

$$p_m(0,\xi) \equiv 0 \quad \text{for } |\xi| \geq 1.$$

By using Taylor expansion and (A-2)$'$, we have

$$p_m(x,\xi) = \sum_{|\beta|=3} \frac{3x^\beta}{\beta!} \int_0^1 (1-\theta)^3 (\partial_x^\beta p_m)(\theta x, \xi)\, d\theta.$$

Changing the variable x to y so that $x = \nu y$ where $0 < \nu \leq 1$, we write

$$v_\nu(y) = u(\nu y) \quad \text{for } u \in C_0^\infty.$$

Let B be a unit ball centered at 0, $\chi(x) \in C_0^\infty(B)$ with $\chi(x) = 1$ in $|x| \leq 2/3$, and choose $0 < \nu_0 \leq 1$ so that $\nu_0 B \subset U$, where U is a neighborhood of 0 in ($\tilde{\text{A}}$-1). For ν with $0 < \nu \leq \nu_0$ we put

$$p_{m,\nu}(x,\xi) = \chi\left(\frac{x}{\nu}\right) p_m(x,\xi)\Psi(\nu\xi),$$

where $\Psi(\xi)$ is the symbol used in Section 3. Then

$$\begin{aligned} p_{m,\nu}(x,D)u|_{x=\nu y} &= (2\pi)^{-n} \int \left(\int e^{i\nu(y-\tilde{y})\cdot\xi} p_{m,\nu}\left(\frac{\nu(y+\tilde{y})}{2}, \xi\right) v_\nu(\tilde{y}) \nu^n \, d\tilde{y} \right) d\xi \\ &= (2\pi)^{-n} \int \left(\int e^{i(y-\tilde{y})\cdot\eta} p_{m,\nu}\left(\frac{\nu(y+\tilde{y})}{2}, \frac{\eta}{\nu}\right) v_\nu(\tilde{y}) \, d\tilde{y} \right) d\eta \\ &\equiv q_\nu(y, D_y) v_\nu(y). \end{aligned}$$

Thus we have

$$\begin{aligned} q_\nu(y,\eta) &= p_{m,\nu}\left(\nu y, \frac{\eta}{\nu}\right) \\ &= \chi(y) p_m\left(\nu y, \frac{\eta}{\nu}\right)\Psi(\eta) \\ &= \nu^{-m} \chi(y) p_m(\nu y, \eta)\Psi(\eta) \\ &= \nu^{-m+3} \sum_{|\beta|=3} \frac{3y^\beta}{\beta!} \chi(y) \int_0^1 (1-\theta)^3 (\partial_x^\beta p_m)(\theta \nu y, \eta)\, d\theta \Psi(\eta), \end{aligned}$$

and set

$$\tilde{q}_\nu(y,\eta) = \nu^{m-3} q_\nu(y,\eta).$$

Then

$$\tilde{q}_\nu(y,\eta) \geq 0 \quad \text{for } (y,\eta) \in \boldsymbol{R}^n \times \boldsymbol{R}^n.$$

Further we have

$$|\tilde{q}^{(\alpha)}_{\nu(\beta)}(y,\eta)| \leq C_{\alpha,\beta} \langle \eta \rangle^{m-|\alpha|},$$

where the $C_{\alpha,\beta}$ are independent of ν. Therefore by Proposition 2.2

$$(\tilde{q}_\nu(y, D_y) v_\nu, v_\nu) \geq -C \|v_\nu\|^2_{m/2-1},$$

for $0 < \nu \leq \nu_0$ if $u \in C_0^\infty(U)$ and $v_\nu(y) = u(\nu y)$. Then

$$\begin{aligned}(\tilde{q}_\nu(y, D_y) v_\nu, v_\nu) &= \nu^{m-3}((p_{m,\nu}(x, D)u|_{x=\nu y}, u(\nu y)) \\ &= \nu^{m-3-n}((p_{m,\nu}(x, D)u, u) \quad \text{for } u \in C_0^\infty(U).\end{aligned}$$

On the other hand,

$$\begin{aligned}\|v_\nu\|^2_{m/2-1} &= (2\pi)^{-n} \int \langle \eta \rangle^{m-2} |\hat{v}_\nu(\eta)|^2 \, d\eta \\ &= (2\pi)^{-n} \int \langle \nu\xi \rangle^{m-2} |\hat{v}_\nu(\nu\xi)|^2 \nu^n \, d\xi \\ &= (2\pi)^{-n} \int \langle \nu\xi \rangle^{m-2} |\hat{u}(\xi)|^2 \nu^{-n} \, d\xi.\end{aligned}$$

Since

$$\sqrt{1 + \nu^2|\xi|^2} = \nu \sqrt{(1 + |\xi|^2) + \frac{1}{\nu^2} - 1},$$

we have

$$\langle \nu\xi \rangle^{m-2} \leq (2\nu)^{m-2} \left(\langle \xi \rangle^{m-2} + \left(\frac{1}{\nu^2} - 1 \right)^{m/2-1} \right) \quad \text{if } m \geq 2.$$

If $m < 2$, then

$$\langle \nu\xi \rangle^{m-2} \leq \nu^{m-2} \langle \xi \rangle^{m-2}.$$

Therefore we obtain

$$\|v_\nu\|_{m/2-1}^2 \le \begin{cases} \nu^{m-2-n}2^{m-2}\left(\|u\|_{m/2-1}^2 + \left(\dfrac{1}{\nu^2}-1\right)^{m/2-1}\|u\|_0^2\right) & \text{if } m>2, \\ \nu^{-n}\|u\|_0^2 & \text{if } m=2, \\ \nu^{m-2-n}\|u\|_{m/2-1}^2 & \text{if } m<2. \end{cases}$$

Consequently, we have

$$(p_{m,\nu}(x,D)u,u) \ge -\nu C\|u\|_{m/2-1}^2 - C_\nu\|u\|_{m/2-2}^2.$$

Further, there exists a constant $c>0$ such that

$$\operatorname{Re} p_{m-1}(x,\xi) + \operatorname{Re} p_{m-2}(x,\xi) \ge c|\xi|^{m-2},$$

if $x \in U$ and $|\xi| \ge 1$. Then there is a constant C' such that

$$\operatorname{Re}((p_{m-1}+p_{m-2})(x,D)u,u) \ge c\|u\|_{m/2-1}^2 - C'\|u\|_{m/2-2}^2$$

for $u \in C_0^\infty(U)$. Taking ν so that $\nu C < c/2$, we have

$$\begin{aligned}\operatorname{Re}(p(x,D)u,u) &= (p_{m,\nu}(x,D)u,u) + \left(\operatorname{Op}\left(\chi\left(\frac{x}{\nu}\right)p_m(x,\xi)(1-\Psi(\nu\xi)\right)u,u\right) \\ &\quad + \left(\operatorname{Op}\left(\left(1-\chi\left(\frac{x}{\nu}\right)\right)p_m(x,\xi)\right)u,u\right) + \operatorname{Re}((p(x,D)-p_m(x,D))u,u) \\ &\ge \frac{c}{2}\|u\|_{m/2-1}^2 - C_\nu\|u\|_{m/2-2} \quad \text{if } u \in C_0^\infty\left(\frac{\nu}{2}B\right),\end{aligned}$$

since $\operatorname{Op}(\chi(x/\nu)p_m(x,\xi)(1-\Psi(\nu\xi))$ and $\operatorname{Op}((1-\chi(x/\nu))p_m(x,\xi)$ are in $S^{-\infty}$. Therefore we know that (A-2) holds with U replaced by $\nu/2B$. □

PROPOSITION 4.2. *We assume that* (A-0), (A-1)′ *and*

(A-2)″ there exists a neighborhood U of 0 such that

(i) $p_{m(\beta)}(x,\xi) = 0 \quad (|\beta| = 2)$,

(ii) $\operatorname{Im} p_{m-1(\beta)}(x,\xi) = 0 \quad (|\beta| = 1)$,

if $x \in U$, $|\xi| \ge 1$ and $p_m(x,\xi) = 0$.

Then we have for any $\nu > 0$ there is a constant C_ν such that

$$\sum_{|\beta|=2} |p_{m(\beta)}(x,\xi)|\langle\xi\rangle^{-2} + \sum_{|\beta|=1} |\operatorname{Im} p_{m-1(\beta)}(x,\xi)|\langle\xi\rangle^{-1} \le \nu s(x,\xi) + C_\nu\langle\xi\rangle^{m-3},$$

if $x \in U$.

PROOF. Let

$$V = \{(x,\xi) \in U \times S^{n-1}; p_m(x,\xi) = 0\},$$

and

$$I(x,\xi) = \sum_{|\beta|=2} |p_{m(\beta)}(x,\xi)|\langle\xi\rangle^{-2} + \sum_{|\beta|=1} |\mathrm{Im}\, p_{m-1(\beta)}(x,\xi)|\langle\xi\rangle^{-1},$$

where S^{n-1} denotes the $(n-1)$-dimensional unit sphere. Then

$$I(x,\xi) = 0 \quad \text{in } \bar{V}.$$

Let $\nu > 0$ and V_ν be a neighborhood of $\bar{V}$ in $\boldsymbol{R}^n \times S^{n-1}$ satisfying

$$I(x,\xi) \leq \nu c \quad \text{for } (x,\xi) \in V_\nu,$$

where

$$c = \min_{\substack{x \in \bar{U} \\ |\xi|=1}} \mathrm{Re}\, p_{m-2}(x,\xi) > 0.$$

Then there is a constant $\hat{c}_\nu > 0$ such that

$$p_m(x,\xi) \geq \hat{c}_\nu \quad \text{for } (x,\xi) \in (\bar{U} \times S^{n-1}) \backslash V_\nu.$$

Therefore

$$p_m(x,\xi) \geq (\hat{c}_\nu |\xi|^2)|\xi|^{m-2},$$

if $|\xi| \geq 1$, $(x, \xi/|\xi|) \in (\bar{U} \times S^{n-1}) \backslash V_\nu$. Hence we have

$$I(x,\xi) \leq \nu(p_m(x,\xi) + \mathrm{Re}\, p_{m-1}(x,\xi) + \mathrm{Re}\, p_{m-2}(x,\xi)),$$

if $(x, \xi/|\xi|) \in \bar{U} \times S^{n-1} \backslash V_\nu$ and $|\xi| \gg 1$. This proves Proposition 4.2. □

Thus Proposition 4.1 and 4.2 imply that the operator L_2 defined in Section 1 satisfies (A-0)–(A-4) if it satisfies (B-1) and (B-4). In particular, L_2 is hypoelliptic at 0 under the conditions (B-1) and (B-4).

EXAMPLE 4.3. Let $h_k(x) \in C^\infty$ $(1 \leq k \leq n)$ satisfy $h_k \geq 0$ and

$$h_{k(\beta)}(x) = 0 \quad \text{if } x \in \boldsymbol{R}^n, \quad h_k(x) = 0 \quad \text{and} \quad |\beta| = 2.$$

We assume that there exist constants $C_{kj} > 0$ and $m_{kj} > 0$ such that for any $k,j = 1,\dots,n$

$$h_k(x) \leq C_{kj} h_j(x)^{m_{kj}}.$$

Put

$$p(x,\xi)=\sum_{k=1}^{n}h_k(x)\xi_k^2+1.$$

Then, applying Theorem 1.2 and Proposition 4.2 we can see that $p(x,D)$ is hypoelliptic.

For $\sigma>0$ we put

$$f_\sigma(t)=\begin{cases}\exp\left(-\dfrac{1}{|t|^\sigma}\right) & (t\neq 0),\\ 0 & (t=0).\end{cases}$$

EXAMPLE 4.4. Let $n=2$ and $\sigma>0$. Put

$$p(x,\xi)=x_1^4\xi_1^2+f_\sigma(x_1)\xi_2^2+1.$$

Then $p(x,D)$ is hypoelliptic. Indeed, by Proposition 4.1 $p(x,\xi)$ satisfies (A-0) with $m=2$, (A-1), (A-2) and (A-3) with $r=1$. Note that

$$|f_\sigma'(t)|\le C\sqrt{f_\sigma(t)}\le C(f_\sigma(t)|\xi|^{3/2}+|\xi|^{-3/2})$$

for $|\xi|\neq 0$. Therefore we have

$$\begin{aligned}&f_\sigma(x_1)(\log\langle\xi\rangle)^2+|f_\sigma'(x_1)|\log\langle\xi\rangle\\ &\le\begin{cases}Cx_1^4(1+\xi_1^2)^{1/2}\le C'p(x,\xi)\langle\xi\rangle^{-1} & \text{if } |\xi_1|\ge|\xi_2|,\\ Cp(x,\xi)\langle\xi\rangle^{-1}+Cp(x,\xi)\langle\xi\rangle^{-1/4}+C\langle\xi\rangle^{-1} & \text{if } |\xi_1|\le|\xi_2|.\end{cases}\end{aligned}$$

This implies that $p(x,\xi)$ satisfies (A-4).

EXAMPLE 4.5. Let $n=2$ and $0<\sigma<2$. Put

$$p(x,\xi)=f_\sigma(x_1)\xi_1^2+x_1^4\xi_2^2+1.$$

Let us prove that $p(x,\xi)$ satisfies (A-0)–(A-4). It is obvious that $p(x,\xi)$ satisfies (A-0) with $m=2$, (A-2) and (A-3) with $r=1$. Fix $\nu>0$. Assume that $x_1^4(\log\langle\xi\rangle)^2\ge\nu$. Then we have

$$\begin{aligned}\nu f_\sigma(x_1)\langle\xi\rangle^2&\ge\nu\exp(-\nu^{-\sigma/4}(\log\langle\xi\rangle)^{\sigma/2})\langle\xi\rangle^2\\ &\ge\nu\langle\xi\rangle^{2-\nu^{-\sigma/4}(\log\langle\xi\rangle)^{\sigma/2-1}}\\ &\ge\nu\langle\xi\rangle\end{aligned}$$

if $\langle\xi\rangle \geq \exp(\nu^{-\sigma/(2(2-\sigma))})$. This gives

$$x_1^4(\log\langle\xi\rangle)^2 \leq \nu p(x,\xi) + C_\nu\langle\xi\rangle^{-1} \quad \text{if } |x_1| \leq 1,$$

where C_ν is a constant. Similarly, we have

$$\nu f_\sigma(x_1)\langle\xi\rangle^2 \geq \nu\langle\xi\rangle^{2-\nu^{-\sigma/3}(\log\langle\xi\rangle)^{\sigma/3-1}} \geq \nu\langle\xi\rangle$$

if $|x_1|^3\log\langle\xi\rangle \geq \nu$ and $\langle\xi\rangle \geq \exp(\nu^{-\sigma/(3-\sigma)})$. This gives, with some constant C_ν,

$$|x_1|^3\log\langle\xi\rangle \leq \nu p(x,\xi) + C_\nu\langle\xi\rangle^{-1} \quad \text{if } |x_1| \leq 1.$$

Therefore $p(x,\xi)$ satisfies (A-4), and $p(x,D)$ is hypoelliptic.

EXAMPLE 4.6. Let $n=1$ and $C \in \boldsymbol{R}\backslash\{0\}$. Then

$$p(x,D) = -x^4\partial_x^2 - C^2.$$

does not satisfy (A-2). If we choose

$$u(x) = \begin{cases} x\exp(iCx^{-1}) & (x \neq 0), \\ 0 & (x = 0), \end{cases}$$

then for $\varphi(x) \in C_0^\infty(\boldsymbol{R})$

$$\begin{aligned}\langle p(x,D)u,\varphi\rangle &= \langle(-x^4\partial_x^2 - C^2)u,\varphi\rangle \\ &= -\langle\partial_x^2(x^4u) - 8\partial_x(x^3u) + (12x^2 + C^2)u,\varphi\rangle \\ &= \langle 0,\varphi\rangle.\end{aligned}$$

Therefore

$$p(x,D)u = 0 \quad \text{in } \mathscr{D}'(\boldsymbol{R}).$$

However u is not differentiable at $x=0$, that is,

$$0 \in \operatorname{sing\,supp} u.$$

Hence, $p(x,D)$ is not hypoelliptic at $x=0$.

EXAMPLE 4.7. Let $C \in \boldsymbol{C}$. Then

$$p(x,D) = -x_1^2\Delta + C.$$

does not satisfy (A-4). Put

$$u(x) = (x_1)_+^\lambda = \begin{cases} x_1^\lambda & (x_1 > 0), \\ 0 & (x_1 \leq 0), \end{cases}$$

where $\lambda = (1 + \sqrt{1+4C})/2$ and we take a branch of $\sqrt{1+4C}$ satisfying $\operatorname{Re}\sqrt{1+4C} \geq 0$. Since $\operatorname{Re}\lambda \geq 1/2 > -1$, we have

$$x_1^2 \frac{d^2}{dx_1^2}((x_1)_+^\lambda) = \lambda(\lambda - 1)(x_1)_+^\lambda \quad \text{in } \mathscr{D}'(\boldsymbol{R}).$$

Therefore

$$x_1^2 \partial_{x_1}^2 u(x) = \lambda(\lambda - 1)u(x) \quad \text{in } \mathscr{D}'(\boldsymbol{R}^n).$$

Obviously,

$$\partial_{x_j}^2 u(x) = 0 \quad \text{in } \mathscr{D}'(\boldsymbol{R}), \quad (2 \leq j \leq n).$$

Since $\lambda(\lambda - 1) - C = 0$, we obtain

$$P(x, D)u = 0 \quad \text{in } \mathscr{D}'(\boldsymbol{R}^n).$$

On the other hand, we have

$$0 \in \operatorname{sing\,supp} u.$$

Hence, $p(x, D)$ is not hypoelliptic at $x = 0$.

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