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EXISTENCE OF WEAK SOLUTIONS FOR A PARABOLIC ELLIPTIC-HYPERBOLIC TRICOMI PROBLEM

By

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Abstract. It is well-known that the pioneer of mixed type boundary value problems is F. G. Tricomi (1923) with his Tricomi equation: $yu_{xx} + u_{yy} = 0$. In this paper we consider the more general case of above equation so that

$$Lu \equiv K_1(y)u_{xx} + (K_2(y)u_y)' + ru = f$$

is hyperbolic-elliptic and parabolic, and then prove the existence of weak solutions for the corresponding Tricomi problem by employing the well-known a-b-c energy integral method to establish an a-priori estimate. This result is interesting in fluid mechanics.

The Tricomi Problem

Consider the parabolic elliptic-hyperbolic equation

$$Lu \equiv K_1(y)u_{xx} + (K_2(y)u_y)' + r(x,y)u = f(x,y), \qquad (*)$$

([2], [6]), in a bounded simply-connected domain $D(\subset \Re^2)$ with a piecewise-smooth boundary $G = \partial D = g_1 \cup g_2 \cup g_3$, where $f = f(x,y)$ is continuous, $r = r(x,y)$ (<0) and $K_1 = K_1(y)$ are once-continuously differentiable for $x \in [-1,1]$ and $y \in [-m, M]$ with $-m = \inf\{y : (x,y) \in D\}$, and $M = \sup\{y : (x,y) \in D\}$, and $K_1(y) > 0$ for $y > 0$, $= 0$ for $y = 0$, and < 0 for $y < 0$. Also $K_2 = K_2(y)$ is twice-continuously differentiable in $[-m, M]$, $K_2(y) > 0$ in D. Besides $\lim_{y\to 0} K(y)$ exists, if $K = K(y) = K_1(y)/K_2(y) > 0$ whenever $y > 0$, $= 0$ whenever $y = 0$, and < 0 whenever $y < 0$.

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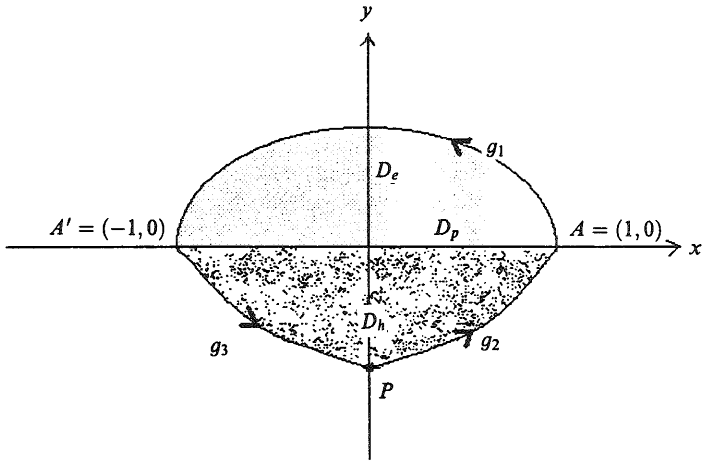

Finally g_1 is "the elliptic arc (for $y > 0$)" connecting points $A' = (-1,0)$ and $A = (1,0)$, g_2 is "the hyperbolic characteristic arc (for $y < 0$)" connecting points $A = (1,0)$ and $P = (0, y_p)$: $\int_0^{y_p} \sqrt{-K(t)}\,dt = -1$ (**e.g.** if $K_1 = y$ and $K_2 = 1$, then $y_p = -(3/2)^{2/3} \cong -1.31$), $g_2(\equiv \mathrm{PA})$: $x = \int_0^y \sqrt{-K(t)}\,dt + 1$, and g_3 is "the hyperbolic characteristic arc (for $y < 0$)" connecting points $A' = (-1,0)$ and $P = (0, y_p)$: $g_3(\equiv A'P)$: $x = -\int_0^y \sqrt{-K(t)}\,dt - 1$.

Denote "the elliptic subregion of D" by D_e (= the space bounded by g_1 and $A'A$), "the hyperbolic subregion of D" by D_h (= the space bounded by g_2, g_3 and AA'), and "the parabolic arc of D" by

$$D_p(\equiv A'A) = \{(x,y) \in D:\ -1 < x < 1, y = 0\}.$$

Note that the **order** of equation (*) does not **degenerate** on the line $y = 0$. But (*) is parabolic for $y = 0$ because $K_1(0) = 0$ and $K_2(0) > 0$ hold simultaneously.

Assume **boundary condition**

$$u = 0 \quad \text{on } g_1 \cup g_2. \tag{**}$$

The Tricomi problem, or Problem (T) consists in finding a function $u = u(x,y)$ which satisfies equation (*) in D and boundary condition (**) on $g_1 \cup g_2$ ([4], [5], [7]).

PRELIMINARIES. Denote $\alpha = (\alpha_1, \alpha_2)$: $\alpha_1,\ \alpha_2 \geq 0$, $|\alpha| = \alpha_1 + \alpha_2$. Also if $p = (x,y) \in \Re^2$, and $\tilde{p} = (\tilde{x}, \tilde{y}) \in \Re^2$, then denote $p^\alpha = x^{\alpha_1} y^{\alpha_2}$, $\langle p, \tilde{p} \rangle = x\tilde{x} + y\tilde{y}$, $|p| = (\langle p, p \rangle)^{1/2}$.

Finally denote

$$D_1 = \frac{\partial}{\partial x}, \quad D_2 = \frac{\partial}{\partial y}, \quad \text{and} \quad (D^\alpha u)(p) = (D_1^{\alpha_1} D_2^{\alpha_2} u)(p)$$

for sufficiently smooth functions $u = u(p) : p = (x, y) \in \Re^2$. Consider **the adjoint equation**

$$L^+ w \equiv K_1(y) w_{xx} + (K_2(y) w_y)' + r(x, y) w = f(x, y), \qquad [*]$$

([1]–[2], [6]), in D, where L^+ is the **formal adjoint operator** of the formal operator L and is $L^+ = L$. (**Note** that equations for characteristics of (*) and [*] are identical). In fact,

$$(K_2(y) w_y)' = K_2(y) w_{yy} + K_2'(y) w_y, \quad \text{and}$$

thus

$$L^+ w = (K_1(y) w)_{xx} + (K_2(y) w)_{yy} - (K_2'(y) w)_y + r(x, y) w$$
$$= Lw, \quad \text{because } (K_2(y) w)_{yy} = (K_2(y) w_y)' + (K_2'(y) w)_y.$$

Note in general that if

$$Lu \equiv \sum_{i,j=1}^{2} a_{ij}(p) D_i D_j u + \sum_{i=1}^{2} a_i(p) D_i u + a(p) u, \quad \text{then}$$

$$L^+ w \equiv \sum_{i,j=1}^{2} D_i D_j (a_{ij}(p) w) - \sum_{i=1}^{2} D_i (a_i(p) w) + a(p) w.$$

Assume **adjoint boundary condition**

$$w = 0 \quad \text{on } g_1 \cup g_3. \qquad [**]$$

Denote

$$C^2(\bar{D}) = \{u(p) \mid p = (x, y) \in \bar{D}(= D \cup G): \ u = u(p)$$
$$\text{is twice-continuously differentiable in } \bar{D}\}.$$

This space is complete normed space with norm

$$\|u\|_{C^2(\bar{D})} = \max\{|D^a u(p)| \mid p \in \bar{D}: \ |a| \le 2\}.$$

Also denote

$$L^2(D) = \left\{ u \,\middle|\, \int_D |u(p)|^2\, dp < \infty \right\}.$$

The **norm** of space $L^2(D)$ is

$$\|u\|_0 = \|u\|_{L^2(D)} = \left(\int_D |u(p)|^2\, dp \right)^{1/2},$$

where $p = (x, y)$, and $dp = dxdy$.

Besides denote

$$D(L) = \{u \in C^2(\bar{D}) \colon \; u = 0 \;\text{ on }\; g_1 \cup g_2\},$$

which is the **domain** of the formal operator L, and

$$D(L^+) = \{w \in C^2(\bar{D}) \colon \; w = 0 \;\text{ on }\; g_1 \cup g_3\},$$

which is the **domain** of the adjoint operator L^+.

Finally denote

$$W_2^2(D) = \{u \mid D^a u(\cdot) \in L^2(D),\; |a| \le 2\}$$

which is **the complete normed Sobolev space** with norm

$$\|u\|_2 = \|u\|_{W_2^2(D)} = \left(\|u\|^2_{L^2(D)} + \sum_{|\alpha|=2} \|D^\alpha u\|^2_{L^2(D)} \right)^{1/2},$$

or equivalently: $\|u\|_2 = (\sum_{|\alpha| \le 2} \|D^\alpha u\|^2_{L^2(D)})^{1/2}$,

$$W_2^2(D, bd) = \overline{D(L)}_{\|\cdot\|_2},$$

which is the closure of domain $D(L)$ with norm $\|\cdot\|_2$, and

$$W_2^2(D, bd^+) = \overline{D(L^+)}_{\|\cdot\|_2},$$

which is the closure of domain $D(L^+)$ with norm $\|\cdot\|_2$, or equivalently:

$$W_2^2(D, bd^+) = \{w \in W_2^2(D) \colon \; \langle Lu, w \rangle_0 = \langle u, L^+ w \rangle_0 \text{ for all } u \in W_2^2(D, bd)\}$$

on the corresponding **norms**.

DEFINITION. A function $u = u(p) \in L^2(D)$ is **a weak solution** of Problem (T) if

$$\langle f, w\rangle_0 = \langle u, L^+w\rangle_0 \quad ([4]_{(2)}, \text{p. } 86\text{–}106)$$

holds for all $w \in W_2^2(D, bd^+)$ ([4]$_{(2)}$, p. 86–106).

Criterion ([1]). (i). A necessary and sufficient condition for **the existence of a weak solution** of Problem (T) is that the following a-priori estimate

$$\|w\|_0 \leq C\|L^+w\|_0, \tag{AP}$$

holds for all $w \in W_2^2(D, bd^+)$, and for some $C = \text{const.} > 0$ ([4]$_{(2)}$, p. 86–106).

(ii). A sufficient condition **for the existence of a weak solution** of Problem (T) is that the following a-priori estimate

$$\|w\|_1 \leq C\|L^+w\|_0, \tag{$[AP]$}$$

holds for all $w \in W_2^2(D, bd^+)$, and for some $C = \text{const.} > 0$.

Also **note** that both **the Hahn-Banach Theorem** and **the Riesz Representation Theorem** would play ([4]$_{(2)}$, p. 92–95) an important role in this paper if above criterion were **not** employed. For **the justification of the definition of weak solutions** we apply Green's theorem ([4]$_{(2)}$, p. 95–98) and classical techniques in order to show that $f = Lu$ in D and $u = 0$ on $g_1 \cup g_2$.

A-Priori estimate ($[AP]$)

We apply the $a - b - c$ classical energy integral method and use adjoint boundary condition [**]. Then **claim** that the a-priori estimate $[AP]$ holds for all $w \in W_2^2(D, bd^+)$, and for some $C = \text{const.} > 0$.

In fact, we investigate

$$J^+ = 2\langle M^+w, L^+w\rangle_0 = \iint_D 2M^+wL^+w\,dxdy \tag{1}$$

where

$$M^+w = a^+(x,y)w + b^+(x,y)w_x + c^+(x,y)w_y \quad \text{in } D,$$

with **choices**:

$$a^+ = -\frac{1}{2}, \quad \text{and } b^+ = x - c_1 \quad \text{in } D, \quad \text{and } c^+ = \begin{cases} y + c_2 & \text{for } y \geq 0 \\ c_2 & \text{for } y \leq 0 \end{cases}, \tag{2}$$

where $c_1 = 1 + c_0$, and c_0, c_2: are positive constants.

Consider **the ordinary identities**:

$$2aK_1ww_{xx} = (2aK_1ww_x)_x - 2aK_1w_x^2 - (a_xK_1w_x^2)_x + a_{xx}K_1w^2,$$

$$2aK_2ww_{yy} = (2aK_2ww_y)_y - 2aK_2w_y^2 - ((aK_2)_yw^2)_y + (aK_2)_{yy}w^2,$$

$$2bK_1w_xw_{xx} = (bK_1w_x^2)_x - b_xK_1w_x^2,$$

$$2bK_2w_xw_{yy} = (2bK_2w_xw_y)_y - (bK_2w_y^2)_x + b_xK_2w_y^2 - 2(bK_2)_yw_xw_y,$$

$$2cK_1w_yw_{xx} = (2cK_1w_xw_y)_x - (cK_1w_x^2)_y + (cK_1)_yw_x^2 - 2K_1c_xw_xw_y,$$

$$2cK_2w_yw_{yy} = (cK_2w_y^2)_y - (cK_2)_yw_y^2,$$

$$2arww = 2arw^2, \quad 2brww_x = (brw^2)_x - (br)_xw^2,$$

$$2crww_y = (crw^2)_y - (cr)_yw^2, \quad 2atww_y = (atw^2)_y - (at)_yw^2,$$

$$2btw_xw_y = 2btw_xw_y, \quad 2ctw_yw_y = 2ctw_y^2,$$

where t ($\equiv$ coefficient of w_y in L^+w), or

$$t = K_2'(y). \tag{3}$$

Then employing above identities and Green's theorem, and setting $t = K_2'(y)$ we obtain from (1) and [*] that

$$J^+ = \iint_D 2(a^+w + b^+w_x + c^+w_y)[K_1(y)w_{xx} + K_2(y)w_{yy} + rw + tw_y]\,dxdy$$
$$= I_D^+ + I_{1G}^+ + I_{2G}^+ + I_{3G}^+, \tag{4}$$

where

$$I_D^+ = \iint_D (A^+w_x^2 + B^+w_y^2 + C^+w^2 + 2D^+w_xw_y)\,dxdy,$$

$$I_{1G}^+ = \oint_{G(=\partial D)} \{2a^+w(K_1w_xv_1 + K_2w_yv_2)\}\,ds,$$

$$I_{2G}^+ = \oint_{G(=\partial D)} \{-[K_1a_x^+v_1 + (a^+K_2)_yv_2] + [(b^+v_1 + c^+v_2)r] + [(a^+v_2)t]\}w^2\,ds,$$

and

$$I_{3G}^+ = \oint_G (\tilde{A}^+w_x^2 + \tilde{B}^+w_y^2 + 2\tilde{D}^+w_xw_y)\,ds,$$

with

$$A^+ = -2a^+K_1 - b_x^+K_1 + (c^+K_1)_y,$$

$$B^+ = -2a^+K_2 + b_x^+K_2 - (c^+K_2)_y + 2c^+t,$$

$$C^+ = [2a^+r + K_1a_{xx}^+ + (a^+K_2)_{yy}] - [(b^+r)_x + (c^+r)_y] - [(a^+t)_y],$$

$$D^+ = -[K_1c_x^+ + (b^+K_2)_y - b^+t], \quad \text{and}$$

$$\tilde{A}^+ = (b^+\nu_1 - c^+\nu_2)K_1, \quad \tilde{B}^+ = (-b^+\nu_1 + c^+\nu_2)K_2,$$

$$\tilde{D}^+ = b^+K_2\nu_2 + c^+K_1\nu_1, \quad \text{where}$$

$$\nu = (\nu_1, \nu_2) = \left(\frac{dy}{ds}, -\frac{dx}{ds}\right), \quad (ds > 0), \tag{5}$$

is the outer unit normal vector on the boundary G of the mixed domain D.

Note that **in $D, y \geq 0$** (if $a^+ = -1/2$, $b^+ = x - c_1$, $c^+ = y + c_2$):

$$A^+ = K_1 - (K_1) + ((y + c_2)K_1)_y = K_1 + (y + c_2)K_1',$$

$$B^+ = K_2 + (K_2) - ((y + c_2)K_2)_y + 2(y + c_2)t = K_2 + (y + c_2)K_2',$$

$$C^+ = \left[-r - \frac{1}{2}K_2''\right] - [((x - c_1)r)_x + ((y + c_2)r)_y] - \left[-\frac{1}{2}K_2''\right]$$

$$= -[3r + (x - c_1)r_x + (y + c_2)r_y], \quad \text{and}$$

$$D^+ = -[((x - c_1)K_2)_y - (x - c_1)t]$$

$$= -[(x - c_1)K_2' - (x - c_1)K_2'] = 0,$$

because from (3): $t = K_2'(y)$.

Similarly **in $D, y \leq 0$** (if $a^+ = -1/2$, $b^+ = x - c_1$, $c^+ = c_2$):

$$A^+ = K_1 - (K_1) + (c_2K_1)_y = c_2K_1',$$

$$B^+ = K_2 + (K_2) - (c_2K_2)_y + 2c_2t = 2K_2 + c_2K_2',$$

$$C^+ = \left[-r - \frac{1}{2}K_2''\right] - [((x - c_1)r)_x + (c_2r)_y] - \left[-\frac{1}{2}K_2''\right]$$

$$= -[2r + (x - c_1)r_x + c_2r_y], \quad \text{and}$$

$$D^+ = -[((x - c_1)K_2)_y - (x - c_1)t] = 0,$$

because from (3): $t = K_2'(y)$.

Therefore

$$I_D^+ = I_{1D}^+ + I_{2D}^+ + I_0^+, \tag{6}$$

where $Q = A^+ w_x^2 + B^+ w_y^2 + 2D^+ w_x w_y = Q(u_x, u_y)$,

$$I_{1D}^+ = \iint_{D, y \geq 0} Q(w_x, w_y)\, dxdy, \text{ or}$$

$$I_{1D}^+ = \iint_{D, y \geq 0} [(K_1 + (y + c_2)K_1')w_x^2 + (K_2 + (y + c_2)K_2')w_y^2]\, dxdy, \tag{6$_1$}$$

$$I_{2D}^+ = \iint_{D, y \leq 0} Q(w_x, w_y)\, dxdy, \text{ or}$$

$$I_{2D}^+ = \iint_{D, y \leq 0} [(c_2 K_1')w_x^2 + (2K_2 + c_2 K_2')w_y^2]\, dxdy, \tag{6$_2$}$$

and

$$I_0^+ = \iint_D C^+ w^2\, dxdy, \quad \text{or}$$

$$I_0^+ = \begin{cases} -\iint_{D, y \geq 0} [3r + (x - c_1)r_x + (y + c_2)r_y]w^2\, dxdy \\ -\iint_{D, y \leq 0} [2r + (x - c_1)r_x + c_2 r_y]w^2\, dxdy. \end{cases} \tag{6$_3$}$$

On G: claim that

$$I_{1G}^+ > 0. \tag{7}$$

In fact,

$$I_{1(g_1 \cup g_3)}^+ = -\int_{g_1 \cup g_3} \{w(K_1 w_x \nu_1 + K_2 w_y \nu_2)\}\, ds = 0, \tag{7$_1$}$$

because $w = 0$ on $g_1 \cup g_3$ from [**].

Also that

$$I_{1g_2}^+ = -\int_{g_2} \{w(K_1 w_x \nu_1 + K_2 w_y \nu_2)\}\, ds > 0. \tag{7$_2$}$$

In fact, **on** g_2:

$$dx = \sqrt{-K}\, dy, \quad \text{or } \nu_2 = -\sqrt{-K}\nu_1,$$

because $dx = -\nu_2\, ds$ and $dy = \nu_1\, ds$ from (5).

Also

$$\begin{aligned} dw &= w_x\,dx + w_y\,dy = (-w_x v_2 + w_y v_1)\,ds \\ &= (w_x\sqrt{-K} + w_y)v_1\,ds \ \text{(with } K = K_1/K_2) \\ &= \frac{\sqrt{-K_1}w_x + \sqrt{K_2}w_y}{\sqrt{K_2}} v_1\,ds \\ &= \frac{K_1 w_x - \sqrt{-K_1K_2}w_y}{-\sqrt{-K_1K_2}} v_1\,ds \\ &= \frac{K_1 w_x v_1 + K_2 w_y v_2}{-\sqrt{-K_1K_2}}\,ds \ \text{(because: } K_2 v_2 = -\sqrt{-K_1K_2}v_1) \end{aligned}$$

or

$$(K_1 w_x v_1 + K_2 w_y v_2)\,ds|_{g_2} = -\sqrt{-K_1K_2}\,dw. \tag*{$(7)_3$}$$

Therefore from $(7)_3$ and by integration by parts we get that

$$I^+_{1g_2} = \frac{1}{2}\int_{g_2} \sqrt{-K_1K_2}\,d(w^2) = -\frac{1}{2}\int_{g_2} (\sqrt{-K_1K_2})' w^2\,dy,$$

because $w = 0$ at the end-points of g_2 (as $w = 0$ on g_1 and $w = 0$ on g_3).

But

$$dy = v_1\,ds > 0 \quad \text{on } \boldsymbol{g_2}.$$

Thus

$$I^+_{1g_2} = \frac{1}{4}\int_{g_2} \frac{(K_1K_2)'}{\sqrt{-K_1K_2}} w^2\,dy > 0 \tag*{$(7)_4$}$$

from condition $[R_{1b}]$, completing the proof of $(7)_2$ and thus of (7) (from $(7)_1$).

Claim now that

$$I^+_{2G} > 0. \tag*{(8)}$$

In fact,

$$I^+_{2(g_1\cup g_3)} = \int_{g_1\cup g_3} \left\{ \left[\frac{1}{2}K_2' v_2\right] + [(b^+ v_1 + c^+ v_2)r] + \left[-\frac{1}{2}K_2' v_2\right] \right\} w^2\,ds, \quad \text{or}$$

$$I^+_{2(g_1\cup g_3)} = \int_{g_1\cup g_3} \{[(b^+ v_1 + c^+ v_2)r]w^2\}\,ds = 0, \tag*{$(8)_1$}$$

because $w = 0$ on $g_1 \cup g_3$ from [**] and $t = K_2'$ from (3).

Also that

$$I^+_{2g_2} = \int_{g_2} \left\{ \left[\frac{1}{2} K_2' v_2 \right] + [((x - c_1)v_1 + c_2 v_2) r] + \left[-\frac{1}{2} K_2' v_2 \right] \right\} w^2 \, ds,$$

or

$$I^+_{2g_2} = \int_{g_2} \{[(x - c_1)v_1 + c_2 v_2] r\} w^2 \, ds > 0, \tag{8$_2$}$$

from **condition** $[R_{1a}]$ and the fact that $(x - c_1)v_1 + c_2 v_2 < 0$ on $\boldsymbol{g_2}$ (as **on** $g_2 : v_1 > 0$, $v_2 < 0$ and $x - c_1 = \int_0^y \sqrt{-K(t)}\, dt - c_0 < 0$) completing the proof of (8), where

$$I^+_{2G} = I^+_{2(g_1 \cup g_3)} + I^+_{2g_2} = I^+_{2g_2} (> 0).$$

Claim then that

$$I^+_{3G} = \oint_G \tilde{Q}^+(w_x, w_y) \, ds > 0, \tag{9}$$

where

$$\tilde{Q}^+(w_x, w_y) = \tilde{A}^+ w_x^2 + \tilde{B}^+ w_y^2 + 2\tilde{D}^+ w_x w_y$$

is quadratic form with respect to w_x, and w_y on G.

In fact, **note** that **on** $\boldsymbol{g_1}$ (if $a^+ = -1/2$, $b^+ = x - c_1$, $c^+ = y + c_2$):

$$\tilde{A}^+ = [(x - c_1)v_1 - (y + c_2)v_2] K_1, \quad \tilde{B}^+ = [-(x - c_1)v_1 + (y + c_2)v_2] K_2,$$

$$\tilde{D}^+ = (x - c_1) K_2 v_2 + (y + c_2) K_1 v_1.$$

From adjoint boundary condition [**] we get

$$0 = dw|_{g_1} = w_x \, dx + w_y \, dy, \quad \text{or}$$

$$w_x = N^+ v_1, \; w_y = N^+ v_2, \tag{9a}$$

where $N^+ =$ normalizing factor. Therefore

$$I^+_{3g_1} = \int_{g_1} \tilde{Q}^+(w_x, w_y) \, ds = \int_{g_1} (N^+)^2 [(x - c_1)v_1 + (y + c_2)v_2] H \, ds, \tag{10}$$

where

$$H = K_1 v_1^2 + K_2 v_2^2 \; (> 0 \text{ on } g_1). \tag{10a}$$

It is clear from (10)–(10a) and **condition** $[R_2]$ that

$$I^+_{3g_1} = \int_{g_1} (N^+)^2 [(x - c_1)\, dy - (y + c_2)\, dx] H \geq 0. \tag{10b}$$

Similarly **on** g_3 (if $a^+ = -1/2$, $b^+ = x - c_1$, $c^+ = c_2$):

$$I^+_{3g_3} = \int_{g_3} \tilde{Q}^+(w_x, w_y)\, ds = \int_{g_3} (N^+)^2[(x - c_1)v_1 + c_2 v_2]H\, ds, \quad \text{or}$$

$$I^+_{3g_3} = \int_{g_3} (N^+)^2[(x - c_1)\, dy - c_2\, dx]H = 0, \tag{11}$$

because

$$H = 0 \quad \text{on } g_3, \tag{11a}$$

as g_3 is **characteristic**.

Finally **claim** that **on** g_2 (if $a^+ = -1/2$, $b^+ = x - c_1$, $c^+ = c_2$):

$$I^+_{3g_2} = \int_{g_2} \tilde{Q}^+(w_x, w_y)\, ds > 0. \tag{12}$$

In fact, $\tilde{Q}^+ = \tilde{Q}^+(w_x, w_y)$ is **non-negative definite on** g_2. It is clear that

$$\tilde{A}^+ = [(x - c_1)v_1 - c_2 v_2]K_1 > 0 \quad \textbf{on } g_2,$$

because of

$$(x - c_1)|_{g_2} = \int_0^y \sqrt{-K(t)}\, dt - c_0 < 0 \quad \text{on } g_2,$$

$$v_1 = \frac{dy}{ds}\bigg|_{g_2} > 0, \quad v_2 = -\frac{dx}{ds}\bigg|_{g_2} < 0, \quad K_1|_{g_2} < 0,$$

$v_2 = -\sqrt{-K}v_1$ on g_2, and of **condition** $[R_6]$. In fact,

$$[(x - c_1)v_1 - c_2 v_2]|_{g_2} = \left[\left(\int_0^y \sqrt{-K(t)}\, dt - c_0\right) + \sqrt{-K}c_2\right]v_1$$

$$= \left(\int_0^y \sqrt{-K(t)}\, dt + c_2\sqrt{-K} - c_0\right)v_1 > 0 \quad \text{on } g_2$$

from **condition** $[R_6]$. Therefore

$$\tilde{A}^+ = \left(\int_0^y \sqrt{-K(t)}\, dt + c_2\sqrt{-K} - c_0\right)v_1 K_1 > 0 \quad \text{on } g_2. \tag{12a}$$

Also

$$\tilde{B}^+ = [-(x - c_1)v_1 + c_2 v_2]K_2, \quad \text{or}$$

$$\tilde{B}^+ = -\left(\int_0^y \sqrt{-K(t)}\, dt + c_2\sqrt{-K} - c_0\right)v_1 K_2 > 0 \quad \text{on } g_2 \tag{12b}$$

because of **condition** $[R_6]$, $v_1|_{g_2} > 0$, $K_2|_{g_2} > 0$, and of above facts. **Note** that

$$\tilde{A}^+ = (-K)\tilde{B}^+ \quad \text{on } g_2. \tag{[12a]}$$

Besides

$$\tilde{D}^+ = (x - c_1)K_2 v_2 + c_2 K_1 v_1, \quad \text{or}$$

$$\tilde{D}^+ = \left[-\left(\int_0^y \sqrt{-K(t)}\,dt - c_0\right)K_2\sqrt{-K} + c_2 K_1\right]v_1, \quad \text{or}$$

$$\tilde{D}^+ = -\sqrt{-K_1 K_2}\left(\int_0^y \sqrt{-K(t)}\,dt + c_2\sqrt{-K} - c_0\right)v_1 \quad \text{on } g_2, \tag{12c}$$

because

$$-K_1/K_2\sqrt{-K} = \sqrt{-K} \quad \text{and} \quad K_2\sqrt{-K} = \sqrt{-K_1 K_2}.$$

Note that

$$\tilde{D}^+ = \sqrt{-K}\tilde{B}^+ \quad \text{on } g_2, \tag{[12c]}$$

because $\sqrt{-K_1 K_2} = \sqrt{-K}K_2$.

Finally from [12a] and [12c], we get

$$\tilde{A}^+\tilde{B}^+ - (\tilde{D}^+)^2 = 0 \quad \text{on } g_2. \tag{[12d]}$$

Therefore the quadratic form $\tilde{Q}^+$ is

$$\tilde{Q}^+ = \tilde{Q}^+(w_x, w_y) = (\sqrt{-K}w_x + w_y)^2(\tilde{B}^+) > 0 \quad \text{on } g_2, \quad \text{or}$$

$$\tilde{Q}^+\,ds = -(\sqrt{-K}w_x + w_y)^2\left(\int_0^y \sqrt{-K(t)}\,dt + c_2\sqrt{-K} - c_0\right)K_2\,dy, \quad \text{or}$$

$$I^+_{3g_2} = -\int_{g_2} (\sqrt{-K}w_x + w_y)^2\left(\int_0^y \sqrt{-K(t)}\,dt + c_2\sqrt{-K} - c_0\right)K_2\,dy > 0, \tag{[12]}$$

because of **condition** $[R_6]$, $dy(= v_1\,ds)|_{g_2} > 0$, and $K_2 > 0$ on g_2, completing the proof of (12).

Therefore

$$I^+_G = I^+_{1G} + I^+_{2G} + I^+_{3G}, \quad \text{or} \tag{13}$$

$$I_G^+ = \frac{1}{4}\int_{g_2} \frac{(K_1K_2)'}{\sqrt{-K_1K_2}} w^2\, dy$$

$$+ \int_{g_2} \{[(x-c_1)v_1 + c_2v_2]r\} w^2\, ds$$

$$+ \int_{g_1} (N^+)^2[(x-c_1)\, dy - (y+c_2)\, dx]H$$

$$- \int_{g_2} (\sqrt{-K}w_x + w_y)^2 \left(\int_0^y \sqrt{-K(t)}\, dt + c_2\sqrt{-K} - c_0\right) K_2\, dy. \qquad (14)$$

But **on** g_2(: $dx = \sqrt{-K}\, dy$)

$$[(x-c_1)v_1 + c_2v_2]\, ds = (x-c_1)\, dy - c_2\, dx = [(x-c_1) - c_2\sqrt{-K}]\, dy$$

$$= \left(\int_0^y \sqrt{-K(t)}\, dt - c_2\sqrt{-K} - c_0\right) dy\ (< 0). \qquad (14a)$$

Thus

$$I_G^+ = \int_{g_1} (N^+)^2[(x-c_1)\, dy - (y+c_2)\, dx]H$$

$$+ \int_{g_2} \left\{ w^2\left[\frac{1}{4}\frac{(K_1K_2)'}{\sqrt{-K_1K_2}} + r\left(\int_0^y \sqrt{-K(t)}\, dt - c_2\sqrt{-K} - c_0\right)\right]\right.$$

$$\left. - \left[(\sqrt{-K}w_x + w_y)^2 \left(\int_0^y \sqrt{-K(t)}\, dt + c_2\sqrt{-K} - c_0\right) K_2\right]\right\} dy > 0, \qquad (15)$$

where $H = K_1v_1^2 + K_2v_2^2$ (> 0 on g_1), and $N^+ =$ normalizing factor: $w_x = N^+v_1$, $w_y = N^-v_2$ (on g_1).

Note from (15) that the two conditions ($[R_{1a}]$–$[R_{1b}]$) could be replaced by the following **condition** $[\boldsymbol{R_1}]$ **on** g_2:

$$[R_1]\ :\ (K_1K_2)' + 4r\sqrt{-K_1K_2}\left(\int_0^y \sqrt{-K(t)}\, dt - c_2\sqrt{-K} - c_0\right) > 0. \qquad (16)$$

Similarly

$$I_D^+ = I_{D,y\geq 0}^+ + I_{D,y\leq 0}^+, \quad \text{or} \qquad (17)$$

$$I_D^+ = \iint_{D,y\geq 0} \{-(3r + (x - c_1)r_x + (y + c_2)r_y)w^2$$

$$+ (K_1 + (y + c_2)K_1')w_x^2 + (K_2 + (y + c_2)K_2')w_y^2\}\, dxdy$$

$$+ \iint_{D,y\leq 0} \{-(2r + (x - c_1)r_x + c_2 r_y)w^2$$

$$+ (c_2 K_1')w_x^2 + (2K_2 + c_2 K_2')w_y^2\}\, dxdy. \tag{18}$$

It is clear now from (4), (15), and (18) that

$$J^+ = I_D^+ + I_G^+ > I_D^+, \tag{19}$$

$$\mu a^2 + \frac{1}{\mu} b^2 \geq 2|ab|, \quad \mu > 0. \tag{20}$$

But from (1) we get

$$2M^+ wL^+ w = 2a^+ wL^+ w + 2b^+ w_x L^+ w + 2c^+ w_y L^+ w. \tag{21}$$

Therefore from (1), (20) and (21) we find

$$J^+ \leq \iint_D 2|M^+ wL^+ w|\, dxdy$$

$$\leq \iint_D \{2|a^+ w|\,|L^+ w| + 2|b^+ w_x|\,|L^+ w| + 2|c^+ w_y|\,|L^+ w|\}\, dxdy$$

$$\leq \iint_D \left\{ \left[\mu_1 (a^+ w)^2 + \frac{1}{\mu_1}(L^+ w)^2\right] + \left[\mu_2 (b^+ w_x)^2 + \frac{1}{\mu_2}(L^+ w)^2\right] \right.$$

$$\left. + \left[\mu_3 (c^+ w_y)^2 + \frac{1}{\mu_3}(L^+ w)^2\right] \right\} dxdy, \quad \text{or}$$

$$J^+ \leq \iint_D T^+(w, w_x, w_y)\, dxdy + \left(\frac{1}{\mu_1} + \frac{1}{\mu_2} + \frac{1}{\mu_3}\right) \iint_D (L^+ w)^2\, dxdy, \tag{22}$$

where $\mu_i = \text{const.} > 0 (i = 1, 2, 3)$, and

$$T^+ = T^+(w, w_x, w_y) = \mu_1 (a^+)^2 w^2 + \mu_2 (b^+)^2 (w_x)^2 + \mu_3 (c^+)^2 (w_y)^2.$$

Denote

$$C_1 = \sqrt{\frac{1}{\mu_1} + \frac{1}{\mu_2} + \frac{1}{\mu_3}} \quad (> 0). \tag{23}$$

Thus from (19) and (22)–(23) we get

$$I_D^+ < J^+ \leq \iint_D T^+(w, w_x, w_y)\,dxdy + C_1^2\|L^+w\|_0^2, \quad \text{or}$$

$$I_D^+ - \iint_D T^+(w, w_x, w_y)\,dxdy < C_1^2\|L^+w\|_0^2. \tag{24}$$

Therefore from (2), (18) and (24) we find

$$\begin{aligned}
&\iint_{D,y\geq 0}\Big\{-\Big[(3r + (x-c_1)r_x + (y+c_2)r_y) + \frac{1}{4}\mu_1\Big]w^2\\
&+ [(K_1 + (y+c_2)K_1') - \mu_2(x-c_1)^2]w_x^2\\
&+ [(K_2 + (y+c_2)K_2') - \mu_3(y+c_2)^2]w_y^2\Big\}\,dxdy\\
&+ \iint_{D,y\leq 0}\Big\{-\Big[(2r + (x-c_1)r_x + c_2r_y) + \frac{1}{4}\mu_1\Big]w^2\\
&+ [(c_2K_1') - \mu_2(x-c_1)^2]w_x^2\\
&+ [(2K_2 + c_2K_2') - \mu_3(c_2)^2]w_y^2\Big\}\,dxdy\\
&< C_1^2\|L^+w\|_0^2.
\end{aligned} \tag{25}$$

But

$$\|w\|_1^2 = \left(\iint_{D,y\geq 0} + \iint_{D,y\leq 0}\right)(w^2 + w_x^2 + w_y^2)\,dxdy. \tag{26}$$

Thus from (25)–(26) and conditions ($[R_3]$–$[R_4]$–$[R_5]$) we get

$$C_2^2\|w\|_1^2 < C_1^2\|L^+w\|_0^2, \quad \text{or}$$

$$\|w\|_1^2 < C^2\|L^+w\|_0^2,$$

with $C = C_1/C_2 = \text{const.} > 0$, completing the proof of the **a-priori estimate** $[AP]$.

Note that

$$C_2 = \sqrt{\min(\delta_{11},\delta_{21},\delta_{31}) + \min(\delta_{12},\delta_{22},\delta_{32})}\ (> 0), \tag{27}$$

where

$$\delta_{ij} = \text{const.} > 0 \ (i = 1,2,3; j = 1,2) \ \text{ in conditions } ([R_3]\text{-}[R_4]\text{-}[R_5]).$$

Therefore by above Criterion ([1]) the following **Existence Theorem** holds.

Existence Theorem

Consider Problem (T) with parabolic elliptic-hyperbolic equation:

$$Lu \equiv K_1(y)u_{xx} + (K_2(y)u_y)' + r(x,y)u = f(x,y),$$

and boundary condition: $u = 0$ on $g_1 \cup g_2$. Also consider the simply-connected domain $D(\subset \Re^2)$ bounded by a piecewise-smooth boundary $G = \partial D = g_1 \cup g_2 \cup g_3$: curve g_1 (for $y > 0$) connecting $A' = (-1,0)$ and $A = (1,0)$, and characteristics g_2, g_3 (for $y < 0$) such that g_2: $x = \int_0^y \sqrt{-K(t)}dt + 1$, g_3: $x = -\int_0^y \sqrt{-K(t)}dt - 1$, and $K = K_1/K_2$: $\lim_{y \to 0} K(y)$ exists, $K_1(y) > 0$ whenever $y > 0$, $= 0$ whenever $y = 0$, and < 0 whenever $y < 0$, as well as $K_2(y) > 0$ in D.

Assume **conditions**:

$$[R_{1a}]: \ r < 0 \quad \text{on } g_2,$$

$$[R_{1b}]: \ (K_1K_2)' > 0 \quad \text{on } g_2,$$

$$[R_{1c}]: \ K_i' > 0 \ (i = 1,2) \quad \text{in } D,$$

$$[R_2]: \ (x - c_1)dy - (y + c_2)dx \geq 0: \text{ “}\textbf{star-likedness}\text{” } \quad \text{on } g_1,$$

$$[R_3]: \ \begin{cases} 4(3r + (x - c_1)r_x + (y + c_2)r_y) + \mu_1 \leq -4\delta_{11} < 0 & \text{for } y \geq 0 \\ 4(2r + (x - c_1)r_x + c_2 r_y) + \mu_1 \leq -4\delta_{12} < 0 & \text{for } y \leq 0, \end{cases}$$

$$[R_4]: \ \begin{cases} K_1 + (y + c_2)K_1' - \mu_2(x - c_1)^2 \geq \delta_{21} > 0 & \text{for } y \geq 0 \\ c_2K_1' - \mu_2(x - c_1)^2 \geq \delta_{22} > 0 & \text{for } y \leq 0, \end{cases}$$

$$[R_5]: \ \begin{cases} K_2 + (y + c_2)K_2' - \mu_3(y + c_2)^2 \geq \delta_{31} > 0 & \text{for } y \geq 0 \\ 2K_2 + c_2K_2' - \mu_3(c_2)^2 \geq \delta_{32} > 0 & \text{for } y \leq 0, \end{cases}$$

where δ_{ij} are positive constants $(i = 1,2,3; j = 1,2)$, and

$$[R_6]: \ \int_0^y \sqrt{-K(t)}\,dt + c_2\sqrt{-K(y)} - c_0 < 0 \quad \text{on } g_2,$$

where $K_i (i = 1,2)$, r, and f are sufficiently smooth, and $c_1 = 1 + c_0$, and c_0, c_2, and μ_i $(i = 1,2,3)$ are positive constants.

Then there exists **a weak solution** of Problem (T) in D.

SPECIAL CASE: In D take

$$K_1 = y \quad \text{and } K_2 = y - ky_p(>0), \quad \text{where } k = \text{constant} > 2 \quad \text{and}$$

$$y_p = \text{constant } (<0): \int_0^{y_p} \sqrt{-\frac{t}{t-ky_p}}\,dt = -1(y_p < t < 0), \quad \text{or equivalently}$$

$$y_p = 1 \Bigg/ \left(\sqrt{k-1} - k\tan^{-1}\frac{1}{\sqrt{k-1}}\right)(<0) \quad \text{for } k > 2.$$

Then conditions $[R_{1b}]$, $[R_4]$, $[R_5]$ and $[R_6]$ hold on $y = 0$ and in general in D. **Note** that substituting $\sqrt{-t/(t-ky_p)} = \varphi$, one gets that

$$\int_0^y \sqrt{-\frac{t}{t-ky_p}}\,dt = ky_p \tan^{-1}\sqrt{-\frac{y}{y-ky_p}} + \sqrt{-y(y-ky_p)},$$

where

$$\int \frac{2\varphi^2}{(1+\varphi^2)^2}\,d\varphi = \tan^{-1}\varphi - \frac{\varphi}{1+\varphi^2} + c.$$

NOTE that conditions ($[R_{1a}]$–$[R_{1b}]$) could be substituted by condition $[R_1]$ (16).

OPEN: If $r = 0$, then (25) does **not** yield existence of weak solution.

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