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LC-DECOMPOSABILITY AND THE AR-PROPERTY IN LINEAR METRIC SPACES

By

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Abstract. We investigate the AR-property for convex sets in non-locally convex linear metric spaces. We introduce the notion of LC-decomposability for convex sets and prove that any LC-decomposable convex set is an AR.

1. Introduction

Detecting the AR-property for convex sets in linear metric spaces is of great importance since Dobrowolski and Torunczyk [4] proved the following theorem:

THEOREM A. (i) *A complete separable linear metric space X is homeomorphic to Hilbert space if and only if X is an AR.*

(ii) *A compact convex set X in a linear metric space is homeomorphic to Hilbert cube if and only if X is an AR.*

For about fifteen years many efforts were made to find out whether the assumption of AR-property in Dobrowolski-Torunczyk's theorem is essential. This question has been answered partly by Cauty [3], who recently proved the following theorem:

THEOREM B. *There exists a σ-compact linear metric space which is not an AR.*

By a theorem of Torunczyk [12], the completion of any non-AR-linear metric space is still a non-AR-space. Therefore Theorem B shows that the AR-property assumption in Theorem A (i) is essential. However, it is unknown

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whether the AR-property assumption can be removed from Theorem A (ii). This is still one of the most interesting (and difficult!) questions in the theory of non-locally convex linear metric spaces.

By Theorem B, convex sets in linear metric spaces may be not AR-spaces. So it is essential to establish conditions for convex sets to be AR's. And the results in [7] and [8] become valuable because of Cauty's theorem.

In [7] it was shown that if a convex set X in a linear metric space can be pushed into its locally convex subsets by arbitrarily small maps, then X is an AR. In this paper, we genelize the result of [7] by demonstrating that if a convex set X can be broken into finite convex sets, each of them can be pushed into its locally convex subsets by arbitrarily small maps, then X is an AR.

Following [7], a subset X in a linear metric space is an *LC-set* if for every $\varepsilon > 0$ there exists a $\delta = \delta(\varepsilon, X)$ such that for any finite set $A \subset X$ with diam $A < \delta$ we have diam(conv A) $< \varepsilon$.

Observe that any set in a locally convex linear metric space is an LC-set.

We say that a subset X in a linear metric space E is a *strongly LC-set* if $[X]$ is an LC-set, where $[X] = \{\lambda x : \lambda \in [0, 1] \text{ and } x \in X\} \subset E$.

Let X be a subset in a linear metric space and $\varepsilon > 0$. We say that X is an *ε-LC-set* if there exists a strongly LC-subset Y of X such that

$$\|x - [Y]\| < 3^{-1}\delta(\varepsilon, [Y]) \quad \text{for every } x \in X. \tag{1}$$

We say that a finite family $\{A_1, \ldots, A_n\}$ of subsets in a linear metric space X is *linearly independent* if for every $x_i \in \operatorname{span} A_i, i = 1, \ldots, n$, the set $\{x_1, \ldots, x_n\} \setminus \{\theta\}$, where θ denotes the zero element of X, is a linearly independent subset of X.

Let X and Y be subsets in a linear metric space. We say that X and Y are *topologically summable* if whenever U is an open subset of X and V is an open subset of Y, the set $U + V$ is open in $X + Y$.

DEFINITION. *We say that a convex set X in a linear metric space is LC-decomposable if $\theta \in X$, and for every $\varepsilon > 0$ there exists positive numbers ε_i, $i = 1, \ldots, n$, with $\sum_{i=1}^n \varepsilon_i \leq \varepsilon$, and linearly independent, topologically summable, ε_i-LC-subsets X_i of X such that $X = conv(X_1 \cup \cdots \cup X_n)$.*

Our result in this paper is the following:

THEOREM 1. *Any LC-decomposable convex set is an AR.*

NOTATION AND CONVENTIONS. In this paper, all maps are assumed to be continuous. By a linear metric space we mean a topological vector space X which is metrizable. The zero element of X is denoted by θ. We equip X with an F-norm $\|\cdot\|$ such that, see [11]

$$\|\lambda x\| \leq \|x\| \quad \text{for every } x \in X \text{ and } \lambda \in \boldsymbol{R} \text{ with } |\lambda| \leq 1.$$

Let A be a subset of a linear metric space X. By span A we denote the linear subspace of X spanned by A and by conv A we denote the convex hull of A in X. We also use the following notation:

$$[A] = [0,1]A = \{\lambda x : \lambda \in [0,1], x \in A\} = \operatorname{conv}\{A \cup \{\theta\}\};$$

$$\|x - A\| = \inf\{\|x - y\| : y \in A\} \quad \text{for } x \in X;$$

$$\operatorname{diam} A = \sup\{\|x - y\| : x, y \in A\}.$$

For undefined notation, see [1], [2] and [11].

2. The key for the proof

In our proof of Theorem 1, we use some ideas from [7] [8] and [10]. The following characterization of ANR-spaces, established in [6], is the key for our proof of the main result in this paper.

Let X be a metric space. For a given open cover $\mathscr{U}$ of X, let $\mathscr{N}(\mathscr{U})$ denote the *nerve* of $\mathscr{U}$. The nerve $\mathscr{N}(\mathscr{U})$ of $\mathscr{U}$ is the simplicial complex

$$\{\sigma : \sigma = \langle U_1, \dots, U_n \rangle,\ U_i \in \mathscr{U},\ n \in \boldsymbol{N}\}$$

made up of all $\sigma = \langle U_1, \dots, U_n \rangle$ for which $\bigcap_{i=1}^{N} U_i \neq \varnothing$. The simplicial complex $\mathscr{N}(\mathscr{U})$ will be endowed with the Whitehead topology (see [1] or [5] for a discussion). Denote

$$\operatorname{mesh} \mathscr{U} = \sup\{\operatorname{diam} U : U \in \mathscr{U}\}.$$

Let $\{\mathscr{U}_n\}$ be a sequence of open covers of a metric space X. We say that $\{\mathscr{U}_n\}$ is a *zero sequence* if mesh $\mathscr{U}_n \to 0$ as $n \to \infty$.

Finally, define

$$\mathscr{U} = \bigcup_{n=1}^{\infty} \mathscr{U}_n \quad \text{and} \quad \mathscr{K}(\mathscr{U}) = \bigcup_{n=1}^{\infty} \mathscr{N}(\mathscr{U}_n \cup \mathscr{U}_{n+1}),$$

and for any $\sigma \in \mathscr{K}(\mathscr{U})$, let

$$n(\sigma) = \sup\{n \in \boldsymbol{N} : \sigma \in \mathscr{N}(\mathscr{U}_n \cup \mathscr{U}_{n+1})\}.$$

Observe that

$$\mathcal{N}(\mathcal{U}_n \cup \mathcal{U}_{n+1}) \cap \mathcal{N}(\mathcal{U}_{n+1} \cup \mathcal{U}_{n+2}) = \mathcal{N}(\mathcal{U}_{n+1}) \quad \text{for every } n \in N.$$

We say that a map $f : \mathcal{U} \to X$ is a *selection* if $f(U) \in U$ for every $U \in \mathcal{U}$. The proof of Theorem 1 is based on the following:

THEOREM 2 [6] (See also [9]). *A metric space X with no isolated points is an ANR if and only if there is a zero sequence $\{\mathcal{U}_n\}$ of open covers of X such that for any selection $g : \mathcal{U} \to X$, there exists a map $f : \mathcal{K}(\mathcal{U}) \to X$ so that $diam(f(\sigma_k) \cup g(\sigma_k^0)) \to 0$ if $n(\sigma_k) \to \infty$ for any sequence of simplices $\{\sigma_k\}$ in $\mathcal{K}(\mathcal{U})$, where σ^0 denote the set of all vertices of σ.*

Now, assume that X is an LC-decomposable convex set. To show that X is an AR, we aim to verify the conditions of Theorem 2. Our first step is to describe a sequence $\{\mathcal{U}_n\}$ of open covers of X as stated in Theorem 2.

Let $\{\varepsilon_n\} = \{2^{-n}\}$. By the LC-decomposability of X, for every $n \in N$ there exist positive numbers ε_i^n, $i = 1, \dots, m(n)$, with

$$\sum_{i=1}^{m(n)} \varepsilon_i^n \le 2^{-n} \tag{2}$$

and linearly independent, topologically summable, ε_i^n-LC-subsets G_i^n of X, $i = 1, \dots, m(n)$, such that $X = \operatorname{conv}\left(\bigcup_{i=1}^{m(n)} G_i^n\right)$.

By definition for each $i = 1, \dots, m(n)$ there exists a strongly LC-subset $F_i^n \subset G_i^n$ such that

$$\|x - [F_i^n]\| < 3^{-1}\delta_i^n \quad \text{for every } x \in G_i^n,$$

where

$$\delta_i^n = \delta(\varepsilon_i^n, [F_i^n]) \quad \text{for } i = 1, \dots, m(n).$$

Denote

$$X_i^n = [G_i^n] \text{ and } Y_i^n = [F_i^n] \quad \text{for } i = 1, \dots, m(n). \tag{3}$$

Then $X = \operatorname{conv}\left(\bigcup_{i=1}^{m(n)} X_i^n\right)$ and Y_i^n is an LC-set for every $i = 1, \dots, m(n)$. We claim that

CLAIM 1. $\|x - Y_i^n\| < 3^{-1}\delta_i^n$ *for every $x \in X_i^n$.*

PROOF. For every $x \in X_i^n$, we have $x = \lambda g$ for some $g \in G_i^n$ and $\lambda \in [0, 1]$. Take $f \in Y_i^n$ such that

$$\|g - Y_i^n\| < 3^{-1}\delta_i^n.$$

Then $\lambda f \in Y_i^n$ and

$$\|x - \lambda f\| = \|\lambda g - \lambda f\| \le \|g - f\| \le \|g - Y_i^n\| < 3^{-1}\delta_i^n.$$

The claim is proved.

Observe that for any finite set $A \subset Y_i^n$, $i = 1, \dots, m(n)$, with

$$\operatorname{diam} A < \delta_i^n \quad \text{we have } \operatorname{diam}(\operatorname{conv} A) < \varepsilon_i^n. \tag{4}$$

For every $i = 1, \dots, m(n)$, let $\mathscr{W}_i^n$ be an open cover of X_i^n such that

$$\operatorname{diam} W < 6^{-1}\delta_i^n \quad \text{for every } W \in \mathscr{W}_i^n. \tag{5}$$

Denote

$$V(W_1^n, \dots, W_{m(n)}^n) = W_1^n + \cdots + W_{m(n)}^n, \quad \text{where } W_i^n \in \mathscr{W}_i^n, i = 1, \dots, m(n). \tag{6}$$

Let

$$\mathscr{U}_n = \left\{U = V(W_1^n, \dots, W_{m(n)}^n) \cap X : W_i^n \in \mathscr{W}_i^n, i = 1, \dots, m(n)\right\}. \tag{7}$$

Since $X_1^n, \dots, X_{m(n)}^n$ are topologically summable, $V = V(W_1^n, \dots, W_{m(n)}^n)$, see (6), is open in $X_1^n + \cdots + X_{m(n)}^n$. Since $\theta \in X_i^n, i = 1, \dots, m(n)$, see (3), we get

$$X = \operatorname{conv}\left(\bigcup_{i=1}^{m(n)} X_i^n\right) \subset X_1^n + \cdots + X_{m(n)}^n.$$

Therefore $U = V \cap X$ is open in X for every $U \in \mathscr{U}_n$.

Our aim is to prove that the sequence $\{\mathscr{U}_n\}$ of open covers of X, defined by (7), satisfies the conditions of Theorem 2. We first show:

LEMMA 1. *$\{\mathscr{U}_n\}$ is a zero sequence of open covers of X.*

PROOF. As we have seen, U is open in X for every $U \in \mathscr{U}_n$. Let us prove that $\mathscr{U}_n$ covers X for every $n \in N$. For a given point $x \in X$, take $x_i \in X_i^n$, $\lambda_i \ge 0$, $i = 1, \dots, m(n)$, with $\sum_{i=1}^{m(n)} \lambda_i = 1$, such that $x = \sum_{i=1}^{m(n)} \lambda_i x_i$. Note that $\lambda_i x_i \in X_i^n$ for $i = 1, \dots, m(n)$. Take $W_i^n \in \mathscr{W}_i^n$ so that $\lambda_i x_i \in W_i^n$ for $i = 1, \dots, m(n)$. Let $V = V(W_1^n, \dots, W_{m(n)}^n)$, see (6). Then $U = V \cap X \in \mathscr{U}_n$ and $x \in U$, see (7). Consequently, $\mathscr{U}_n$ covers X.

Now, we shall show that $\{\mathcal{U}_u\}$ is a zero sequence. In fact, we are going to prove

$$\operatorname{diam} U < 2^{-n} \quad \text{for every } U \in \mathcal{U}_n. \tag{8}$$

In fact, given $U \in \mathcal{U}_n$ we have $U = V \cap X$, where

$$V = V(W_1^n, \ldots, W_{m(n)}^n) = W_1^n + \cdots + W_{m(n)}^n, \quad \text{see (6).}$$

Therefore, for every $x, y \in V$, $x = \sum_{i=1}^{m(n)} x_i$, $y = \sum_{i=1}^{m(n)} y_i$, where x_i, $y_i \in W_i^n$, for $i = 1, \ldots, m(n)$. Observe that $\delta_i^n \leq \varepsilon_i^n$, for $i = 1, \ldots, m(n)$. Therefore from (2) and (5) we get

$$\|x - y\| \leq \sum_{i=1}^{m(n)} \|x_i - y_i\| \leq \sum_{i=1}^{m(n)} \operatorname{diam} W_i^n$$
$$< \sum_{i=1}^{m(n)} 6^{-1}\delta_i^n < \sum_{i=1}^{m(n)} \varepsilon_i^n \leq 2^{-n}.$$

Consequently $\operatorname{diam} V < 2^{-n}$. Since

$$\operatorname{diam} U = \operatorname{diam}(V \cap X) \leq \operatorname{diam} V < 2^{-n}$$

the inequality (8) is established. The lemma is proved.

Let $U_j \in \mathcal{U}_n$, $j = 1, \ldots, k$, where

$$U_j = V(W_1^n(j), \ldots, W_{m(n)}^n(j)) \cap X = (W_1^n(j) + \cdots + W_{m(n)}^n(j)) \cap X. \tag{9}$$

Then we have

LEMMA 2. *If $\bigcap_{j=1}^k U_j \neq \varnothing$, then $\bigcap_{j=1}^k W_i^n(j) \neq \varnothing$ for every $i = 1, \ldots, m(n)$.*

PROOF. For every $x \in \bigcap_{j=1}^k U_j$, we have $x = \sum_{i=1}^{m(n)} x_i(j)$, where $x_i(j) \in W_i^n(j)$ for $j = 1, \ldots, k$ and $i = 1, \ldots, m(n)$, see (9). Then for every $j = 1, \ldots, k$ we have

$$\sum_{i=1}^{m(n)} (x_i(j) - x_i(1)) = \theta.$$

Observe that $x_i(j) - x_i(1) \in \operatorname{span} X_i^n$ for every $i = 1, \ldots, m(n)$. By the linear independence of $\{X_i^n, i = 1, \ldots, m(n)\}$ we conclude that

$$x_i(j) = x_i(1) \quad \text{for every } j = 1, \ldots, k \quad \text{and} \quad i = 1, \ldots, m(n).$$

Consequently, letting

$$y_i = x_i(j) = x_i(1) \quad \text{for } i = 1, \dots, m(n),$$

we get

$$y_i \in \bigcap_{j=1}^{k} W_i^n(j) \quad \text{for every } i = 1, \dots, m(n).$$

The lemma is proved.

3. Proof of the main result

In this section, we prove Theorem 1. Since X is contractible, it suffices to show that X is an ANR, see [2]. We are going to verify the conditions of Theorem 2 for the sequence $\{\mathcal{U}_n\}$, defined in Section 2, see (7).

By Lemma 1, $\{\mathcal{U}_n\}$ is a zero sequence of open covers of X. Let $\mathcal{U} = \bigcup_{n=1}^{\infty} \mathcal{U}_n$ and let $g : \mathcal{U} \to X$ be a selection. For every $U \in \mathcal{U}$ we have $U \in \mathcal{U}_n$ for some $n \in N$. Hence $U = V \cap X$, where

$$V = V(W_1^n, \dots, W_{m(n)}^n) = W_1^n + \cdots + W_{m(n)}^n. \tag{10}$$

Since $g(U) \in X = \operatorname{conv}\left(\bigcup_{i=1}^{m(n)} X_i^n\right)$, we have

$$g(U) = \sum_{i=1}^{m(n)} \lambda_i x_i, \quad \text{where } x_i \in X_i^n, \lambda_i \geq 0 \quad \text{and} \quad \sum_{i=1}^{m(n)} \lambda_i = 1. \tag{11}$$

We claim that

$$\lambda_i x_i \in W_i^n \quad \text{for every } i = 1, \dots, m(n). \tag{12}$$

In fact, since $g(U) \in V = W_1^n + \cdots + W_{m(n)}^n$, we have $g(U) = \sum_{i=1}^{m(n)} z_i$, where $z_i \in W_i^n$ for $i = 1, \dots, m(n)$. Therefore

$$\sum_{i=1}^{m(n)} (z_i - \lambda_i x_i) = \theta.$$

Observe that $z_i - \lambda_i x_i \in \operatorname{span} X_i^n$ for every $i = 1, \dots, m(n)$. By the linear independence of $\{X_i^n, i = 1, \dots, m(n)\}$ we have $\lambda_i x_i = z_i \in W_i^n$ for every $i = 1, \dots, m(n)$. The claim is established.

Since $x_i \in X_i^n$, $i = 1, \dots, m(n)$, by Claim 1 there exist $y_i \in Y_i^n$, $i = 1, \dots, m(n)$ such that

$$\|x_i - y_i\| < 3^{-1} \delta_i^n \quad \text{for every } i = 1, \dots, m(n). \tag{13}$$

We define

$$f(U)=\sum_{i=1}^{m(n)}\lambda_i y_i. \tag{14}$$

(Observe that $f:\mathscr{U}\to X$ may not be a selection: Theorem 2 requires $g:\mathscr{U}\to X$ be a selection, but it does not require $f:\mathscr{U}\to X$ to be so.)

From (2) (4) (11) (13) and (14) we get

$$\begin{aligned}\|f(U)-g(U)\| &\le \sum_{i=1}^{m(n)}\|\lambda_i x_i-\lambda_i y_i\| \\ &\le \sum_{i=1}^{m(n)}\|x_i-y_i\| < \sum_{i=1}^{m(n)} 3^{-1}\delta_i^n \\ &\le \sum_{i=1}^{m(n)} 3^{-1}\varepsilon_i^n < 2^{-n}\end{aligned} \tag{15}$$

for every $U\in\mathscr{U}_n$.

Now, using the convexity of X we extend $f:\mathscr{U}\to X$ affinely to a map, which is still denoted by f, $f:\mathscr{K}(\mathscr{U})\to X$. We claim that f satisfies the required conditions.

Let $\sigma=\langle U_1,\dots,U_k\rangle\in\mathscr{K}(\mathscr{U})=\bigcup_{n=1}^{\infty}\mathscr{N}(\mathscr{U}_n\cup\mathscr{U}_{n+1})$. Take $p\in N$ so that

$$U_1,\dots,U_p\in\mathscr{U}_{n(\sigma)} \quad\text{and}\quad U_{p+1},\dots,U_k\in\mathscr{U}_{n(\sigma)+1}.$$

Let $\sigma=\langle\sigma_0,\sigma_1\rangle$, where

$$\sigma_0=\langle U_1,\dots,U_p\rangle \quad\text{and}\quad \sigma_1=\langle U_{p+1},\dots,U_k\rangle. \tag{16}$$

Our next step is to compute $\operatorname{diam} f(\sigma_i)$ for $i=0,1$. Let

$$g(U_j)=\sum_{i=1}^{m(n(\sigma))}\lambda_i(j)x_i(j) \quad\text{and}\quad f(U_j)=\sum_{i=1}^{m(n(\sigma))}\lambda_i(j)y_i(j) \tag{17}$$

where

$$\lambda_i(j)x_i(j)\in W_i^{n(\sigma)}(j),\, y_i(j)\in Y_i^{n(\sigma)},\, \lambda_i(j)\ge 0,\quad i=1,\dots,m(n),\ j=1,\dots,p$$

and

$$\sum_{i=1}^{m(n(\sigma))}\lambda_i(j)=1 \quad\text{for every } j=1,\dots,p.$$

Observe that $U_j = V_j \cap X$, $j = 1, \dots, p$, where

$$V_j = V(W_1^{n(\sigma)}(j), \dots, W_{m(n(\sigma))}^{n(\sigma)}(j)) = W_1^{n(\sigma)}(j) + \cdots + W_{m(n(\sigma))}^{n(\sigma)}(j). \tag{18}$$

Since $\bigcap_{j=1}^p U_j \neq \varnothing$, from Lemma 2 we obtain

$$\bigcap_{j=1}^{p} W_i^{n(\sigma)}(j) \neq \varnothing \quad \text{for every } i = 1, \dots, m(n(\sigma)).$$

Therefore from (5) we get

$$\operatorname{diam} \bigcup_{j=1}^{p} W_i^{n(\sigma)}(j) < 2(6^{-1}\delta_i^{n(\sigma)}) = 3^{-1}\delta_i^{n(\sigma)}, \tag{19}$$

for every $i = 1, \dots, m(n(\sigma))$. Denote

$$A_i = \{\lambda_i(j)y_i(j) : j = 1, \dots, p\} \quad \text{for } i = 1, \dots, m(n(\sigma)). \tag{20}$$

Since $\theta \in Y_i^{n(\sigma)}$, see (3), it follows that

$$A_i \subset Y_i^{n(\sigma)} \quad \text{for } i = 1, \dots, m(n(\sigma)). \tag{21}$$

We claim that

Claim 2. *diam $A_i < \delta_i^{n(\sigma)}$ for every $i = 1, \dots, m(n(\sigma))$.*

Proof. From (5) (12) (13) and (19) we obtain

$$\begin{aligned}
\|\lambda_i(j)y_i(j) - \lambda_i(j')y_i(j')\| &\leq \|\lambda_i(j)y_i(j) - \lambda_i(j)x_i(j)\| \\
&\quad + \|\lambda_i(j)x_i(j) - \lambda_i(j')x_i(j')\| \\
&\quad + \|\lambda_i(j')x_i(j') - \lambda_i(j')y_i(j')\| \\
&\leq \|y_i(j) - x_i(j)\| + \operatorname{diam} \bigcup_{j=1}^{p} W_i^{n(\sigma)}(j) \\
&\quad + \|y_i(j') - x_i(j')\| \\
&< 3^{-1}\delta_i^{n(\sigma)} + 3^{-1}\delta_i^{n(\sigma)} + 3^{-1}\delta_i^{n(\sigma)} = \delta_i^{n(\sigma)},
\end{aligned}$$

which proves the claim.

From (4) and from Claim 2 it follows that

$$\operatorname{diam}(\operatorname{conv} A_i) < \varepsilon_i^{n(\sigma)} \quad \text{for every } i = 1, \dots, m(n(\sigma)). \tag{22}$$

For every $x \in \sigma_0$, we have $x = \sum_{j=1}^{p} \alpha_j U_j$ where $\alpha_j \geq 0$ and $\sum_{j=1}^{p} \alpha_j = 1$. Then from (17) and (22) we obtain

$$\begin{aligned}
\|f(x) - f(U_1)\| &= \left\| \sum_{j=1}^{p} \alpha_j (f(U_j) - f(U_1)) \right\| \\
&= \left\| \sum_{j=1}^{p} \alpha_j \sum_{i=1}^{m(n(\sigma))} (\lambda_i(j) y_i(j) - \lambda_i(1) y_i(1)) \right\| \\
&= \left\| \sum_{i=1}^{m(n(\sigma))} \sum_{j=1}^{p} \alpha_j (\lambda_i(j) y_i(j) - \lambda_i(1) y_i(1) \right\| \\
&\leq \sum_{i=1}^{m(n(\sigma))} \left\| \sum_{j=1}^{p} \alpha_j (\lambda_i(j) y_i(j) - \lambda_i(1) y_i(1)) \right\| \\
&\leq \sum_{i=1}^{(m(n(\sigma))} \operatorname{diam}(\operatorname{conv} A_i) < \sum_{i=1}^{m(n(\sigma))} \varepsilon_i^{n(\sigma)} < 2^{-n(\sigma)}.
\end{aligned}$$

Similarly for every $x \in f(\sigma_1)$ we have

$$\|x - f(U_{p+1})\| < 2^{-n(\sigma)-1}.$$

(Observe that $U_i \in \mathscr{U}_{n(\sigma)+1}$ for $i = p+1, \ldots, k$.)

Now for every $x \in \sigma$ we have $x = \alpha x_0 + (1-\alpha) x_1$, where $x_i \in \sigma_i$ for $i = 0, 1$ and $\alpha \in [0, 1]$. Let $y = \alpha U_1 + (1-\alpha) U_{p+1}$. Then we get

$$\begin{aligned}
\|f(x) - f(y)\| &= \|\alpha(f(x_0) - f(U_1)) + (1-\alpha)(f(x_1) - f(U_{p+1}))\| \\
&\leq \|f(x_0) - f(U_1)\| + \|f(x_1) - f(U_{p+1})\| \qquad (23) \\
&< 2^{-n(\sigma)} + 2^{-n(\sigma)-1} < 2^{-n(\sigma)+1}.
\end{aligned}$$

Since g is a selection, from (8) and (15) we get

$$\begin{aligned}
\|f(y) - f(U_1)\| &= \|\alpha f(U_1) + (1-\alpha) f(U_{p+1}) - f(U_1)\| \\
&= \|(1-\alpha)(f(U_1) - f(U_{p+1}))\| \leq \|f(U_1) - f(U_{p+1})\| \\
&\leq \|f(U_1) - g(U_1)\| + \|g(U_1) - g(U_{p+1})\| + \|g(U_{p+1}) - f(U_{p+1})\| \\
&< 2^{-n(\sigma)} + 2^{-n(\sigma)+1} + 2^{-n(\sigma)} = 2^{-n(\sigma)+2}. \qquad (24)
\end{aligned}$$

Therefore from (23) and (24) we obtain

$$\begin{aligned}
\|f(x) - f(U_1)\| &\leq \|f(x) - f(y)\| + \|f(y) - f(U_1)\| \\
&< 2^{-n(\sigma)+1} + 2^{-n(\sigma)+2} < 2^{-n(\sigma)+3}
\end{aligned}$$

for every $x \in \sigma$. Consequently

$$\operatorname{diam} f(\sigma) < 2^{-n(\sigma)+4}. \tag{25}$$

Since g is a selection, from (8) we get

$$\operatorname{diam} g(\sigma^0) < 2^{-n(\sigma)+1}. \tag{26}$$

(Note that σ^0 denotes the set of all vertices of σ, meanwhile σ_0 is the simplex defined by (16).) Hence from (15) (25) and (26) we obtain

$$\begin{aligned}\operatorname{diam}(f(\sigma)) \cup g(\sigma^0)) &\leq \operatorname{diam}(f(\sigma)) + \|f(U_1) - g(U_1)\| + \operatorname{diam}(g(\sigma^0)) \\ &< 2^{-n(\sigma)+4} + 2^{-n(\sigma)} + 2^{-n(\sigma)+1} < 2^{-n(\sigma)+5}.\end{aligned}$$

Therefore

$$\operatorname{diam}\big(f(\sigma) \cup g(\sigma^0)\big) \to 0 \quad \text{as } n(\sigma) \to \infty.$$

Consequently, X is an ANR by Theorem 2 and the proof of Theorem 1 is finished.

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