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ON THE FIRST EIGENVALUE OF NON-ORIENTABLE CLOSED SURFACES

By

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0. Introduction

Let (M, g) be a 2-dimensional non-orientable closed Riemannian manifold. We study the spectrum of the Laplacian for functions on (M, g). We express it by

$$\mathrm{Spec}(M, g) = \{0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \cdots\}.$$

Let $(\tilde{M}, \tilde{g})$ be the orientable Riemannian double cover of (M, g). Our interest is what properties are preserved between (M, g) and $(\tilde{M}, \tilde{g})$. The positive first eigenvalue $\lambda_1(M, g)$ has many geometric informations. We have interests in the influences for the positive first eigenvalue by taking the Riemannian double cover. Generally we have $\lambda_1(M, g) \geq \lambda_1(\tilde{M}, \tilde{g})$. So we study the difference between $\lambda_1(M, g)$ and $\lambda_1(\tilde{M}, \tilde{g})$. Especially we find the cases that $\lambda_1(M, g) = \lambda_1(\tilde{M}, \tilde{g})$ holds good.

It is well-known (cf. [9]) that 2-dimensional closed manifolds are classified as follows.

The Classification Theorem of Closed Surfaces. *A closed surface is homeomorphic to one of the following spaces.*

$$S^2, T^2, \#^n T^2 \ (n \geq 2) : \textit{orientable}$$

$$\boldsymbol{RP}^2, \#^n \boldsymbol{RP}^2 \ (n \geq 2) : \textit{non-orientable}$$

where $\#^n M$ means the connected sum of n-copies of a manifold M. Moreover the double cover of $\#^n \boldsymbol{RP}^2$ $(n \geq 2)$ is homeomorphic to $\#^{n-1} T^2$.

In this paper we show the following results.

THEOREM A. *If M is homeomorphic to $\boldsymbol{RP}^2$, then*

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$$\lambda_1(M,g) > \lambda_1(\tilde{M},\tilde{g})$$

for every metric g on M.

THEOREM B. *If M is homeomorphic to $\#^n\boldsymbol{RP}^2$ $(n \geq 2)$, there exists a metric g on M such that*

$$\lambda_1(M,g) = \lambda_1(\tilde{M},\tilde{g}).$$

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1. Preliminaries

Let us consider the Riemannian double covering $(\tilde{M},\tilde{g})$ of (M,g). We define the isometry $J:(\tilde{M},\tilde{g}) \to (\tilde{M},\tilde{g})$ as follows. For each point p of (M,g), let two points $\tilde{p}_1$, $\tilde{p}_2$ of $(\tilde{M},\tilde{g})$ be the fiber of a point p in (M,g). Then we define J by exchanging two points $\tilde{p}_1$, $\tilde{p}_2$. Let $E(\lambda)$, $E^+(\lambda)$ and $E^-(\lambda)$ be the spaces of C^∞ functions on $\tilde{M}$ such that

$$E(\lambda) = \text{the eigenspace associated with the eigenvalue } \lambda,$$

$$E^+(\lambda) = \{f \in E(\lambda) \mid f \circ J = f\},$$

$$E^-(\lambda) = \{f \in E(\lambda) \mid f \circ J = -f\}.$$

PROPOSITION (P. Buser [3], p. 306). *$E(\lambda)$ is decomposed orthogonally as $E^+(\lambda) \oplus E^-(\lambda)$.*

Since all eigenfunctions on (M,g) are lifted to ones on $(\tilde{M},\tilde{g})$ canonically, the eigenvalues on (M,g) are in $\mathrm{Spec}(\tilde{M},\tilde{g})$. The eigenfunctions on $(\tilde{M},\tilde{g})$ which come from ones on (M,g) are invariant by J. Conversely every $f \in E^+(\lambda)$ is reduced to the eigenfunction on (M,g). The eigenvalues on (M,g) coincide with the ones on $(\tilde{M},\tilde{g})$ satisfying $E^+(\lambda) \neq \{0\}$.

The eigenfunctions on $(\tilde{M},\tilde{g})$ which do not come from ones on (M,g) have non-zero components of $E^-(\lambda)$ under the above decomposition. Thus we concentrate our attension on the non-zero smallest λ such as $E^-(\lambda) \neq \{0\}$. We denote it by ν. Our purpose is to compare $\lambda_1(M,g)$ with ν.

2. Proof of Theorem A

In this section the main tool is the nodal domain theorem (cf. [6] and [7]).

PROPOSITION (S. Y. Cheng [6], p. 186). *Let g be any Riemannian metric on S^2. Then the nodal line of a first eigenfunction is a smooth simple closed curve.*

We assume that (M,g) is homeomorphic to $\boldsymbol{RP}^2$. Then its Riemannian double cover $(\tilde{M},\tilde{g})$ is homeomorphic to S^2. If the first eigenfunction φ on (M,g) lifts to the first eigenfunction $\tilde{\varphi}$ on $(\tilde{M},\tilde{g})$, the nodal set of $\tilde{\varphi}$ is a simple closed curve and the number of the nodal domains is two. We denote one of them by D. Since $\pi_1(\tilde{M}) \cong \{e\}$, the nodal set ∂D is contractible. Then D is homeomorphic to an open 2-disk. Its boundary ∂D is regularly embedded. Then its closure $\bar{D}$ is homeomorphic to a closed 2-disk. Without loss of generality we take D the positive nodal domain. Since $\tilde{\varphi}$ is invariant by the isometry J, we have $J(\bar{D}) = \bar{D}$.

We apply the Brouwer's fixed poin theorem (cf. [8] p. 19) to $J : \bar{D} \to \bar{D}$. Then J has a fixed point. But by the definition of J, it does not have any fixed points. It is a contradiction.

3. Proof of Theorem B

The first eigenvalue λ_1 is characterized by the Rayleigh quotient, that is,

$$\lambda_1(M,g) = \inf \frac{\int_M |\nabla f|^2 \, dv}{\int_M f^2 \, dv}$$

where f runs over all non-vanishing functions orthogonal to constant functions in $L^2(M,g)$.

Our method to prove Proposition B is an analogue of Cheeger's construction in [5] of the deformation of Riemannian metrics g_ε on S^2 such that $\lambda_1(S^2,g_\varepsilon)$ converges to 0 as $\varepsilon \to 0$.

Let us recall the construction.

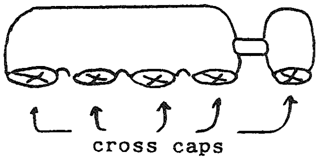
Step 1. Connect two canonical spheres, $(S^2,g)\#(S^2,g)$, by the tube whose radius is ε and length is l. We express it by (S,g_ε).

Step 2. We consider the test functions f_ε which is equal to c on the right-hand bulb, $-c$ on the left-hand bulb and change linearly from c to $-c$ across tube. We choose c so $\int_S f_\varepsilon^2 \, dv = 1$.

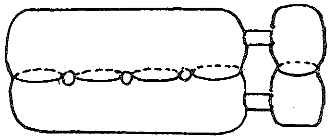
Step 3. By the Rayleigh quotient, $\lambda_1(S,g_\varepsilon)$ converges to 0 as $\varepsilon \to 0$.

Now we proceed this method to $\#^n \boldsymbol{RP}^2$ for $n \geq 2$. We consider $\#^n \boldsymbol{RP}^2$ as $(\#^{n-1}\boldsymbol{RP}^2)\#\boldsymbol{RP}^2$. We take a suitable metric on $\#^n \boldsymbol{RP}^2$ in such a way that

$$\frac{\mathrm{Vol}(\#^{n-1}\boldsymbol{RP}^2)}{\mathrm{Vol}(\boldsymbol{RP}^2)} = n - 1.$$



$(M, g_\varepsilon) = (\#^{n-1}RP^2)\#RP^2$



$(\tilde{M}, \tilde{g}_\varepsilon)$

We connect $\#^{n-1}\boldsymbol{RP}^2$ and $\boldsymbol{RP}^2$ by the tube whose radius is ε and length is l. We denote it by (M, g_ε). We consider the test functions f_ε which are equal to 1 on $\#^{n-1}\boldsymbol{RP}^2$ and $-(n-1)$ on $\boldsymbol{RP}^2$, and decrease from 1 to $-(n-1)$ across the tube. We may take f_ε such that they are orthogonal to constant functions in $L^2(M, g_\varepsilon)$ and $|\nabla f_\varepsilon| \le d$ where d is a constant depending only on l and n. It follows from the Rayleigh quotient that

$$\lim_{\varepsilon \to 0} \lambda_1(M, g_\varepsilon) = 0.$$

Let $(\tilde{M}, \tilde{g}_\varepsilon)$ be the double cover of (M, g_ε). Then $\lambda_1(M, g_\varepsilon)$ belongs to in $\mathrm{Spec}(\tilde{M}, \tilde{g}_\varepsilon)$. We denote it by $\lambda(\varepsilon)$ in brief. In Sect.1, we treat $E^-(\lambda)$. We denote by $\nu(\varepsilon)$ the non-zero first $E^-(\lambda)$-type eigenvalue on $(\tilde{M}, \tilde{g}_\varepsilon)$. We only have to compare $\lambda(\varepsilon)$ with $\nu(\varepsilon)$.

PROPOSITION (C. Anné and B. Colbois [2]). *Let (M_1, g_1) and (M_2, g_2) be two connected orientable Riemannian manifolds of the same dimension. We connected them by two tubes whose radii are both ε and lengths l_1 and l_2, respectively. We denote it by (M, g_ε) with a little smoothing at the connected parts. We express* $\mathrm{Spec}(M, g_\varepsilon)$ *as*

$$\mathrm{Spec}(M, g_\varepsilon) = \{\lambda_0(\varepsilon) < \lambda_1(\varepsilon) \le \lambda_2(\varepsilon) \le \cdots\}.$$

Let

$$\{\mu_0 \le \mu_1 \le \mu_2 \le \mu_3 \le \cdots\}$$

be the union of $\mathrm{Spec}(M_1, g_1)$, $\mathrm{Spec}(M_2, g_2)$, $\mathrm{Spec}_D([0, l_1], \mathrm{can})$ *and* $\mathrm{Spec}_D([0, l_2], \mathrm{can})$ (*the D of the* Spec_D *means the Dirichlet condition*) *counting with multiplicity.*

Then for any n we have

$$\lim_{\varepsilon \to 0} \lambda_n(\varepsilon) = \mu_n.$$

Especially $\lambda_0(\varepsilon) = 0$, $\mu_0 = \mu_1 = 0$ *and* $\mu_2 > 0$.

REMARK. In [2] they treat p-forms on (M,g). It is necessary for them to assume that $\dim M \geq 3$. But for functions the arguments there hold good in $\dim M = 2$. As for it see also [1].

We apply the above proposition to $(\tilde{M},\tilde{g}_\varepsilon)$. Since $\lim_{\varepsilon\to 0}\lambda(\varepsilon)=0$, $\nu(\varepsilon)$ converges to some positive value which is bigger than or equal to μ_2. The positive value depends on the equipped metrics on $\#^{n-1}\boldsymbol{RP}^2$, $\boldsymbol{RP}^2$ and the lengths of the tubes l_1, l_2. By the continuity of the eigenvalue in the parameter ε, there exists $\varepsilon_0 > 0$ such that

$$\lambda(\varepsilon) < \nu(\varepsilon)$$

for $0<\varepsilon<\varepsilon_0$.

Hence for all the metrics g_ε on M such that $0<\varepsilon<\varepsilon_0$, we have

$$\lambda_1(M,g_\varepsilon)=\lambda(\varepsilon)=\lambda_1(\tilde{M},\tilde{g}_\varepsilon).$$

4. Example

Here we give an example which clarifies Theorem B. Let $K(a,b)$ be the flat Klein bottle as the quotient space of $(\boldsymbol{R}^2,\mathrm{can})$ identifying by $(x,y)\mapsto(x,y+b)$ and $(x,y)\mapsto(x+a/2,-y)$. The double cover of $K(a,b)$ is the flat torus $T(a,b)$ as the quotient space of $(\boldsymbol{R}^2,\mathrm{can})$ identifying by $(x,y)\mapsto(x+a,y)$ and $(x,y)\mapsto(x,y+b)$.

The spectra of $K(a,b)$ and $T(a,b)$ are given in [4] as follows:

$$\mathrm{Spec}(K(a,b))=\left\{4\pi^2\left(\frac{m^2}{a^2}+\frac{n^2}{b^2}\right): m,n\in\boldsymbol{Z}\text{ and } n\neq 0 \text{ for } m:\text{odd}\right\},$$

$$\mathrm{Spec}(T(a,b))=\left\{4\pi^2\left(\frac{m^2}{a^2}+\frac{n^2}{b^2}\right): m,n\in\boldsymbol{Z}\right\}.$$

Then we have

$$\lambda_1(K(a,b))=\begin{cases}4\dfrac{\pi^2}{b^2} & \text{for } \dfrac{a}{2}\leq b,\\[2mm] 16\dfrac{\pi^2}{a^2} & \text{for } b\leq\dfrac{a}{2},\end{cases}$$

$$\lambda_1(T(a,b))=\begin{cases}4\dfrac{\pi^2}{b^2} & \text{for } a\leq b,\\[2mm] 4\dfrac{\pi^2}{a^2} & \text{for } b\leq a.\end{cases}$$

By comparing $\lambda_1(K(a,b))$ with $\lambda_1(T(a,b))$, we have

$$\lambda_1(K(a,b)) = \lambda_1(T(a,b))$$

for $a \leq b$.

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