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A SHORT INTERVALS RESULT FOR $2n$-TWIN PRIMES IN ARITHMETIC PROGRESSIONS

By

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1. Introduction

In 1937 I. M. Vinogradov [11] proved that for every sufficiently large odd integer N the equation

$$N = p_1 + p_2 + p_3 \tag{1}$$

has solutions in prime numbers p_1, p_2, p_3. Vinogradov's estimate for linear trigonometric sums over primes enabled Chudakov, Estermann and Van der Corput (for example, see [1]) to prove in 1937 that almost all even integers $2n$ can be written as a sum of two primes (Goldbach's problem):

$$2n = p_1 + p_2. \tag{2}$$

The arguments used in the proof of (2) show that a similar result holds for the $2n$-twin primes problem (for example, see [3]):

$$2n = p_1 - p_2. \tag{3}$$

Zulauf [12], [13] obtained asymptotic formulas for the number of the prime solutions of the equations (1) and (2) with $p_1 \equiv l \,(\mathrm{mod}\, k)$, $(l,k) = 1$. Zulauf's formulas hold uniformly for $k \le L^D$, where $L = \log N$ and $D > 0$ is some fixed constant. Recently Tolev [9] established the following result. Let us consider integers k, l such that $(k,l) = 1$ and denote

$$J_{k,l}(N) = \sum_{\substack{p_1+p_2+p_3=N \\ p_1 \equiv l \,(\mathrm{mod}\, k)}} \log p_1 \log p_2 \log p_3.$$

Then, for every $A > 0$ there exists $B = B(A) > 0$ such that

$$\sum_{k \le \sqrt{N} L^{-B}} \max_{(l,k)=1} \left| J_{k,l}(N) - \frac{N^2}{2\varphi(k)} \mathfrak{S}_{k,l}(N) \right| \ll N^2 L^{-A}. \tag{4}$$

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In [9] the main tool is the Hardy—Littlewood circle method. Tolev applied the Bombieri—Vinogradov theorem to estimate the contribution of the major arcs and some arguments belonging to Mikawa to treat the minor arcs.

In 1992, using a variant of Linnik's dispersion method, Mikawa [4] proved the following result on $2n$-twin primes in arithmetic progressions, that is on the equation (3), with $p_2 \equiv l \pmod{k}$, $(l,k)=1$.

Let

$$(5) \quad E(N,k,l,2n) = \begin{cases} \displaystyle\sum_{\substack{m,h \le N \\ m-h=2n \\ h \equiv l \,(\mathrm{mod}\, k)}} \Lambda(m)\Lambda(h) - H(N,k,2n), & \text{if } (2n+l,k)=1 \\ \displaystyle\sum_{\substack{m,h \le N \\ m-h=2n \\ h \equiv l \,(\mathrm{mod}\, k)}} \Lambda(m)\Lambda(h), & \text{otherwise} \end{cases}$$

where Λ is von Mangoldt's function and

$$H(N,k,2n) = 2\prod_{p>2}\left(1-\frac{1}{(p-1)^2}\right)\prod_{\substack{p|nk \\ p>2}}\left(\frac{p-1}{p-2}\right)\frac{N-|2n|}{\varphi(k)}.$$

Then, for every $A>0$ there exists $B=B(A)>0$ such that

$$(6) \quad \sum_{k \le \sqrt{N}L^{-B}} \max_{(l,k)=1} \sum_{0<2n\le N} |E(N,k,l,2n)| \ll N^2L^{-A},$$

where the implied constant depends only on A.

Arguing in a standard way, it is easy to see that (4) is implied by (6). In this paper we establish a short intervals result for the equation (3) with $p_1 \equiv l \pmod{k}, (l,k)=1$, by following the method of [9] and combining some arguments of [5] and [7] in order to treat the minor arcs. More precisely, let

$$I_{k,l}(2n) = \sum_{\substack{p_2<2n \\ p_1-p_2=2n \\ p_1 \equiv l \,(\mathrm{mod}\, k)}} \log p_1 \log p_2,$$

$$\mathfrak{S}_{k,l}(2n) = \begin{cases} 2\displaystyle\prod_{p>2}\left(1-\frac{1}{(p-1)^2}\right)\prod_{\substack{p|nk \\ p>2}}\left(\frac{p-1}{p-2}\right), & \text{if } (2n-l,k)=1 \\ 0, & \text{otherwise.} \end{cases}$$

Our result is the following:

THEOREM 1. *Let $A > 0$, $0 < \varepsilon < 2/3$ be arbitrary constants and let $N^{1/3+\varepsilon} \le H \le N$. There exists a constant $B = B(A) > 0$ such that*

$$\sum_{k \le \mathscr{K}(N)} \max_{(l,k)=1} \sum_{N<2n\le N+H} \left| I_{k,l}(2n) - \frac{2n}{\varphi(k)} \mathfrak{S}_{k,l}(2n) \right| \ll HNL^{-A},$$

where $\mathscr{K}(N) = \min(HL^{-2A-6}, \sqrt{N}L^{-B})$.

The implied constant depends only on A and ε.

Now let us denote

$$\mathscr{E} = \sum_{k \le \mathscr{K}(N)} \max_{(l,k)=1} \sum_{N<2n\le N+H} \left| I_{k,l}(2n) - \frac{2n}{\varphi(k)} \mathfrak{S}_{k,l}(2n) \right|,$$

$$I^*_{k,l}(2n) = I^*_{k,l}(2n, N, Y) = \sum_{\substack{p_1 - p_2 = 2n \\ N < p_1 \le N+Y \\ p_2 \le Y \\ p_1 \equiv l \,(\mathrm{mod}\, k)}} \log p_1 \log p_2,$$

$$P^*(2n) = P^*(2n, N, Y) = \sum_{\substack{m_1 - m_2 = 2n \\ N < m_1 \le N+Y \\ m_2 \le Y}} 1.$$

If $Y = N$ and $N < 2n \le N + H$ we have $I^*_{k,l}(2n) = I_{k,l}(2n) + \mathcal{O}(HL/k)$, for $(l,k) = 1$, and $P^*(2n) = 2n + \mathcal{O}(H)$. Moreover, since $\mathfrak{S}_{k,l}(2n) \ll \log^3 L$ (see (22) below), then, for $Y = N$, we see that

$$\mathscr{E} \ll \sum_{k \le \mathscr{K}(N)} \max_{(l,k)=1} \sum_{N<2n\le N+H} \left| I^*_{k,l}(2n) - \frac{P^*(2n)}{\varphi(k)} \mathfrak{S}_{k,l}(2n) \right| + H^2L^2.$$

Hence Theorem 1 will be implied by the following:

THEOREM 2. *Let $A > 0$, $0 < \varepsilon < 2/5$, $7/12 < \theta \le 1$ be arbitrary constants and let $Y = N^\theta$, $Y^{1/3+\varepsilon} \le H \le Y$. There exist some positive constants $\delta, B = B(A)$ such that*

$$\sum_{k \le \mathscr{K}(Y)} \max_{(l,k)=1} \sum_{N<2n\le N+H} \left| I^*_{k,l}(2n) - \frac{P^*(2n)}{\varphi(k)} \mathfrak{S}_{k,l}(2n) \right| \ll HYL^{-A},$$

where $\mathscr{K}(Y) = \min(HL^{-2A-6}, \mathscr{H}(Y))$ and

$$\mathscr{H}(Y) = \mathscr{H}(Y, A, \varepsilon, \theta) = \begin{cases} YN^{-1/2}L^{-B} & \text{if } 3/5 + \varepsilon < \theta \le 1, \\ YN^{-11/20-\delta} & \text{if } 7/12 < \theta. \end{cases}$$

Here the implied constant depends only on A, ε for $3/5+\varepsilon<\theta\le 1$, while it depends on A, ε, δ and θ for $7/12<\theta$. The values of $\mathscr{H}(Y)$ are due to the short intervals version of the Bombieri-Vinogradov theorem (see [8]) we use to estimate the contribution of the major arcs. Note that $\mathscr{K}(Y)=\mathscr{H}(Y)$, whenever $\theta\le 3/4$.

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2. Notation

Let N be a sufficiently large integer and $\varepsilon>0$ be a sufficiently small real number. Let $A>0$ be an arbitrary constant and $B=A+5C+68$ with $C=3/\varepsilon(8A+47)$. Suppose that K is a positive real number such that $K\le\mathscr{K}(Y)$.

The lowercase letter p, with or without subscript, will always denote prime numbers. Let (m,n) denote the greatest common divisor and let $[m,n]$ denote the least common multiple of m and n. We shall use the convention that a congruence, $m\equiv n \pmod k$, will be written as $m\equiv n\ (k)$. As usual $\mu(n)$ is Möbius' function, $\varphi(n)$ is Euler's function and $\tau(n)$ denotes the number of positive divisors of n. Moreover $e(x)=\exp(2\pi ix)$ and $\|\beta\|$ is the distance of the real number β from the nearest integer.

Let c denote a positive real constant, which will not be necessarily the same in all instances. For example, this convention allows us to write

$$(\log x)e^{-c\sqrt{\log x}}\ll e^{-c\sqrt{\log x}}.$$

We denote

$$\sum_{a=1}^{q}{}^{*}=\sum_{\substack{a=1\\(a,q)=1}}^{q},\quad \sum_{q}^{\square}=\sum_{\mu(q)\neq 0}$$

$$Q=L^{C},\quad \tau=YQ^{-2},$$

$$E_1=\bigcup_{q\le Q}\bigcup_{\substack{a=0\\(a,q)=1}}^{q-1}\left(\frac{a}{q}-\frac{1}{q\tau},\frac{a}{q}+\frac{1}{q\tau}\right),\quad \text{major arcs},$$

$$E_2=\left(-\frac{1}{\tau},1-\frac{1}{\tau}\right)\Big\backslash E_1,\quad \text{minor arcs},$$

$$S_{k,l}(\alpha)=\sum_{\substack{N<p\le N+Y\\p\equiv l\ (k)}}(\log p)e(\alpha p),\quad S(\alpha)=\sum_{p\le Y}(\log p)e(-\alpha p),$$

$$M(\alpha) = \sum_{N<m\le N+Y} e(\alpha m), \quad M_o(\alpha) = \sum_{m\le Y} e(\alpha m),$$

$$\Delta(x, y, h) = \max_{x<r\le x+y} \max_{(m,h)=1} \left| \sum_{\substack{x<p\le r \\ p\equiv m\,(h)}} \log p - \frac{r-x}{\varphi(h)} \right|.$$

3. Outline of the Proof of Theorem 2

We have

$$I_{k,l}^*(2n) = \int_{-1/\tau}^{1-1/\tau} S_{k,l}(\alpha)S(\alpha)e(-2n\alpha)\,d\alpha = I_{k,l}^{(1)}(2n) + I_{k,l}^{(2)}(2n),$$

where

$$I_{k,l}^{(i)}(2n) = \int_{E_i} S_{k,l}(\alpha)S(\alpha)e(-2n\alpha)\,d\alpha, \quad i = 1, 2.$$

Consequently, if we denote

(7) $$\mathscr{E}^* = \sum_{k\le K} \max_{(l,k)=1} \sum_{N<2n\le N+H} \left| I_{k,l}^*(2n) - \frac{P^*(2n)}{\varphi(k)} \mathfrak{S}_{k,l}(2n) \right|,$$

then

(8) $$\mathscr{E}^* \le \mathscr{E}_1 + \mathscr{E}_2,$$

where

(9) $$\mathscr{E}_1 = \sum_{k\le K} \max_{(l,k)=1} \sum_{N<2n\le N+H} \left| I_{k,l}^{(1)}(2n) - \frac{P^*(2n)}{\varphi(k)} \mathfrak{S}_{k,l}(2n) \right|,$$

(10) $$\mathscr{E}_2 = \sum_{k\le K} \max_{(l,k)=1} \sum_{N<2n\le N+H} |I_{k,l}^{(2)}(2n)|.$$

Theorem 2 will follow from (7)–(10) and from the inequalities

(11) $$\mathscr{E}_1 \ll HYL^{-A}, \quad \mathscr{E}_2 \ll HYL^{-A}.$$

4. The Estimate of $\mathscr{E}_1$

It is clear that

(12) $$I_{k,l}^{(1)}(2n) = \sum_{q\le Q} \sideset{}{^*}\sum_{a=1}^{q} I_{k,l}(a, q),$$

where

$$I_{k,l}(a,q) = \int_{-1/(q\tau)}^{1/(q\tau)} S_{k,l}\left(\frac{a}{q}+\alpha\right) S\left(\frac{a}{q}+\alpha\right) e\left(-2n\left(\frac{a}{q}+\alpha\right)\right) d\alpha. \tag{13}$$

Let us consider $S_{k,l}(a/q+\alpha)$ for a, q, α satisfying

$$q \le Q, \quad (a,q)=1, \quad |\alpha| \le \frac{1}{q\tau} \tag{14}$$

For $(k,l)=1$, arguing as in [9], we have

$$S_{k,l}\left(\frac{a}{q}+\alpha\right) = \frac{c_{k,l}(a,q)}{\varphi([k,q])} M(\alpha) + \mathcal{O}(Q^2 L\Delta), \tag{15}$$

where

$$c_{k,l}(a,q) = \sum_{\substack{m=1 \\ m \equiv l\ ((k,q))}}^{q}{}^{*} e\left(\frac{am}{q}\right) \quad \text{and} \quad \Delta = \Delta(N, Y, [k,q]).$$

Under the condition (14) we see that

$$S\left(\frac{a}{q}+\alpha\right) = \frac{\mu(q)}{\varphi(q)} M_o(-\alpha) + \mathcal{O}(Ye^{-c\sqrt{L}}), \tag{16}$$

(see [10], Lemma 3.1).

Formulas (15), (16) and the trivial estimate

$$\left|S_{k,l}\left(\frac{a}{q}+\alpha\right)\right| \ll \frac{YL}{k}$$

imply that

$$\begin{aligned} S_{k,l}\left(\frac{a}{q}+\alpha\right) S\left(\frac{a}{q}+\alpha\right) e\left(-2n\left(\frac{a}{q}+\alpha\right)\right) \\ = \frac{\mu(q)}{\varphi(q)} \frac{c_{k,l}(a,q)}{\varphi([k,q])} e\left(-2n\frac{a}{q}\right) M(\alpha) M_o(-\alpha) e(-2n\alpha) \\ + \mathcal{O}\left(\frac{Y^2}{k} e^{-c\sqrt{L}}\right) + \mathcal{O}\left(\frac{YQ^2L}{\varphi(q)}\Delta\right). \end{aligned}$$

Therefore, from (13) we find

$$I_{k,l}(a,q) = \frac{\mu(q)}{\varphi(q)}\frac{c_{k,l}(a,q)}{\varphi([k,q])}e\left(-\frac{2na}{q}\right)\int_{-1/q\tau}^{1/q\tau} M(\alpha)M_o(-\alpha)e(-2n\alpha)\,d\alpha$$
$$+\,\mathcal{O}\left(\frac{Y}{k}e^{-c\sqrt{L}}\right) + \mathcal{O}(q^{-2}Q^4L^2\Delta).$$

Hence

$$\sum_{a=1}^{q}{}^{*} I_{k,l}(a,q) = \frac{\mu(q)b_{k,l}(q,2n)}{\varphi([k,q])\varphi(q)}\int_{-1/q\tau}^{1/q\tau} M(\alpha)M_o(-\alpha)e(-2n\alpha)\,d\alpha \tag{17}$$
$$+\,\mathcal{O}\left(\frac{Y}{k}e^{-c\sqrt{L}}\right) + \mathcal{O}(q^{-1}Q^4L^2\Delta),$$

where

$$b_{k,l}(q,2n) = \sum_{a=1}^{q}{}^{*} c_{k,l}(a,q)e\left(-2n\frac{a}{q}\right).$$

Now we show that $b_{k,l}(q,2n)$ is a multiplicative function of q. Let us suppose that $q = q_1q_2$, with $(q_1,q_2) = 1$. Then, writing $a = a_2q_1 + a_1q_2$ and $m = m_2q_1 + m_1q_2$, we have

$$b_{k,l}(q,2n) = \sum_{a=1}^{q}{}^{*} \sum_{\substack{m=1\\ m\equiv l\ ((k,q))}}^{q}{}^{*} e\left(\frac{am}{q}\right)e\left(-2n\frac{a}{q}\right) \tag{18}$$

$$= \sum_{a_1=1}^{q_1}{}^{*}\sum_{a_2=1}^{q_2}{}^{*} e\left(\frac{-2n(a_2q_1+a_1q_2)}{q_1q_2}\right) \sum_{\substack{m=1\\ m\equiv l\ ((k,q_1q_2))}}^{q_1q_2}{}^{*} e\left(\frac{(a_2q_1+a_1q_2)m}{q_1q_2}\right)$$

$$= \sum_{a_1=1}^{q_1}{}^{*}\sum_{a_2=1}^{q_2}{}^{*} e\left(\frac{-2na_1}{q_1}\right)e\left(\frac{-2na_2}{q_2}\right)$$
$$\times \sum_{\substack{m_1=1\ m_2=1\\ m_2q_1+m_1q_2\equiv l\ ((k,q_1)(k,q_2))}}^{q_1\quad q_2}{}^{*}{}^{*} e\left(\frac{a_1m_1q_2}{q_1}\right)e\left(\frac{a_2m_2q_1}{q_2}\right)$$

$$= \sum_{a_1=1}^{q_1}{}^{*}\sum_{a_2=1}^{q_2}{}^{*} e\left(\frac{-2na_1}{q_1}\right)e\left(\frac{-2na_2}{q_2}\right) \sum_{\substack{m_1=1\\ m_1q_2\equiv l\ ((k,q_1))}}^{q_1}{}^{*} e\left(\frac{a_1m_1q_2}{q_1}\right)$$
$$\times \sum_{\substack{m_2=1\\ m_2q_1\equiv l\ ((k,q_2))}}^{q_2}{}^{*} e\left(\frac{a_2m_2q_1}{q_2}\right)$$

$$= \sum_{a_1=1}^{q_1}{}^{*} e\left(\frac{-2na_1}{q_1}\right) \sum_{\substack{m_1=1 \\ m_1 \equiv l\ ((k,q_1))}}^{q_1}{}^{*} e\left(\frac{a_1m_1}{q_1}\right) \sum_{a_2=1}^{q_2}{}^{*} e\left(\frac{-2na_2}{q_2}\right)$$

$$\times \sum_{\substack{m_2=1 \\ m_2 \equiv l\ ((k,q_2))}}^{q_2}{}^{*} e\left(\frac{a_2m_2}{q_2}\right)$$

$$= b_{k,l}(q_1,2n)b_{k,l}(q_2,2n).$$

Moreover, let us suppose that $q = p$ is a prime. Then we have

$$b_{k,l}(p,2n) = \begin{cases} 1 & \text{if } p \nmid 2nk, \\ 1-p & \text{if } p \nmid k,\ p|2n, \\ -1 & \text{if } p|k,\ p \nmid 2n-l, \\ p-1 & \text{if } p|k,\ p|2n-l. \end{cases} \tag{19}$$

In fact, if $p \nmid k$ then $c_{k,l}(a,p) = -1$ and

$$b_{k,l}(p,2n) = -\sum_{a=1}^{p-1} e\left(\frac{-2na}{p}\right) = \begin{cases} 1 & \text{if } p \nmid 2n, \\ 1-p & \text{if } p|2n. \end{cases}$$

If $p|k$ then $c_{k,l}(a,p) = e(al/p)$ and

$$b_{k,l}(p,2n) = \sum_{a=1}^{p-1} e\left(\frac{al}{p}\right) e\left(\frac{-2na}{p}\right) = \begin{cases} -1 & \text{if } p \nmid 2n-l, \\ p-1 & \text{if } p|2n-l. \end{cases}$$

Consequently, from (18) and (19) we see that

$$|\mu(q)b_{k,l}(q,2n)| \le \varphi(q).$$

Since

$$\int_{-1/q\tau}^{1/q\tau} M(\alpha)M_o(-\alpha)e(-2n\alpha)\,d\alpha = P^*(2n) + \mathcal{O}(q\tau),$$

from (12), (17) and $\varphi((k,q))\varphi([k,q]) = \varphi(k)\varphi(q)$ we write

$$I_{k,l}^{(1)}(2n) = \frac{P^*(2n)}{\varphi(k)} \sum_{q \le Q} \frac{\mu(q)b_{k,l}(q,2n)\varphi((q,k))}{\varphi(q)^2} + \mathcal{O}\left(\frac{Y}{k}e^{-c\sqrt{L}}\right) \tag{20}$$

$$+ \mathcal{O}\left(\frac{\tau L}{k}\sum_{q \le Q}(k,q)\right) + \mathcal{O}\left(Q^4L^2\sum_{q \le Q} q^{-1}\Delta\right).$$

Now observe that $I^*_{k,l}(2n) \le L^2$ whenever $(2n-l,k) > 1$. Hence, by the definition of $\mathfrak{S}_{k,l}(2n)$, the contribution of $2n$'s such that $(2n-l,k) > 1$ to $\mathscr{E}^*$ is $\ll \mathscr{K}(Y)HL^2$, which is admissible for (11). Consequently, we will assume that $(2n-l,k) = 1$. From (19), we get

$$\sum_{Q_1 \le q \le Q_2} \frac{\mu(q)b_{k,l}(q,2n)\varphi((q,k))}{\varphi(q)^2} \le \sum_{d|k} d \sideset{}{^\square}\sum_{\substack{Q_1 \le q \le Q_2 \\ (q,k)=d}} \frac{|b_{k,l}(q,2n)|}{\varphi(q)^2} \tag{21}$$

$$= \sideset{}{^\square}\sum_{q|k} \frac{d|b_{k,l}(d,2n)|}{\varphi(d)^2} \sideset{}{^\square}\sum_{\substack{Q_1/d \le q \le Q_2/d \\ (q,k)=1}} \frac{|b_{k,l}(q,2n)|}{\varphi(q)^2}$$

$$= \sideset{}{^\square}\sum_{d|k} \frac{d}{\varphi(d)^2} \sideset{}{^\square}\sum_{\substack{h|2n \\ (h,k)=1}} \frac{|b_{k,l}(h,2n)|}{\varphi(h)^2} \sideset{}{^\square}\sum_{\substack{Q_1/dh \le q \le Q_2/dh \\ (q,k)=1 \\ (q,2n)=1}} \frac{1}{\varphi(q)^2}$$

$$\ll \sum_{d|k} \frac{d}{\varphi(d)^2} \sum_{\substack{h|2n \\ (h,k)=1}} \frac{1}{\varphi(h)} \min\left(\frac{dh}{Q_1}, 1\right),$$

using the elementary inequality

$$\sum_{q>Z} \frac{1}{\varphi(q)^2} \ll Z^{-1}.$$

Hence

$$\sum_{q=1}^{+\infty} \frac{\mu(q)b_{k,l}(q,2n)\varphi((q,k))}{\varphi(q)^2}$$

is absolutely convergent for $(2n-l,k) = 1$. Moreover, from (21) we see that

$$\sum_{q \le Q_2} \frac{\mu(q)b_{k,l}(q,2n)\varphi((q,k))}{\varphi(q)^2} \ll \frac{k}{\varphi(k)} \frac{2n}{\varphi(2n)} \log L \ll \log^3 L. \tag{22}$$

By Euler's identity ([2], Th. 286) and by (19) we find

$$\sum_{q=1}^{+\infty} \frac{\mu(q)b_{k,l}(q,2n)\varphi((q,k))}{\varphi(q)^2} = \mathfrak{S}_{k,l}(2n), \quad \text{if } (2n-l,k) = 1. \tag{23}$$

Since $Y \ll P^*(2n) \ll Y$, from (20) and (23) we write

$$I_{k,l}^{(1)}(2n) = P^*(2n)\frac{\mathfrak{S}_{k,l}(2n)}{\varphi(k)} + \mathcal{O}\left(\frac{Y}{\varphi(k)}\sum_{q>Q}\frac{|\mu(q)b_{k,l}(q,2n)|\varphi((q,k))}{\varphi(q)^2}\right)$$

$$+ \mathcal{O}\left(\frac{Y}{k}e^{-c\sqrt{L}}\right) + \mathcal{O}\left(\frac{\tau L}{k}\sum_{q\le Q}(k,q)\right) + \mathcal{O}\left(Q^4L^2\sum_{q\le Q}q^{-1}\Delta\right).$$

Then we obtain

$$\mathcal{E}_1 \ll YL\Sigma_1 + H\tau L\Sigma_2 + HQ^4L^2\Sigma_3 + HYe^{-c\sqrt{L}}, \tag{24}$$

where

$$\Sigma_1 = \sum_{N<2n\le N+H}\sum_{k\le K}\frac{1}{k}\max_{\substack{(l,k)=1\\(2n-l,k)=1}}\sum_{q>Q}\frac{|\mu(q)b_{k,l}(q,2n)|\varphi((q,k))}{\varphi(q)^2},$$

$$\Sigma_2 = \sum_{k\le K}\sum_{q\le Q}\frac{(k,q)}{k},$$

$$\Sigma_3 = \sum_{k\le K}\sum_{q\le Q}q^{-1}\Delta(N,Y,[k,q]).$$

Let us estimate Σ_1. From (21) we have

$$\Sigma_1 = \sum_{N<2n\le N+H}\sum_{k\le K}\frac{1}{k}\sum_{d|k}\frac{d}{\varphi(d)^2}\sum_{\substack{h|2n\\(h,k)=1}}\frac{1}{\varphi(h)}\min\left(\frac{dh}{Q},1\right) \tag{25}$$

$$\ll \sum_{k\le K}\frac{1}{k}\sum_{d|k}\frac{d}{\varphi(d)^2}\sum_{\substack{h\le N+H\\(h,k)=1}}\frac{1}{\varphi(h)}\min\left(\frac{dh}{Q},1\right)\sum_{\substack{N<2n\le N+H\\2n\equiv 0(h)}}1$$

$$\ll HQ^{-1}\sum_{k\le K}\frac{1}{k}\sum_{d|k}\frac{d^2}{\varphi(d)^2}\sum_{h\le Q/d}\frac{1}{\varphi(h)}$$

$$+ H\sum_{k\le K}\frac{1}{k}\sum_{d|k}\frac{d}{\varphi(d)^2}\sum_{Q/d<h\le N+H}\frac{1}{h\varphi(h)}$$

$$\ll HQ^{-1}L\sum_{k\le K}\frac{1}{k}\sum_{d|k}\frac{d^2}{\varphi(d)^2} \ll HQ^{-1}L^2\sum_{k\le K}\frac{\tau(k)}{k} \ll HQ^{-1}L^4,$$

where we have used the elementary inequality

$$\sum_{k\le Z}\frac{\tau(k)}{k}\ll \log^2 Z.$$

For the sum Σ_2 we get

$$\Sigma_2 \ll \sum_{d\le Q} d\sum_{k\le K}\frac{1}{k}\sum_{\substack{q\le Q\\(k,q)=d}} 1 \tag{26}$$

$$\ll \sum_{d\le Q}\sum_{k\le K/d}\frac{1}{k}\sum_{q\le Q/d} 1 \ll QL\sum_{d\le Q}\frac{1}{d}\ll QL^2.$$

Now we estimate Σ_3:

$$\Sigma_3 = \sum_{h\le QK}\omega(h)\Delta(N,Y,h),\quad \text{with}\quad \omega(h)=\sum_{k\le K}\sum_{\substack{q\le Q\\ [k,q]=h}} q^{-1} \tag{27}$$

It is easy to see that $\omega(h)\ll L^2$. Using the short intervals version of Bombieri—Vinogradov's theorem given by the Corollary in [8], from (27) and the definitions of K and Q, we obtain

$$\Sigma_3 \ll YQL^{66-B}. \tag{28}$$

By (24), (25), (26), (28), the definitions of Q, τ and B we find

$$\mathscr{E}_1 \ll HYL^{-A}.$$

5. The Estimate of $\mathscr{E}_2$

Clearly, for every $k\le K$ there exist an integer l_k such that $(l_k,k)=1$ and

$$\mathscr{E}_2=\sum_{k\le K}\sum_{N<2n\le N+H}\left|\int_{E_2} S(\alpha)S_{k,l_k}(\alpha)e(-2n\alpha)\,d\alpha\right|.$$

We use Cauchy's inequality to get

$$\mathscr{E}_2^2 \le \left(H\sum_{k\le K}\frac{1}{k}\right)\left(\sum_{k\le K} k\sum_{N<2n\le N+H}\left|\int_{E_2} S(\alpha)S_{k,l_k}(\alpha)e(-2n\alpha)\,d\alpha\right|^2\right) \tag{29}$$

$$\ll HL\mathscr{D},\quad \text{say}.$$

By the well-known estimates $\sum_{V<h\le V+U} e(h\alpha)\ll \min(U,\|\alpha\|^{-1})$ and $ab\ll a^2+b^2$ we have

$$(30)\qquad \mathscr{D} = \sum_{k\le K} k \sum_{N<2n\le N+H} \left| \int_{E_2} S(\alpha) S_{k,l_k}(\alpha) e(-2n\alpha)\, d\alpha \right|^2$$

$$\ll \sum_{k\le K} k \int_{E_2}\int_{E_2} |S(\alpha)S_{k,l_k}(\alpha)S(\xi)S_{k,l_k}(\xi)| \min\left(H, \frac{1}{\|\xi-\alpha\|}\right) d\alpha d\xi$$

$$\ll \sum_{k\le K} k \int_{E_2}\int_{E_2} (|S(\alpha)S_{k,l_k}(\xi)|^2 + |S_{k,l_k}(\alpha)S(\xi)|^2) \min\left(H, \frac{1}{\|\xi-\alpha\|}\right) d\alpha d\xi$$

$$\ll \sum_{k\le K} k \int_{E_2}\int_{E_2} |S_{k,l_k}(\xi)S(\alpha)|^2 \min\left(H, \frac{1}{\|\xi-\alpha\|}\right) d\alpha d\xi$$

$$\ll \sum_{k\le K} k \int_{E_2} |S(\alpha)|^2 \left(\int_{-1/2}^{1/2} |S_{k,l_k}(\alpha+\eta)|^2 \min\left(H, \frac{1}{|\eta|}\right) d\eta \right) d\alpha.$$

Let us denote

$$w_k(h) = \sum_{\substack{N<p_1,p_2\le N+Y \\ p_1\equiv p_2\equiv l_k\,(k) \\ p_1-p_2=h}} \log p_1 \log p_2$$

and observe that

$$(31)\qquad w_k(h) \ll L^2(Y/k+1).$$

Then we write

$$|S_{k,l_k}(\alpha+\eta)|^2 = \sum_{\substack{|h|\le Y \\ k|h}} w_k(h) e((\alpha+\eta)h).$$

Hence from (30), (31) and the estimate

$$\int_{-1/2}^{1/2} e(h\eta) \min\left(H, \frac{1}{\eta}\right) d\eta \ll \min\left(\log H, \frac{H}{|h|}\right)$$

we obtain

$$(32)\qquad \mathscr{D} \ll \sum_{k\le K} k \sum_{\substack{|h|\le Y \\ k|h}} w_k(h) \int_{E_2} |S(\alpha)|^2 e(h\alpha)\, d\alpha \int_{-1/2}^{1/2} e(h\eta) \min\left(H, \frac{1}{|\eta|}\right) d\eta$$

$$\ll KY^2L^5 + YL^2 \sum_{0<h\le Y} \min\left(\log H, \frac{H}{h}\right) \left| \int_{E_2} |S(\alpha)|^2 e(h\alpha)\, d\alpha \right| \sum_{\substack{k\le K \\ k|h}} 1.$$

Now we write

$$(33)\qquad \sum_{0<h\le Y}\min\left(\log H,\frac{H}{h}\right)\left|\int_{E_2}|S(\alpha)|^2e(h\alpha)d\alpha\right|\sum_{\substack{k\le K\\ k|h}}1$$

$$\ll \sum_{1\le r\le Y/H}\ \sum_{r-1<h/H\le r}\tau(h)\min\left(\log H,\frac{1}{r-1}\right)\left|\int_{E_2}|S(\alpha)|^2e(h\alpha)\,d\alpha\right|$$

$$\ll L\max_{0\le t\le Y}\sum_{t<h\le t+H}\tau(h)\left|\int_{E_2}|S(\alpha)|^2e(h\alpha)\,d\alpha\right|$$

$$\ll L\max_{0\le t\le Y}\left(\sum_{t<h\le t+H}\tau(h)^2\sum_{t<h\le t+H}\left|\int_{E_2}|S(\alpha)|^2e(h\alpha)\,d\alpha\right|^2\right)^{1/2},$$

by Cauchy's inequality.

Arguing as in §5 of [7] (see the definition of the constant C) we have that

$$(34)\qquad \sum_{t<h\le t+H}\left|\int_{E_2}|S(\alpha)|^2e(h\alpha)\,d\alpha\right|^2\ll HY^2L^{-4A-11}.$$

Moreover, we see that

$$\max_{0\le t\le Y}\sum_{t<h\le t+H}\tau(h)^2\ll\max_{0\le t\le H}\sum_{t<h\le t+H}\tau(h)^2+\max_{H\le t\le Y}\sum_{t<h\le t+H}\tau(h)^2$$

$$\ll\sum_{h\le 2H}\tau(h)^2+\max_{H\le t\le Y}\sum_{t<h\le t+H}\tau(h)^2$$

Hence, using the estimate

$$\sum_{d\le Z}\tau(d)^2\ll Z\log^3 Z$$

for the first sum and Lemma 1 in [6] for the second one, we get

$$(35)\qquad \max_{0\le t\le Y}\sum_{t<h\le t+H}\tau(h)^2\ll HL^3.$$

The second inequality of (11) follows from (29) and (32)–(35), provided $K\ll HL^{-2A-6}$. Theorem 2 is completely proved.

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