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SPACELIKE MINIMAL SURFACES IN 4-DIMENSIONAL LORENTZIAN SPACE FORMS

By

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Abstract. We give a necessary and sufficient condition for the existence of spacelike minimal surfaces in 4-dimensional Lorentzian space forms, which is a generalization of the Ricci condition for minimal surfaces in 3-dimensional Riemannian space forms.

1. Introduction

Let $N^n(c)$ and $N_1^n(c)$ denote the n-dimensional simply connected Riemannian space form and Lorentzian space form of constant curvature c, respectively. Every minimal surface in $N^3(c)$ may be seen as a minimal surface in $N^4(c)$. We note that $N^3(c)$ is naturally included in $N_1^4(c)$, and every minimal surface in $N^3(c)$ may be seen also as a spacelike minimal surface in $N_1^4(c)$. So minimal surfaces in $N^3(c)$ can be generalized into two ways, that is, minimal surfaces in $N^4(c)$ and spacelike minimal surfaces in $N_1^4(c)$. Then it seems interesting to compare the geometry of (spacelike) minimal surfaces in $N^4(c)$ and $N_1^4(c)$.

Let M be a minimal surface in $N^3(c)$ with induced metric ds^2 and Gaussian curvature K. Then M satisfies the Ricci condition, that is, the metric $d\hat{s}^2 = \sqrt{c-K}\, ds^2$ is flat at points where $K < c$. Conversely, let M be a 2-dimensional simply connected Riemannian manifold with metric ds^2 and Gaussian curvature K $(< c)$. If M satisfies the Ricci condition, then there exists an isometric minimal immersion of M into $N^3(c)$ (cf. [4]). Hence, the Ricci condition is a necessary and sufficient condition for the existence of minimal surfaces in $N^3(c)$.

In [2, Th. 1], the Ricci condition is generalized for minimal surfaces in $N^4(c)$. In this paper, we give another generalization for spacelike minimal surfaces in $N_1^4(c)$.

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THEOREM. (i) *Let M be a spacelike minimal surface in $N_1^4(c)$. We denote by K, K_ν and Δ the Gaussian curvature, the normal curvature and the Laplacian of M, respectively. Then*

$$\Delta \log\{(c-K)^2 + K_\nu^2\} = 8K \tag{1}$$

at points where $(c-K)^2 + K_\nu^2 > 0$, and

$$\Delta \tan^{-1}\left(\frac{K_\nu}{c-K}\right) = -2K_\nu \tag{2}$$

at points where $K \neq c$.

(ii) *Conversely, let M be a 2-dimensional simply connected Riemannian manifold with Gaussian curvature K $(\neq c)$ and Laplacian Δ. If K_ν is a function on M satisfying* (1) *and* (2), *then there exists an isometric minimal immersion of M into $N_1^4(c)$ with normal curvature K_ν.*

REMARK. The condition (1) is equivalent to that the metric

$$d\hat{s}^2 = \{(c-K)^2 + K_\nu^2\}^{1/4}\, ds^2$$

is flat at points where $(c-K)^2 + K_\nu^2 > 0$. Here ds^2 is the induced metric on M.

Using the divergence theorem for (1) and (2), we get the following corollaries.

COROLLARY 1. *Let M be a compact spacelike minimal surface in $N_1^4(c)$ with Gaussian curvature K and normal curvature K_ν.*

(i) *If $(c-K)^2 + K_\nu^2 > 0$ on M, then M is of genus* 1.

(ii) *If K is constant, then $K = c$ or $K = 0$.*

COROLLARY 2. *Let M be a compact spacelike minimal surface in $N_1^4(c)$ with Gaussian curvature K and normal curvature K_ν.*

(i) *If $K \neq c$ on M, then $\int_M K_\nu\, dM = 0$.*

(ii) *If $K \neq c$ and K_ν does not change sign on M, then $K_\nu = 0$.*

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2. Preliminaries

In this section, we recall the method of moving frames for spacelike surfaces in $N_1^4(c)$. Unless otherwise stated, we shall use the following convention on the ranges of indices:

$$1 \le A, B, \cdots \le 4, \quad 1 \le i, j, \cdots \le 2, \quad 3 \le \alpha, \beta, \cdots \le 4.$$

Let $\{e_A\}$ be an oriented local orthonormal frame field in $N_1^4(c)$, and $\{\omega^A\}$ be the dual coframe. Here the Lorentzian metric of $N_1^4(c)$ is given by

$$ds^2 = (\omega^1)^2 + (\omega^2)^2 + (\omega^3)^2 - (\omega^4)^2.$$

We can define the connection forms $\{\omega_B^A\}$ by

$$de_B = \sum_A \omega_B^A e_A.$$

Then

$$\omega_B^A + \omega_A^B = 0, \quad \omega_4^A = \omega_A^4, \quad \text{where } 1 \le A, B \le 3. \tag{3}$$

The structure equations are given by

$$d\omega^A = -\sum_B \omega_B^A \wedge \omega^B, \tag{4}$$

$$d\omega_B^A = -\sum_C \omega_C^A \wedge \omega_B^C + \frac{1}{2} \sum_{C,D} R_{BCD}^A \omega^C \wedge \omega^D, \tag{5}$$

$$R_{BCD}^A = c\varepsilon_B(\delta_C^A \delta_{BD} - \delta_D^A \delta_{BC}), \tag{6}$$

where $\varepsilon_B = 1$ for $1 \le B \le 3$ and $\varepsilon_4 = -1$.

Let M be an oriented spacelike surface in $N_1^4(c)$, that is, the induced metric on M is Riemannian. We choose the frame $\{e_A\}$ so that $\{e_i\}$ are tangent to M. Then $\omega^\alpha = 0$ on M. In the following, our argument will be restricted on M. By (4)

$$0 = -\sum_i \omega_i^\alpha \wedge \omega^i.$$

So there is a symmetric tensor h_{ij}^α such that

$$\omega_i^\alpha = \sum_j h_{ij}^\alpha \omega^j, \tag{7}$$

where h_{ij}^α are the components of the second fundamental form of M.

The Gaussian curvature K and the normal curvature K_ν of M are given by

$$d\omega_2^1 = K\omega^1 \wedge \omega^2, \quad d\omega_4^3 = K_\nu \omega^1 \wedge \omega^2. \tag{8}$$

Then by (3), (5), (6) and (7) we have

$$K = c + h_{11}^3 h_{22}^3 - (h_{12}^3)^2 - h_{11}^4 h_{22}^4 + (h_{12}^4)^2, \tag{9}$$

and

$$K_\nu = -(h_{11}^3 h_{12}^4 - h_{12}^3 h_{11}^4 + h_{12}^3 h_{22}^4 - h_{22}^3 h_{12}^4). \tag{10}$$

The mean curvature vector H of M is given by

$$H = \frac{1}{2}\sum_{i,\alpha} h_{ii}^\alpha e_\alpha.$$

The surface M is called minimal if $H = 0$ on M.

3. Proof of Theorem

(i) As M is a spacelike minimal surface in $N_1^4(c)$, using the notations in Section 2, we may write

$$\omega_1^3 = s\omega^1 + t\omega^2, \quad \omega_2^3 = t\omega^1 - s\omega^2, \quad \omega_1^4 = u\omega^1 + v\omega^2, \quad \omega_2^4 = v\omega^1 - u\omega^2. \tag{11}$$

By (9) and (10)

$$K = c - s^2 - t^2 + u^2 + v^2, \quad K_\nu = -2(sv - tu). \tag{12}$$

Using (4), (5), (6) and (11) we have

$$\begin{aligned} d\omega_1^3 &= ds \wedge \omega^1 - s\omega_2^1 \wedge \omega^2 + dt \wedge \omega^2 - t\omega_1^2 \wedge \omega^1 \\ &= -\omega_2^3 \wedge \omega_1^2 - \omega_4^3 \wedge \omega_1^4 \\ &= -(t\omega^1 - s\omega^2) \wedge \omega_1^2 - \omega_4^3 \wedge (u\omega^1 + v\omega^2). \end{aligned}$$

So, using the notation like

$$ds = s_1\omega^1 + s_2\omega^2, \quad dt = t_1\omega^1 + t_2\omega^2,$$

$$\omega_2^1 = (\omega_2^1)_1\omega^1 + (\omega_2^1)_2\omega^2 = -\omega_1^2, \quad \omega_4^3 = (\omega_4^3)_1\omega^1 + (\omega_4^3)_2\omega^2 = \omega_3^4,$$

we get

$$2s(\omega_2^1)_1 + 2t(\omega_2^1)_2 - v(\omega_4^3)_1 + u(\omega_4^3)_2 = -s_2 + t_1.$$

Similarly, from the exterior derivative of ω_2^3, ω_1^4 and ω_2^4,

$$2s(\omega_2^1)_2 - 2t(\omega_2^1)_1 - v(\omega_4^3)_2 - u(\omega_4^3)_1 = s_1 + t_2,$$

$$2u(\omega_2^1)_1 + 2v(\omega_2^1)_2 - t(\omega_4^3)_1 + s(\omega_4^3)_2 = -u_2 + v_1,$$

$$2u(\omega_2^1)_2 - 2v(\omega_2^1)_1 - t(\omega_4^3)_2 - s(\omega_4^3)_1 = u_1 + v_2.$$

Therefore we have

$$
\begin{pmatrix} s & -t & v & u \\ t & s & -u & v \\ u & -v & t & s \\ v & u & -s & t \end{pmatrix}
\begin{pmatrix} 2\omega_2^1 \\ 2(*\omega_2^1) \\ -\omega_4^3 \\ -(*\omega_4^3) \end{pmatrix}
=
\begin{pmatrix} *ds + dt \\ *dt - ds \\ *du + dv \\ *dv - du \end{pmatrix}, \tag{13}
$$

where $*$ denotes the Hodge star operator on M.

Set

$$
A = \begin{pmatrix} s & -t & v & u \\ t & s & -u & v \\ u & -v & t & s \\ v & u & -s & t \end{pmatrix},
$$

and

$$
\begin{aligned}
X &= s^2 - t^2 - u^2 + v^2, \quad & Y &= 2(st - uv), \\
Z &= s^2 + t^2 + u^2 + v^2, \quad & W &= 2(su + tv).
\end{aligned} \tag{14}
$$

Let A_{ij} $(1 \le i, j \le 4)$ denote the cofactors of A. Then, noting (12) and (14), we can see that

$$A_{11} = -A_{43} = s(c-K) - vK_v = sX + tY = sZ - uW, \tag{15}$$

$$A_{21} = A_{33} = t(c-K) + uK_v = -tX + sY = tZ - vW, \tag{16}$$

$$A_{31} = -A_{23} = -u(c-K) + tK_v = -uX - vY = uZ - sW, \tag{17}$$

$$A_{41} = A_{13} = -v(c-K) - sK_v = vX - uY = vZ - tW, \tag{18}$$

and

$$\det A = (c-K)^2 + K_v^2 = X^2 + Y^2 = Z^2 - W^2. \tag{19}$$

By (12)–(19), at points where $(c-K)^2 + K_v^2 > 0$,

$$
\begin{aligned}
2\omega_2^1 &= \frac{1}{\det A}\{A_{11}(*ds + dt) + A_{21}(*dt - ds) + A_{31}(*du + dv) + A_{41}(*dv - du)\} \\
&= \frac{*d\{(c-K)^2 + K_v^2\}}{4\{(c-K)^2 + K_v^2\}} + \frac{X\,dY - Y\,dX}{2(X^2+Y^2)} \\
&= \frac{1}{4} * d\log\{(c-K)^2 + K_v^2\} + \frac{X\,dY - Y\,dX}{2(X^2+Y^2)}.
\end{aligned}
$$

Hence, by the exterior derivative of this equation, together with (8), we get the equation (1).

Similarly, by (12)–(19), at points where $K \neq c$,

$$
\begin{aligned}
-\omega_4^3 &= \frac{1}{\det A}\{A_{13}(*ds+dt)+A_{23}(*dt-ds)+A_{33}(*du+dv)+A_{43}(*dv-du)\} \\
&= \frac{(c-K)(*dK_v)-K_v\{*d(c-K)\}}{2\{(c-K)^2+K_v^2\}}+\frac{Z\,dW-W\,dZ}{2(Z^2-W^2)} \\
&= \frac{1}{2}*d\tan^{-1}\left(\frac{K_v}{c-K}\right)+\frac{Z\,dW-W\,dZ}{2(Z^2-W^2)}.
\end{aligned}
$$

Hence, by the exterior derivative of this equation, together with (8), we get the equation (2).

(ii) We may assume that M is a small neighborhood. Let ds^2 be the metric on M. As noted in the remark in the introduction, the condition (1) implies that the metric

$$d\hat{s}^2 = \{(c-K)^2+K_v^2\}^{1/4}\,ds^2$$

is flat. So there exists a coordinate system (x^1, x^2) such that

$$ds^2 = \{(c-K)^2+K_v^2\}^{-1/4}\{(dx^1)^2+(dx^2)^2\}.$$

Set

$$\omega^i = \{(c-K)^2+K_v^2\}^{-1/8}\,dx^i,$$

so that $\{\omega^i\}$ is an orthonormal coframe field with dual frame $\{e_i\}$. By

$$d\omega^1 = -\omega_2^1\wedge\omega^2, \quad d\omega^2 = -\omega_1^2\wedge\omega^1,$$

we can find that the connection form $\omega_2^1 = -\omega_1^2$ is given by

$$\omega_2^1 = -\omega_1^2 = \frac{1}{8}*d\log\{(c-K)^2+K_v^2\}.$$

As $K \neq c$, we may choose smooth functions s and v so that

$$s^2 - v^2 = c-K, \quad sv = -\frac{1}{2}K_v.$$

Let E be a 2-plane bundle over M with metric $\langle\,,\rangle$ and orthonormal sections $\{e_\alpha\}$ such that

$$\langle e_3, e_3\rangle = 1, \quad \langle e_3, e_4\rangle = 0, \quad \langle e_4, e_4\rangle = -1.$$

Let h be a symmetric section of $\mathrm{Hom}(TM \times TM, E)$ such that

$$(h_{ij}^3) = \begin{pmatrix} s & 0 \\ 0 & -s \end{pmatrix}, \quad (h_{ij}^4) = \begin{pmatrix} 0 & v \\ v & 0 \end{pmatrix},$$

and set

$$\omega_1^3 = -\omega_3^1 = s\omega^1, \quad \omega_2^3 = -\omega_3^2 = -s\omega^2,$$
$$\omega_1^4 = \omega_4^1 = v\omega^2, \quad \omega_2^4 = \omega_4^2 = v\omega^1.$$

We define a compatible connection ${}^\perp\nabla$ of E so that

$${}^\perp\nabla e_3 = \omega_3^4 e_4, \quad {}^\perp\nabla e_4 = \omega_4^3 e_3,$$

where

$$\omega_4^3 = \omega_3^4 = -\frac{1}{2} * d\tan^{-1}\left(\frac{K_v}{c-K}\right).$$

Now, almost reversing the argument in (i) with $t = u = 0$, we can find that $\{\omega_B^A\}$ satisfy the structure equations:

$$d\omega_2^1 = -\omega_3^1 \wedge \omega_2^3 - \omega_4^1 \wedge \omega_2^4 + c\omega^1 \wedge \omega^2,$$

$$d\omega_1^3 = -\omega_2^3 \wedge \omega_1^2 - \omega_4^3 \wedge \omega_1^4, \quad d\omega_2^3 = -\omega_1^3 \wedge \omega_2^1 - \omega_4^3 \wedge \omega_2^4,$$
$$d\omega_1^4 = -\omega_2^4 \wedge \omega_1^2 - \omega_3^4 \wedge \omega_1^3, \quad d\omega_2^4 = -\omega_1^4 \wedge \omega_2^1 - \omega_3^4 \wedge \omega_2^3,$$
$$d\omega_4^3 = -\omega_1^3 \wedge \omega_4^1 - \omega_2^3 \wedge \omega_4^2,$$

which are the integrability conditions. Therefore, by the fundamental theorem, there exists an isometric immersion of M into $N_1^4(c)$, which is minimal and has normal curvature K_v.

4. Some Problems

Refering to our results and the case of minimal surfaces in $N^4(c)$, it should be natural to consider the following problems (cf. [3], [1], [5], [6] and their references).

PROBLEM 1. Classify spacelike minimal surfaces with constant Gaussian curvature in $N_1^4(c)$.

PROBLEM 2. Classify spacelike minimal surfaces with constant normal curvature in $N_1^4(c)$.

PROBLEM 3. Classify spacelike minimal surfaces in $N_1^4(c)$ which are locally isometric to minimal surfaces in $N^3(c)$, or spacelike minimal surfaces in $N_1^3(c)$.

Of course, we may consider the higher codimensional problems. Here we should note that a spacelike minimal surface with constant Gaussian curvature c in $N_1^4(c)$ may not be totally geodesic. These problems will be discussed elsewhere.

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