TSUKUBA J. MATH.
Vol. 21 No. 1 (1997), 113-116

A CHARACTERIZATION OF GEODESIC HYPERSPHERES OF QUATERNIONIC PROJECTIVE SPACE

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Abstract. We study a condition that allows us to characterize geodesic hyperspheres among all real hypersurfaces of quaternionic projective space.

1. Introduction

Along this paper M will denote a connected real hypersurface of the quaternionic projective space QP^m, $m \geq 3$, endowed with the metric g of constant quaternionic sectional curvature 4. Let N be a unit local normal vector field on M and $U_i = -J_iN$, $i = 1, 2, 3$, where $\{J_i\}_{i=1,2,3}$ is a local basis of the quaternionic structure of QP^m, [2]. Let us denote by $\boldsymbol{D}^{\perp} = \mathrm{Span}\{U_1, U_2, U_3\}$ and by $\boldsymbol{D}$ its orthogonal complement in TM.

If A denotes the Weingarten endomorphism of M we have the

THEOREM A, [1]. *Let M be a real hypersurface of QP^m, $m \geq 2$. Then $g(A\boldsymbol{D}, \boldsymbol{D}^{\perp}) = \{0\}$ if and only if M is congruent to an open part of one of the following real hypersurfaces of QP^m:*

i) *a geodesic hypersphere,*

ii) *a tube of some radius r, $0 < r < \pi/2$, around the canonically (totally geodesic) embedded quaternionic projective space QP^k, $k \in \{1, \ldots, m-2\}$,*

iii) *a tube of some radius r, $0 < r < \pi/4$, around the canonically (totally geodesic) embedded projective space CP^m.*

Let us denote by R the curvature tensor of M. In [4] we have proved that there do not exist real hypersurfaces of QP^m, $m \geq 2$, such that $\sigma(R(X, Y)AZ) = 0$, for any X, Y, Z tangent to M, where σ denotes the cyclic sum.

Received January 19, 1995.

The purpose of the present paper is to study a weaker condition than the one considered in [4]. Concretely we propose to study real hypersurfaces of QP^m such that

$$\sigma(R(X, Y)AZ) = 0 \tag{1.1}$$

for any $X, Y, Z \in D$. We shall prove the following

THEOREM. *Let M be a real hypersurface of QP^m, $m \geq 3$. Then M satisfies (1.1) if and only if it is congruent to an open part of a geodesic hypersphere of QP^m.*

2. Preliminaries

Let X be a tangent vector field to M. We write $J_iX = \phi_iX + f_i(X)N$, $i = 1,2,3$, where ϕ_iX is the tangent component of J_iX and $f_i(X) = g(X, U_i)$, $i = 1,2,3$. As $J_i^2 = -Id$, $i = 1,2,3$, where Id denotes the identity endomorphism on TQP^m, we get

$$\phi_i^2X = -X + f_i(X)U_i, \quad f_i(\phi_iX) = 0, \quad \phi_iU_i = 0, \quad i = 1,2,3 \tag{2.1}$$

for any X tangent to M. As $J_iJ_j = -J_jJ_i = J_k$, where (i,j,k) is a cyclic permutation of $(1,2,3)$ we obtain

$$\phi_iX = \phi_j\phi_kX - f_k(X)U_j = -\phi_k\phi_jX + f_j(X)U_k \tag{2.2}$$

and

$$f_i(X) = f_j(\phi_kX) = -f_k(\phi_jX) \tag{2.3}$$

for any X tangent to M, where (i,j,k) is a cyclic permutation of $(1,2,3)$. It is also easy to see that for any X, Y tangent to M and $i = 1,2,3$,

$$g(\phi_iX, Y) + g(X, \phi_iY) = 0, \quad g(\phi_iX, \phi_iY) = g(X, Y) - f_i(X)f_i(Y) \tag{2.4}$$

and

$$\phi_iU_j = -\phi_jU_i = U_k \tag{2.5}$$

(i,j,k) being a cyclic permutation of $(1,2,3)$. Finally from the expression of the curvature tensor of QP^m, $m \geq 2$, we have that the curvature tensor of M is given by

$$(2.6)\quad R(X,Y)Z = g(Y,Z)X - g(X,Z)Y + \sum_{i=1}^{3}\{g(\phi_i Y,Z)\phi_i X - g(\phi_i X,Z)\phi_i Y + 2g(X,\phi_i Y)\phi_i Z\} + g(AY,Z)AX - g(AX,Z)AY$$

for any X, Y, Z tangent to M, see [3].

3. Proof of the Theorem

Let $\{E_1,\dots,E_{4m-4}\}$ be an orthonormal basis of D at any point of M.

If in (1.1) we take $Z = E_j$, $Y = \phi_1 E_j$, from (2.6) and applying the formulas (2.1) to (2.5) we have for any $X \in D$

$$(3.1)\quad \begin{aligned} &\{g(\phi_1 X, AE_j) - g(AX,\phi_1 E_j)\}E_j + \{g(AX,E_j) + g(\phi_1 X, A\phi_1 E_j)\}\phi_1 E_j \\ &+ \{2g(AX,\phi_3 E_j) - g(\phi_3 X, AE_j) + g(\phi_2 X, A\phi_1 E_j)\}\phi_2 E_j + \{g(\phi_2 X, AE_j) \\ &+ g(\phi_3 X, A\phi_1 E_j) - 2g(AX,\phi_2 E_j)\}\phi_3 E_j - 2g(X,E_j)\phi_1 AE_j \\ &- 2g(X,\phi_3 E_j)\phi_2 AE_j + 2g(X,\phi_2 E_j)\phi_3 AE_j + 2g(\phi_1 X, E_j)\phi_1 A\phi_1 E_j \\ &+ 2g(\phi_2 X, E_j)\phi_2 A\phi_1 E_j + 2g(\phi_3 X, E_j)\phi_3 A\phi_1 E_j - \{g(E_j, AE_j) \\ &+ g(\phi_1 E_j, A\phi_1 E_j)\}\phi_1 X - \{g(\phi_3 E_j, AE_j) + g(\phi_2 E_j, A\phi_1 E_j)\}\phi_2 X \\ &+ \{g(\phi_2 E_j, AE_j) - g(\phi_3 E_j, A\phi_1 E_j)\}\phi_3 X + 2\phi_1 AX = 0 \end{aligned}$$

If now we take the scalar product of (3.1) and U_1 and sum on j we obtain

$$(3.2)\quad g(\phi_2 X, AU_2) + g(\phi_3 X, AU_3) = 0$$

for any $X \in D$.

The same reasoning taking in (1.1) $Z = E_j$, $Y = \phi_2 E_j$ and considering the scalar product of the result and U_2 gives us

$$(3.3)\quad g(\phi_1 X, AU_1) + g(\phi_3 X, AU_3) = 0$$

for any $X \in D$.

If we repeat the above computation for $Z = E_j$, $Y = \phi_3 E_j$ and take the U_3-component we get

$$(3.4)\quad g(\phi_1 X, AU_1) + g(\phi_2 X, AU_2) = 0$$

for any $X \in D$. Thus from (3.2), (3.3) and (3.4) we have

$$(3.5)\quad g(\phi_i X, AU_i) = 0, \quad i = 1,2,3$$

for any $X \in \boldsymbol{D}$. Thus $g(A\boldsymbol{D}, \boldsymbol{D}^{\perp}) = \{0\}$ and from Theorem A, M must be congruent to an open part of either i), ii) or iii) appearing in such a Theorem.

Let us consider the case iii) of a tube of radius r, $0 < r < \pi/4$, over CP^m. The principal curvatures on $\boldsymbol{D}$ are $\cot(r)$ and $-\tan(r)$ both with multiplicity $2(m-1)$. As $m \geq 3$ we can consider unit $X, W \in \boldsymbol{D}$ such that Span $\{X, \phi_1 X, \phi_2 X, \phi_3 X)\} \perp$ Span $\{W, \phi_1 W, \phi_2 W, \phi_3 W\}$ and such that X and $\phi_1 X$ are principal with principal curvature $\cot(r)$ and $\phi_2 W$ is principal with principal curvature $-\tan(r)$. Thus if in (1.1) we take $Y = \phi_1 X$ and $Z = \phi_2 W$, by the first identity of Bianchi we should have $-(\tan(r) + \cot(r))R(X, \phi_1 X)\phi_2 W = 0$. But applying (2.6) this implies $(\tan(r) + \cot(r))\phi_3 W = 0$ which is impossible.

In the case ii) of Theorem A we also have two distinct principal curvatures on $\boldsymbol{D}$ and a reasoning similar to the above one proves that this case cannot occur.

On the other hand, geodesic hyperspheres have only one principal curvature on $\boldsymbol{D}$, thus they satisfy (1.1) and this finishes the proof.

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