NORMALITY AND COLLECTIONWISE NORMALITY OF PRODUCT SPACES, II

By

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Abstract. We prove that the product space $X \times Y$ is collectionwise normal if and only if $X \times Y$ is normal in the following cases; (1) X is a collectionwise normal Σ-space and Y is a collectionwise normal first countable P-space, (2) X is the closed image of a normal M-space and Y is a collectionwise normal first countable P-space, (3) X is the closed image of a paracompact M-space and Y is a collectionwise normal P-space. In particular, (2) and (3) essentially generalize K. Chiba's theorems [3].

1. Introduction

Throughout this paper we assume all spaces to be Hausdorff, and all maps to be continuous. For two collectionwise normal spaces X and Y, the result which asserts normality of $X \times Y$ implies its collectionwise normality has been proved in cases Y being metrizable, Lasňev, a paracompact M-space and σ-locally compact paracompact by Okuyama [12], Hoshina [5], Rudin-Starbird [14] and Chiba [3], respectively. In a previous paper [16, Theorem 2.2], the author proved another case that the product of a paracompact Σ-space X and a collectionwise normal P-space Y is collectionwise normal if and only if it is normal; this extends Nagami's theorem [9] with Y being a paracompact σ-space as well as affirmatively answers to the problem posed by Yang [17].

In [2], K. Chiba showed the following; for a collectionwise normal Σ-space X and a paracompact first countable P-space Y, $X \times Y$ is collectionwise normal. If we weaken the paracompactness of Y to the collectionwise normality, even the normality of $X \times Y$ need not be implied. Being suggested by these facts, we obtain the following result.

THEOREM 1.1. *Let X be a collectionwise normal Σ-space and Y a collectionwise normal first countable P-space. Then the product $X \times Y$ is collectionwise normal if and only if $X \times Y$ is normal.*

In [3], K. Chiba showed that normality of $X \times Y$ implies its collectionwise normality in the following cases;

(A) X is the closed image of a normal M-space and Y is a paracompact first countable P-space,

(B) X is the closed image of a paracompact first countable M-space and Y is a collectionwise normal Σ-space.

In the proof of both cases, it needed to show at first the collectionwise normality of $Z \times Y$ which is essential to the proof, where Z is an M-space assumed in (A) or (B). In this paper, we prove further the following two theorems which generalize the Chiba's results above; it should be noted that in our results even the normality of $Z \times Y$, where Z is an M-space in our theorems, need not be implied from our cases.

THEOREM 1.2. *Let X be the closed image of a normal M-space and Y a collectionwise normal first countable P-space. Then the product $X \times Y$ is collectionwise normal if and only if $X \times Y$ is normal.*

THEOREM 1.3. *Let X be the closed image of a paracompact M-space and Y a collectionwise normal P-space. Then the product $X \times Y$ is collectionwise normal if and only if $X \times Y$ is normal.*

2. Preliminaries and key lemmas

Let N be the set of all positive integers.

A space Y is a *P-space* [8] if for any index set Ω and for any collection $\{G(\alpha_1, \ldots, \alpha_n) \mid \alpha_1, \ldots, \alpha_n \in \Omega; n \in N\}$ of open subsets of Y such that

$$G(\alpha_1, \ldots, \alpha_n) \subset G(\alpha_1, \ldots, \alpha_n, \alpha_{n+1}) \quad \text{for } \alpha_1, \ldots, \alpha_n, \alpha_{n+1} \in \Omega,$$

there exists a collection $\{F(\alpha_1, \ldots, \alpha_n) \mid \alpha_1, \ldots, \alpha_n \in \Omega; n \in N\}$ of closed subsets of Y such that the conditions (a), (b) below are satisfied:

(a) $F(\alpha_1, \ldots, \alpha_n) \subset G(\alpha_1, \ldots, \alpha_n)$ for $\alpha_1, \ldots, \alpha_n \in \Omega$,

(b) $Y = \bigcup \{G(\alpha_1, \ldots, \alpha_n) \mid n \in N\} \Rightarrow Y = \bigcup \{F(\alpha_1, \ldots, \alpha_n) \mid n \in N\}$.

A *Σ-space* [10] is a space X having a sequence, called a *Σ-net*, $\{\mathscr{E}_n \mid n \in N\}$ of locally finite closed covers of X which satisfies the following conditions:

(c) $\mathscr{E}_n$ is written as $\{E(\alpha_1, \ldots, \alpha_n) \mid \alpha_1, \ldots, \alpha_n \in \Omega\}$ with an index set Ω,

(d) $E(\alpha_1, \ldots, \alpha_n) = \bigcup \{E(\alpha_1, \ldots, \alpha_n, \alpha_{n+1}) \mid \alpha_{n+1} \in \Omega\}$ for $\alpha_1, \ldots, \alpha_n \in \Omega$,

(e) For every $x \in X$, $C(x)$ is countably compact, and there exists a sequence $\alpha_1, \alpha_2, \ldots \in \Omega$ such that $C(x) \subset V$ with V open implies $C(x) \subset E(\alpha_1, \ldots, \alpha_n) \subset V$ for some n, where $C(x) = \bigcap\{E \mid y \in E \in \mathscr{E}_n, n \in N\}$. We call $\{E(\alpha_1, \ldots, \alpha_n) \mid n \in N\}$ a *local net* of $C(x)$.

The definition of M-spaces is due to Morita [8].

The well-known facts are that every M-space is a Σ-space, every Σ-space is a P-space, every normal P-space is countably paracompact and every normal M-space is collectionwise normal (see Nagata [11]). We also note that not every closed image of a paracompact M-space is a Σ-space (see Gruenhage [4]).

The following lemma will be fundamental for the proof of our results.

LEMMA 2.1.   *Let X be a space. Suppose $\{K_\lambda \mid \lambda \in \Lambda\}$ a discrete collection of closed subsets of X satisfies the following conditions* (i) *and* (ii) *below:*

(i) *for each $n \in N$ there exists a locally finite collection $\mathcal{O}_n = \{O_\lambda^n \mid \lambda \in \Lambda\}$ of open subsets of X such that*

$$K_\lambda \subset \bigcup\{O_\lambda^n \mid n \in N\}$$

for every $\lambda \in \Lambda$,

(ii) *there exists an open subset W_λ of X such that*

$$K_\lambda \subset W_\lambda \quad \text{and} \quad \overline{W_\lambda} \cap \left(\bigcup\{K_\mu \mid \mu \neq \lambda, \mu \in \Lambda\}\right) = \varnothing$$

for every $\lambda \in \Lambda$.

Then there exists a disjoint collection $\{Q_\lambda \mid \lambda \in \Lambda\}$ of open subsets of X such that $K_\lambda \subset Q_\lambda$ for every $\lambda \in \Lambda$.

PROOF.   Suppose $\{K_\lambda \mid \lambda \in \Lambda\}$ satisfies the conditions (i) and (ii). Let $\{\mathcal{O}_n \mid n \in N\}$ and $\{W_\lambda \mid \lambda \in \Lambda\}$ be the collections described in the conditions (i) and (ii). Here we put

$$R_\lambda^n = O_\lambda^n \cap W_\lambda$$

for each $\lambda \in \Lambda$ and $n \in N$. Then $\{R_\lambda^n \mid \lambda \in \Lambda\}$ is a locally finite collection of open subsets of X for every $n \in N$, which satisfies

$$\overline{R_\lambda^n} \cap \left(\bigcup\{K_\mu \mid \mu \neq \lambda, \mu \in \Lambda\}\right) = \varnothing$$

for each $\lambda \in \Lambda$, $n \in N$ and

$$K_\lambda \subset \bigcup\{R_\lambda^n \mid n \in N\}$$

for every $\lambda \in \Lambda$. We define

$$Q_\lambda = \bigcup \left\{ R_\lambda^n - \bigcup \{ \overline{R_\mu^i} \mid \mu \neq \lambda, \mu \in \Lambda; i \leq n \} \mid n \in N \right\}$$

for each $\lambda \in \Lambda$. Then we can easily show that $\{ Q_\lambda \mid \lambda \in \Lambda \}$ is a disjoint collection of open subsets of X such that $K_\lambda \subset Q_\lambda$ for every $\lambda \in \Lambda$. It completes the proof of Lemma 2.1. $\square$

REMARK. X is collectionwise normal if and only if X is normal and every discrete collection $\{ K_\lambda \mid \lambda \in \Lambda \}$ of closed subsets of X satisfies the condition (i) of Lemma 2.1.

We need the following lemmas for our results. Our proofs of the lemmas are based on the proof of [16, Theorem 2.2].

LEMMA 2.2. *Let X be a collectionwise normal Σ-space and Y a collectionwise normal first countable P-space. Let $\{ K_\lambda \mid \lambda \in \Lambda \}$ be a locally finite collection of closed subsets of $X \times Y$. Then $\{ K_\lambda \mid \lambda \in \Lambda \}$ satisfies the condition (i) of Lemma 2.1.*

PROOF. Let $\{ \mathscr{E}_n \mid n \in N \}$ be a Σ-net of X, where we express

$$\mathscr{E}_n = \{ E(\alpha_1, \ldots, \alpha_n) \mid \alpha_1, \ldots, \alpha_n \in \Omega \}.$$

Since X is collectionwise normal and countably paracompact, for each $n \in N$ there exists a locally finite open cover $\{ L(\alpha_1, \ldots, \alpha_n) \mid \alpha_1, \ldots, \alpha_n \in \Omega \}$ of X such that

$$(1) \qquad\qquad E(\alpha_1, \ldots, \alpha_n) \subset L(\alpha_1, \ldots, \alpha_n)$$

for each $\alpha_1, \ldots, \alpha_n \in \Omega$.

Define $\Delta = \{ \Gamma \subset \Lambda \mid \operatorname{Card} \Gamma \text{ is finite} \}$. For each $\alpha_1, \ldots, \alpha_n \in \Omega$, $n \in N$ and $\Gamma \in \Delta$, let us put

$$G_\Gamma(\alpha_1, \ldots, \alpha_n) = \bigcup \left\{ O \mid O \text{ is open in } Y \text{ and} \right.$$

$$\left. (E(\alpha_1, \ldots, \alpha_n) \times O) \cap \left(\bigcup \{ K_\lambda \mid \lambda \notin \Gamma \} \right) = \varnothing \right\}.$$

Then $G_\Gamma(\alpha_1, \ldots, \alpha_n)$ is open in Y and we have

$$G_\Gamma(\alpha_1, \ldots, \alpha_n) \subset G_\Gamma(\alpha_1, \ldots, \alpha_n, \alpha_{n+1})$$

for each $\alpha_1, \ldots, \alpha_n, \alpha_{n+1} \in \Omega$ since $E(\alpha_1, \ldots, \alpha_n) \supset E(\alpha_1, \ldots, \alpha_n, \alpha_{n+1})$. Hence if we put

$$G(\alpha_1, \ldots, \alpha_n) = \bigcup \{ G_\Gamma(\alpha_1, \ldots, \alpha_n) \mid \Gamma \in \Delta \},$$

then $G(\alpha_1, \ldots, \alpha_n)$ is an open subset of Y and we have

$$G(\alpha_1, \ldots, \alpha_n) \subset G(\alpha_1, \ldots, \alpha_n, \alpha_{n+1})$$

for each $\alpha_1, \ldots, \alpha_n, \alpha_{n+1} \in \Omega$. Since Y is a P-space, there exists a collection

$$\{ M(\alpha_1, \ldots, \alpha_n) \mid \alpha_1, \ldots, \alpha_n \in \Omega; n \in N \}$$

of closed subsets of Y such that

$$(2) \qquad\qquad M(\alpha_1, \ldots, \alpha_n) \subset G(\alpha_1, \ldots, \alpha_n)$$

for each $\alpha_1, \ldots, \alpha_n \in \Omega; n \in N$, and

$$(3) \qquad Y = \bigcup \{ G(\alpha_1, \ldots, \alpha_n) \mid n \in N \} \Longrightarrow Y = \bigcup \{ M(\alpha_1, \ldots, \alpha_n) \mid n \in N \}.$$

Here we may assume

$$(4) \qquad\qquad M(\alpha_1, \ldots, \alpha_n) \subset M(\alpha_1, \ldots, \alpha_n, \alpha_{n+1})$$

for each $\alpha_1, \ldots, \alpha_n, \alpha_{n+1} \in \Omega$.

Fix $n \in N$ and $\alpha_1, \ldots, \alpha_n \in \Omega$ arbitrarily. Define

$$P_\Gamma(\alpha_1, \ldots, \alpha_n) = \{ y \in Y \mid (E(\alpha_1, \ldots, \alpha_n) \times \{y\}) \cap K_\lambda \neq \varnothing \text{ if and only if } \lambda \in \Gamma \}$$

for each $\Gamma \in \Delta$. We show that the collection

$$(5) \qquad\qquad \{ M(\alpha_1, \ldots, \alpha_n) \cap P_\Gamma(\alpha_1, \ldots, \alpha_n) \mid \Gamma \in \Delta \}$$

is locally finite in Y. To prove this, let $y \in Y$ and we show that the above collection is locally finite at y. Since $M(\alpha_1, \ldots, \alpha_n)$ is closed in Y, we may assume that $y \in M(\alpha_1, \ldots, \alpha_n)$. Then by (2) we have $y \in G(\alpha_1, \ldots, \alpha_n)$, and hence there exists a $\Gamma_y \in \Delta$ such that $y \in G_{\Gamma_y}(\alpha_1, \ldots, \alpha_n)$. Suppose

$$G_{\Gamma_y}(\alpha_1, \ldots, \alpha_n) \cap P_\Gamma(\alpha_1, \ldots, \alpha_n) \neq \varnothing,$$

then we show $\Gamma \subset \Gamma_y$. To show this, let a $\lambda \in \Gamma$. Select a point

$$z \in G_{\Gamma_y}(\alpha_1, \ldots, \alpha_n) \cap P_\Gamma(\alpha_1, \ldots, \alpha_n).$$

Since $z \in P_\Gamma(\alpha_1, \ldots, \alpha_n)$, we have

$$(E(\alpha_1, \ldots, \alpha_n) \times \{z\}) \cap K_\lambda \neq \varnothing,$$

so

(6) $$(E(\alpha_1, \ldots, \alpha_n) \times G_{\Gamma_y}(\alpha_1, \ldots, \alpha_n)) \cap K_\lambda \neq \varnothing.$$

By the definition of $G_{\Gamma_y}(\alpha_1, \ldots, \alpha_n)$, we have

(7) $$(E(\alpha_1, \ldots, \alpha_n) \times G_{\Gamma_y}(\alpha_1, \ldots, \alpha_n)) \cap \left(\bigcup \{K_\mu \mid \mu \notin \Gamma_y\} \right) = \varnothing.$$

The formulations (6) and (7) show $\lambda \in \Gamma_y$, it follows that $\Gamma \subset \Gamma_y$. And since $G_{\Gamma_y}(\alpha_1, \ldots, \alpha_n)$ is an open neighborhood of y and Card Γ_y is finite, we have shown that the collection (5) above is localy finite at y.

Since Y is collectionwise normal and countably paracompact, there exists a locally finite collection $\{H_\Gamma(\alpha_1, \ldots, \alpha_n) \mid \Gamma \in \Delta\}$ of open subsets of Y such that

$$M(\alpha_1, \ldots, \alpha_n) \cap P_\Gamma(\alpha_1, \ldots, \alpha_n) \subset H_\Gamma(\alpha_1, \ldots, \alpha_n)$$

for each $\Gamma \in \Delta$.

We define

$$O_\lambda^n = \bigcup \{L(\alpha_1, \ldots, \alpha_n) \times H_\Gamma(\alpha_1, \ldots, \alpha_n) \mid \Gamma \in \Delta \text{ and } \lambda \in \Gamma; \ \alpha_1, \ldots, \alpha_n \in \Omega\},$$

for each $n \in N$ and $\lambda \in \Lambda$. Then we can see that $\{O_\lambda^n \mid \lambda \in \Lambda\}$ is a locally finite collection of open subsets of $X \times Y$ for each $n \in N$.

We shall show that $K_\lambda \subset \bigcup \{O_\lambda^n \mid n \in N\}$ for every $\lambda \in \Lambda$. To see this, let $(x, y) \in K_\lambda$. We choose $\alpha_1, \alpha_2, \ldots \in \Omega$ so that

$$\{E(\alpha_1, \ldots, \alpha_n) \mid n \in N\} \text{ is a local net of } C(x).$$

Before everything, we show that $Y = \bigcup \{G(\alpha_1, \ldots, \alpha_n) \mid n \in N\}$. Let any $z \in Y$. Put

$$\Gamma_{xz} = \{\mu \in \Lambda \mid (C(x) \times \{z\}) \cap K_\mu \neq \varnothing\}.$$

Since $C(x) \times \{z\}$ is countably compact and $\{K_\mu \mid \mu \in \Lambda\}$ is locally finite, Γ_{xz} is finite, that is, $\Gamma_{xz} \in \Delta$. Moreover, since $C(x) \times \{z\}$ is countably compact and Y is first countable, there exist open subsets O and O' of X and Y, respectively, such that

$$C(x) \times \{z\} \subset O \times O' \subset X \times Y - \bigcup \{K_\mu \mid \mu \notin \Gamma_{xz}\}.$$

From the property of the local net, there exists an $n \in N$ such that

$$C(x) \subset E(\alpha_1, \ldots, \alpha_n) \subset O.$$

Therefore

$$(E(\alpha_1, \ldots, \alpha_n) \times O') \cap \left(\bigcup \{K_\mu \mid \mu \notin \Gamma_{xz}\} \right) = \varnothing.$$

Thus we can verify $z \in O' \subset G_{\Gamma_{xz}}(\alpha_1, \ldots, \alpha_n) \subset G(\alpha_1, \ldots, \alpha_n)$. Hence we have $Y = \bigcup \{G(\alpha_1, \ldots, \alpha_n) \mid n \in N\}$.

It follows from (3) that

$$Y = \bigcup \{M(\alpha_1, \ldots, \alpha_n) \mid n \in N\}.$$

So there exists an $n \in N$ such that $y \in M(\alpha_1, \ldots, \alpha_n)$. Let

$$\Gamma_{xy} = \{\mu \in \Lambda \mid (C(x) \times \{y\}) \cap K_\mu \neq \varnothing\}.$$

Likewise the matter shown above, we have $\Gamma_{xy} \in \Delta$, and there exist open subsets O_x and O_y of X and Y, respectively, such that

$$C(x) \times \{y\} \subset O_x \times O_y \subset X \times Y - \bigcup \{K_\mu \mid \mu \notin \Gamma_{xy}\}.$$

From the property of the local net, there exists an $m \in N$ such that

$$C(x) \times \{y\} \subset E(\alpha_1, \ldots, \alpha_m) \times \{y\} \subset O_x \times O_y,$$

where we can select $m \geq n$. So we have

$$(8) \qquad (E(\alpha_1, \ldots, \alpha_m) \times \{y\}) \cap \left(\bigcup \{K_\mu \mid \mu \notin \Gamma_{xy}\}\right) = \varnothing.$$

Moreover, by the definition of Γ_{xy} and the fact $C(x) \subset E(\alpha_1, \ldots, a_m)$, we have

$$(9) \qquad (E(\alpha_1, \ldots, \alpha_m) \times \{y\}) \cap K_\mu \neq \varnothing \quad \text{for every } \mu \in \Gamma_{xy}.$$

It follows from the formulations (8) and (9) that

$$y \in P_{\Gamma_{xy}}(\alpha_1, \ldots, \alpha_m).$$

By (4), $y \in M(\alpha_1, \ldots, \alpha_n) \subset M(\alpha_1, \ldots, \alpha_m)$. So we have

$$(10) \qquad y \in M(\alpha_1, \ldots, \alpha_m) \cap P_{\Gamma_{xy}}(\alpha_1, \ldots, \alpha_m) \subset H_{\Gamma_{xy}}(\alpha_1, \ldots, \alpha_m).$$

It follows from (1) and (10) that

$$(x, y) \in L(\alpha_1, \ldots, \alpha_m) \times H_{\Gamma_{xy}}(\alpha_1, \ldots, \alpha_m).$$

Since $(x, y) \in K_\lambda$, it is clear that $\lambda \in \Gamma_{xy}$. Thus we have $(x, y) \in O_\lambda^m$, which proves that $K_\lambda \subset \bigcup \{O_\lambda^n \mid n \in N\}$. It follows that $\{K_\lambda \mid \lambda \in \Lambda\}$ satisfies the condition (i) of Lemma 2.1. This completes the proof of the lemma. $\qquad \square$

If $C(x)$ in the proof of Lemma 2.2 is compact, then without using the first countability of Y the similar proof to the above yields the following lemma.

LEMMA 2.3.   *Let X be a paracompact Σ-space and Y a collectionwise normal P-space. Let $\{K_\lambda \mid \lambda \in \Lambda\}$ be a locally finite collection of closed subsets of $X \times Y$. Then $\{K_\lambda \mid \lambda \in \Lambda\}$ satisfies the condition* (i) *of Lemma* 2.1.

3.   The proofs of Theorems

Let us proceed to the proofs of Theorems.

PROOF OF THEOREM 1.1.   It is shown by Remark of Lemma 2.1 and Lemma 2.2.   □

PROOF OF THEOREM 1.2.   Let Z be a normal M-space and f a closed continuous map from Z onto X. Then we can express that $X = \bigcup \{X_i \mid i \geq 0\}$, where X_i is closed discrete for every $i \geq 1$ and $f^{-1}(x)$ is countably compact for each $x \in X_0$ (see Nagata [11]).

First we observe that for a subset A of $Z \times Y$ the following equality holds:

$$(11) \qquad (f \times 1_Y)(\bar{A}) \cap (X_0 \times Y) = \overline{(f \times 1_Y)(A)} \cap (X_0 \times Y).$$

Let $\{D_\lambda \mid \lambda \in \Lambda\}$ be a discrete collection of closed subsets of $X \times Y$. For each $i \geq 1$, we can take a discrete collection $\{U_\lambda^i \mid \lambda \in \Lambda\}$ of open subsets of $X \times Y$ such that

$$D_\lambda \cap (X_i \times Y) \subset U_\lambda^i$$

for every $\lambda \in \Lambda$, because X and Y are collectionwise normal.
Let

$$F_\lambda = D_\lambda - \bigcup \{U_\lambda^i \mid i \geq 1\}$$

for each $\lambda \in \Lambda$. Then F_λ is closed in $X \times Y$ and we have $F_\lambda \subset X_0 \times Y$.

Here $\{(f \times 1_Y)^{-1}(F_\lambda) \mid \lambda \in \Lambda\}$ is a discrete collection of closed subsets of $Z \times Y$.

Since Z is a normal M-space (therefore a collectionwise normal Σ-space), it follows that $\{(f \times 1_Y)^{-1}(F_\lambda) \mid \lambda \in \Lambda\}$ satisfies the condition (i) of Lemma 2.1 because of Lemma 2.2.

Since $X \times Y$ is normal, for each $\lambda \in \Lambda$, there exists an open subset W_λ of $X \times Y$ such that

$$F_\lambda \subset W_\lambda \subset \overline{W_\lambda} \subset X \times Y - \bigcup \{F_\mu \mid \mu \neq \lambda, \mu \in \Lambda\}.$$

Then $\{(f \times 1_Y)^{-1}(W_\lambda) \mid \lambda \in \Lambda\}$ is the required collection for $\{(f \times 1_Y)^{-1}(F_\lambda) \mid \lambda \in \Lambda\}$ to satisfy the condition (ii) of Lemma 2.1. Hence $\{(f \times 1_Y)^{-1}(F_\lambda) \mid \lambda \in \Lambda\}$ satisfies the condition (ii) of Lemma 2.1.

By Lemma 2.1, we can take a disjoint collection $\{Q_\lambda \mid \lambda \in \Lambda\}$ of open subsets of $Z \times Y$ such that

$$(f \times 1_Y)^{-1}(F_\lambda) \subset Q_\lambda$$

for each $\lambda \in \Lambda$.

Define

$$V_\lambda = X \times Y - \overline{(f \times 1_Y)(Z \times Y - Q_\lambda)}$$

for each $\lambda \in \Lambda$. It is clear that $\{V_\lambda \mid \lambda \in \Lambda\}$ is a disjoint collection of open subsets of $X \times Y$. Since $F_\lambda \subset X_0 \times Y$, we can show that

$$F_\lambda \subset V_\lambda$$

for each $\lambda \in \Lambda$ by (11). By the normality of $X \times Y$, there exists a discrete collection $\{U_\lambda^0 \mid \lambda \in \Lambda\}$ of open subsets of $X \times Y$ such that

$$F_\lambda \subset U_\lambda^0 \subset V_\lambda$$

for each $\lambda \in \Lambda$.

The collection $\{U_\lambda^i \mid \lambda \in \Lambda, i \geq 0\}$ has the properties that

$$D_\lambda \subset \bigcup \{U_\lambda^i \mid i \geq 0\}$$

for each $\lambda \in \Lambda$ and that $\{U_\lambda^i \mid \lambda \in \Lambda\}$ is discrete for each $i \geq 0$. Namely $\{D_\lambda \mid \lambda \in \Lambda\}$ satisfies the condition (i) of Lemma 2.1, and also satisfies the condition (ii) of Lemma 2.1 because of the normality of $X \times Y$. Therefore we can get a disjoint collection $\{B_\lambda \mid \lambda \in \Lambda\}$ of open subsets of $X \times Y$ such that $D_\lambda \subset B_\lambda$ for each $\lambda \in \Lambda$ by Lemma 2.1. Hence $X \times Y$ is collectionwise normal, which completes the proof of Theorem 1.2. □

PROOF OF THEOREM 1.3. Let Z be a paracompact M-space and f a closed continuous map from Z onto X. We can express $X = \bigcup \{X_i \mid i \geq 0\}$ that has the properties of the proof of Theorem 1.2. Let $\{D_\lambda \mid \lambda \in \Lambda\}$ be a discrete collection of closed subsets of $X \times Y$. The proof of this theorem is similar to that of Theorem 1.2, because we can use Lemma 2.3 instead of Lemma 2.2. So we can take a discrete collection $\{U_\lambda^i \mid \lambda \in \Lambda\}$ of open subsets of $X \times Y$ for each $i \geq 0$ such that

$$D_\lambda \subset \bigcup \{U_\lambda^i \mid i \geq 0\}$$

for each $\lambda \in \Lambda$ as the proof of Theorem 1.2. Hence $\{D_\lambda \mid \lambda \in \Lambda\}$ satisfies the condition (i) of Lemma 2.1. Since $X \times Y$ is normal, $\{D_\lambda \mid \lambda \in \Lambda\}$ satisfies the condition (ii) of Lemma 2.1. Therefore we can get a disjoint collection $\{B_\lambda \mid \lambda \in \Lambda\}$ of open subsets of $X \times Y$ such that $D_\lambda \subset B_\lambda$ for each $\lambda \in \Lambda$. Hence $X \times Y$ is collectionwise normal, which completes the proof of Theorem 1.3.   □

REMARK. Since ω_1 is a collectionwise normal first countable M-space and $\omega_1 + 1$ is a compact space, the assumption of the normality of $X \times Y$ of Theorems 1.1, 1.2 and 1.3 can not be dropped.

4. Some problems

It is well-known by Nagami [10, Corollary 4.2] that for a paracompact Σ-space X and a paracompact P-space Y the product $X \times Y$ is paracompact. If we replace the paracompactness of X and Y by the collectionwise normality, then even the normality of $X \times Y$ need not implied in general. Thus the following problem naturally arises. [16, Theorem 2.2] or Theorem 1.1 can be regarded as a partial answer to this problem.

PROBLEM 4.1. *Let X be a collectionwise normal Σ-space and Y a collectionwise normal (or a paracompact) P-space. Is it true that $X \times Y$ is collectionwise normal if and only if $X \times Y$ is normal?*

Corresponding to Nagami's result above, we have the following theorem.

THEOREM 4.2. *Let X be the closed image of a paracompact M-space and Y a paracompact P-space. Then $X \times Y$ is paracompact if and only if $X \times Y$ normal.*

PROOF. First we note the fact that for spaces X and Y given in the theorem $X \times Y$ is normal iff $X \times Y$ is countably paracompact; the proof is similar to Bešlagić and Chiba [1, Section 5]. Assume $X \times Y$ is normal, therefore it is countably paracompact. Let K be a compact space. Then $Y \times K$ is a paracompact P-space, and by the fact above the countably paracompact space $(X \times Y) \times K = X \times (Y \times K)$ is normal. Hence $X \times Y$ is paracompact from Tamano's theorem [15, Theorem 2].   □

In view of this result, under the similar consideration to Problem 4.1, the following problem also arises. A partial answer to this problem is Theorem 1.2 or Theorem 1.3.

PROBLEM 4.3. *Let X be the closed image of a normal M-space and Y a collectionwise normal (or a paracompact) P-space. Is it true that $X \times Y$ is collectionwise normal if and only if $X \times Y$ is normal?*

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