TSUKUBA J. MATH.
Vol. 24 No. 1 (2000), 69–80

SPACES OF UPPER SEMI-CONTINUOUS MULTI-VALUED FUNCTIONS ON SEPARABLE METRIC SPACES

By

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Abstract. Let $X = (X, d)$ be a metric space. By $\mathrm{USCC}(X, \mathbf{I})$, we denote the space of upper semi-continuous multi-valued functions $\varphi : X \to \mathbf{I} = [0, 1]$ such that each $\varphi(x)$ is a closed interval. Each $\varphi \in \mathrm{USCC}(X, \mathbf{I})$ can be identified with its graph, which is a closed subset of $X \times \mathbf{I}$. The space $\mathrm{USCC}(X, \mathbf{I})$ admits the Hausdorff metric induced by the product metric on $X \times \mathbf{I}$. In this paper, by proving the converse of Fedorchuk's result, we show that $\mathrm{USCC}(X, \mathbf{I})$ is homeomorphic to the Hilbert cube $Q = [-1, 1]^{\omega}$ if and only if X is infinite, locally connected and compact. In case X is a dense subset of a locally connected metric space Y such that $Y \backslash X$ is locally non-separating in Y, $\mathrm{USCC}(X, \mathbf{I})$ can be regarded as a subspace of $\mathrm{USCC}(Y, \mathbf{I})$. It is also proved that the pair $(\mathrm{USCC}(Y, \mathbf{I}), \mathrm{USCC}(X, \mathbf{I}))$ is homeomorphic to (Q, s) if and only if $X \neq Y$, X is G_{δ} in Y, and Y is compact, where $s = (-1, 1)^{\omega} \subset Q$.

Introduction

Let $X = (X, d)$ be a metric space. By $(2^X)_m$, we denote the hyperspace of non-empty bounded closed subsets of X with the Hausdorff metric d_{H} defined by d (cf. [Ku, p. 214]). Let 2^X be the totality of non-empty closed subsets of X. In case X is unbounded, $2^X \neq (2^X)_m$ and d_{H} is not a metric on the whole 2^X (e.g., $d_{\mathrm{H}}(\{x\}, X) = \infty$ for any $x \in X$) but d_{H} induces a topology on 2^X. This topology depends on the metric d (cf. [SU_2, § 1]).

1991 *Mathematics Subject Classification*. 54C60, 57N20, 58C06, 58D17.

Key words and phrases. The space of upper semi-continuous multi-valued functions, the hyperspace of non-empty closed sets, the Hausdorff metric, the Hilbert cube, the pseudo-interior, locally non-separating, Property S.
Received November 24, 1998
Revised June 29, 1999

We endow the product space $X \times \mathbb{R}$ with the metric

$$\rho((x,t),(x',t')) = \max\{d(x,x'), |t-t'|\}.$$

Let $\varphi : X \to \mathbb{R}$ be a multi-valued function such that each $\varphi(x)$ is compact. Then, φ is upper semi-continuous (u.s.c.) if and only if the graph of φ is closed in $X \times \mathbb{R}$, whence we can regard $\varphi \in 2^{X\times\mathbb{R}}$. By $\mathrm{USC}_B(X)$, we denote the space of bounded u.s.c. multi-valued functions $\varphi : X \to \mathbb{R}$ such that each $\varphi(x)$ is non-empty and compact, where $\varphi : X \to \mathbb{R}$ is *bounded* means that the image $\varphi(X) = \bigcup_{x\in X}\varphi(x)$ is bounded. The space $\mathrm{USC}_B(X)$ is now regarded as a subspace of $2^{X\times\mathbb{R}}$. One should note that $\mathrm{USC}_B(X) \not\subset (2^{X\times\mathbb{R}})_m$ in general, but $\rho_{\mathrm{H}}(\varphi,\psi) < \infty$ can be defined for each $\varphi, \psi \in \mathrm{USC}_B(X)$ because φ and ψ are bounded. Let $\mathrm{USC}(X,\mathbf{I})$ be the subspace of $\mathrm{USC}_B(X)$ consisting of all $\varphi \in \mathrm{USC}_B(X)$ with the image $\varphi(X) \subset \mathbf{I}$. By $\mathrm{USCC}_B(X)$, we denote the subspace of $\mathrm{USC}_B(X)$ consisting of all $\varphi \in \mathrm{USCC}_B(X)$ such that each $\varphi(x)$ is connected (i.e., a closed interval). Let $\mathrm{USCC}(X,\mathbf{I}) = \mathrm{USCC}_B(X) \cap \mathrm{USC}(X,\mathbf{I})$.

In case X is compact, every u.s.c. multi-valued function $\varphi : X \to \mathbb{R}$ is bounded, so we denote $\mathrm{USC}_B(X) = \mathrm{USC}(X)$ and $\mathrm{USCC}_B(X) = \mathrm{USCC}(X)$. In this case, every admissible metric for X induces the same topology for $\mathrm{USC}_B(X)$, that is, the topology for $\mathrm{USC}_B(X)$ does not depend on the metric d. In case X is non-compact, it depends on the metric d (see the end of Introduction).

Fedorchuk [Fe$_{1,2}$] proved that if X is an infinite locally connected compact metric space then $\mathrm{USCC}(X,\mathbf{I})$ is homeomorphic to ($\approx$) the Hilbert cube $Q = [-1,1]^\omega$ and $\mathrm{USCC}(X) \approx Q\backslash\{0\}$ ($\approx Q \times [0,1)$) (cf. [SU$_1$, Appendix]). In this paper, by showing the converse of this result, we have the following:

THEOREM 1. *For a metric space X, the following are equivalent*:
(a) $\mathrm{USCC}(X,\mathbf{I}) \approx Q$;
(b) $\mathrm{USCC}_B(X) \approx Q\backslash\{0\}$ ($\approx Q \times [0,1)$);
(c) *X is infinite, locally connected and compact.*

In case X is a dense subset of a metric space Y, we have the natural isometric embedding $e_Y : \mathrm{USC}_B(X) \to \mathrm{USC}_B(Y)$ defined by $e_Y(\varphi) = \mathrm{cl}_{Y\times\mathbb{R}}\,\varphi$. Then $e_Y(\mathrm{USC}(X,\mathbf{I})) \subset USC(Y,\mathbf{I})$. But, in general,

$$e_Y(\mathrm{USCC}_B(X)) \not\subset \mathrm{USCC}_B(Y) \quad \text{nor} \quad e_Y(\mathrm{USCC}(X,\mathbf{I})) \not\subset \mathrm{USCC}(Y,\mathbf{I}).$$

For example, let $Y = \mathbf{S}^1$ be the unit circle of Euclidean plane $\mathbb{R}^2$ with the usual metric, $X = \mathbf{S}^1\backslash\{(1,0)\}$, and $f : X \to \mathbb{R}$ be the map defined by $f(x,y) = y$ if $x \le 0$ and $f(x,y) = y/|y|$ if $x > 0$. Then $e_Y(f)(1,0) = \{-1,1\}$ is not connected.

In case Y is locally connected, it will be shown that

$$e_Y(\mathrm{USCC}_B(X)) \subset \mathrm{USCC}_B(Y) \quad \text{and/or} \quad e_Y(\mathrm{USCC}(X, \mathbf{I})) \subset \mathrm{USCC}(Y, \mathbf{I})$$

if and only if the complement $Y \backslash X$ is *locally non-separating* in Y, that is, $U \cap X \neq \emptyset$ is connected for each non-empty connected open set U in Y (Proposition 2). Let $s = (-1, 1)^\omega$ be the pseudo-interior of Q, which is homeomorphic to the separable Hilbert space ℓ_2. We generalize Theorem 1 to pairs as follows:

THEOREM 2. *Let X be a dense subset of a locally connected metric space Y with the locally non-separating complement in Y. Then the following are equivalent*:

(a) $(\mathrm{USCC}(Y, \mathbf{I}), e_Y(\mathrm{USCC}(X, \mathbf{I}))) \approx (Q, s)$;

(b) $(\mathrm{USCC}_B(Y), e_Y(\mathrm{USCC}_B(X))) \approx (Q \times [0, 1), s \times [0, 1))$;

(c) *$X \neq Y$, X is G_δ in Y and Y is compact.*

In the above, it should be observed that if Y is locally connected and $Y \backslash X$ is locally non-separating in Y then X is dense in Y.

A metric space $X = (X, d)$ (or a metric d) has *Property S* if X is covered by finitely many connected sets with arbitrarily small diameters. It should be remarked that a metric space with Property S is totally bounded, hence *a complete metric space with Property S is compact*. The subspace of 2^X consisting of compacta is denoted by $\exp(X)$. In case X is compact, $\exp(X) = 2^X$. In [Cu], Curtis proved that X admits a Peano compactification $\tilde{X}$ such that $(\exp(\tilde{X}), \exp(X)) \approx (Q, s)$ if and only if X is connected, locally connected, completely metrizable, nowhere locally compact and admits a metric d with Property S. We have the following version of this Curtis' result:

THEOREM 3. *A metrizable space X has a metrizable compactification $\tilde{X}$ such that*

$$(\mathrm{USCC}(\tilde{X}, \mathbf{I}), e_{\tilde{X}}(\mathrm{USCC}(X, \mathbf{I}))) \approx (Q, s)$$

if and only if X is completely metrizable, non-compact and admits a metric with Property S.

One should note that some admissible metric d for X cannot be extended to $\tilde{X}$ even if d has Property S. For example, let $X = (0, 1)$ and $\tilde{X} = [0, 1]$. Then, $X \approx \mathbf{S}^1 \backslash \{(1, 0)\}$. The metric on X inherited from $\mathbf{S}^1$ has Property S but cannot be extended to $\tilde{X}$. The following is a direct consequence of Theorems 2 and 3:

COROLLARY 1. *Let X be completely metrizable, non-compact and admits a metric with Property S. Then X admits a metric which induces the topology on* $\mathrm{USCC}_B(X)$ *such that* $\mathrm{USCC}(X,\mathbf{I}) \approx \mathrm{USCC}_B(X) \approx \ell_2$. □

In the above, the topology of $\mathrm{USCC}(X,\mathbf{I})$ is not defined by using a complete metric on X. In [SU₂], it is proved that the spaces $\mathrm{USCC}_B(X)$ and $\mathrm{USCC}(X,\mathbf{I})$ are homeomorphic to a *non-separable* Hilbert space for a uniformly locally connected, non-compact and complete metric space X (*even if X is separable*). One should observe that $\mathrm{USCC}_B(\mathbb{R})$ is non-separable but $\mathrm{USCC}_B((0,1))$ is separable, where $\mathbb{R}$ and $(0,1)$ have the usual metrics.

Proofs of Theorems

We start with the following:

PROPOSITION 1. *For a locally compact metric space X,* $\mathrm{USCC}(X,\mathbf{I})$ *is closed in* $2^{X\times\mathbf{I}}$ *if and only if X is locally connected.*

PROOF. The "if" part is Proposition 1.1 in [SU₂], where the local compactness of X need not be assumed.

To see the "only if" part, assume that X is not locally connected. Then some $x_0 \in X$ has a compact neighborhood B_0 such that any neighborhood of x_0 contained in B_0 is not connected. Let $\delta = d(x_0, X\backslash B_0) > 0$. Then we have disjoint non-empty closed sets A_1 and B_1 in X such that $B_0 = A_1 \cup B_1$, $d(x_0, A_1) < 2^{-1}\delta$ and $x_0 \in B_1$. In fact, since B_0 is compact, the intersection of clopen sets in B_0 containing x_0 is the component of B_0, which is not a neighborhood of x_0. Then we have a clopen set B_1 in B_0 and $x_1 \in B_0\backslash B_1$ with $d(x_0, x_1) < 2^{-1}\delta$, whence $A_1 = B_0\backslash B_1$ and B_1 satisfy the condition. Using the same argument inductively, we have disjoint non-empty closed sets A_n and B_n in X, $n \in \mathbb{N}$, such that $B_{n-1} = A_n \cup B_n, d(x_0, A_n) < 2^{-n}\delta$ and $x_0 \in B_n$. For each $n \in \mathbb{N}$, let

$$\varphi_n = \bigcup_{i=1}^{n} A_i \times \{0\} \cup B_n \times \{1\} \cup (X\backslash \mathrm{int}_X B_0) \times \mathbf{I} \in \mathrm{USCC}(X,\mathbf{I}).$$

Note that $\varphi_n(\mathrm{int}_X B_0) = \{0,1\}$. Since $2^{B_0\times\mathbf{I}} = \exp(B_0 \times \mathbf{I})$ is compact, $(\varphi_n|B_0)_{n\in\mathbb{N}}$ has a subsequence $(\varphi_{n_i}|B_0)_{i\in\mathbb{N}}$ converging to some $\varphi' \in 2^{B_0\times\mathbf{I}}$. Then $(\varphi_{n_i})_{i\in\mathbb{N}}$ converges to $\varphi = \varphi' \cup (X\backslash \mathrm{int}_X B_0) \times \mathbf{I}$ in $2^{X\times\mathbf{I}}$. Since $(x_0, 0) \in \varphi_n$ for all $n \in \mathbb{N}$, we have $(x_0, 0) \in \varphi$. For each $n \in \mathbb{N}$, choose $x_n \in A_n$ so that $d(x_n, x_0) < 2^{-n}\delta$.

Since $\rho((x_0,1),(x_n,1)) < 2^{-n}\delta$ and $(x_n,1) \in \varphi_n$, we have $(x_0,1) \in \varphi$. However $(x_0,1/2) \notin \varphi$ because $\operatorname{int}_X B_0 \times (0,1) \cap \varphi_n = \varnothing$ for any $n \in \mathbb{N}$. This means that $\varphi \cap \{x_0\} \times \mathbf{I}$ (i.e., $\varphi(x_0)$) is not connected, hence $\varphi \notin \mathrm{USCC}(X,\mathbf{I})$. This is a contradiction. □

For a metric space X, there exists the natural closed embedding i_X: $X \to \mathrm{USCC}(X,\mathbf{I})$ defined as follows:

$$i_X(x) = X \times \{0\} \cup \{x\} \times \mathbf{I} \subset X \times \mathbf{I} \quad \text{for each } x \in X,$$

whence each $i_X(x) \in \mathrm{USCC}(X,\mathbf{I})$ is defined by

$$i_X(x)(y) = \begin{cases} \{0\} & \text{if } y \neq x, \\ \mathbf{I} & \text{if } y = x. \end{cases}$$

Observe that $\rho_{\mathrm{H}}(i_X(x), i_X(x')) = d(x,x')$ if $d(x,x') < 1$, hence i_X is locally isometric. It is easy to see that $i_X(X)$ is closed in $\mathrm{USCC}(X,\mathbf{I})$.

Proof of Theorem 1. The implications (c) ⇒ (a) and (c) ⇒ (b) are Fedorchuk's results [$\mathrm{Fe}_{1,2}$] (cf. [SU_1, Appendix]).

(a) ⇒ (c): By using the embedding i_X above, X can be embedded in $\mathrm{USCC}(X,\mathbf{I})$ as a closed set, hence X is compact. By Proposition 1, X is locally connected. If X is a singleton, the space $\mathrm{USCC}(X,\mathbf{I})$ is homeomorphic to the hyperspace of subcontinua (i.e., closed subintervals) of $\mathbf{I}$, so $\mathrm{USCC}(X,\mathbf{I}) \approx \mathbf{I}^2$ (cf. [Du, §3]). Hence, if X is finite then $\mathrm{USCC}(X,\mathbf{I}) \approx \mathbf{I}^{2n}$, where n is the number of points of X. Therefore, X must be infinite.

(b) ⇒ (c): Since $\mathrm{USCC}_B(X)$ is locally compact, $\varphi_0 = X \times \{0\} \in \mathrm{USCC}_B(X)$ has a compact neighborhood N in $\mathrm{USCC}_B(X)$. Choose $\delta > 0$ so that every $\varphi \in \mathrm{USCC}_B(X)$ with $\rho_{\mathrm{H}}(\varphi,\varphi_0) < \delta$ belongs to N. Then, $\mathrm{USCC}(X,[0,\delta]) \subset N$ and $\mathrm{USCC}(X,[0,\delta])$ is closed in $\mathrm{USCC}_B(X)$. Hence, $\mathrm{USCC}(X,\mathbf{I}) \approx \mathrm{USCC}(X,[0,\delta])$ is compact. As seen in the above, it follows that X is compact and locally connected. Since

$$\mathrm{USCC}_B(X) = \mathrm{USCC}(X) \approx \mathrm{USCC}(X,(0,1)) \subset \mathrm{USCC}(X,\mathbf{I}),$$

$\mathrm{USCC}(X,\mathbf{I})$ is infinite-dimensional, which implies that X is infinite. □

By $C_B(X)$, we denote the Banach space of bounded continuous real-valued functions of X with the sup-norm and let $C(X,\mathbf{I}) = \{f \in C_B(X) \mid f(X) \subset \mathbf{I}\}$. Although $C_B(X) \subset \mathrm{USCC}_B(X)$ as sets, the Banach space $C_B(X)$ is not a subspace of $\mathrm{USCC}_B(X)$ in case X is non-compact (cf. [FK, Remark 3.6] and Supplement).

In [SU_2, Corollary 1.5], it is also shown that if X is locally connected and has no isolated points then the closures of $C(X,\mathbf{I})$ and $C_B(X)$ in $2^{X\times\mathbf{I}}$ are $\mathrm{USCC}(X,\mathbf{I})$ and $\mathrm{USCC}_B(X)$, respectively. In case X is locally compact, the converse also holds by Proposition 1.

COROLLARY 2. *For a locally compact metric space X,*

$$\mathrm{cl}_{2^{X\times\mathbf{I}}} C(X,\mathbf{I}) = \mathrm{USCC}(X,\mathbf{I}) \quad \textit{and/or} \quad \mathrm{cl}_{2^{X\times\mathbf{R}}} C_B(X) = \mathrm{USCC}_B(X)$$

if and only if X is locally connected and has no isolated point. □

Next, we show the following:

PROPOSITION 2. *Let X be a dense subset of a locally connected metric space Y. Then, the following are equivalent:*

(a) $e_Y(\mathrm{USCC}(X,\mathbf{I})) \subset \mathrm{USCC}(Y,\mathbf{I})$;

(b) $e_Y(\mathrm{USCC}_B(X)) \subset \mathrm{USCC}_B(Y)$;

(c) $Y\backslash X$ *is locally non-separating in* Y.

PROOF. (c) ⇒ (b): Suppose $e_Y(\mathrm{USCC}_B(X)) \not\subset \mathrm{USCC}_B(Y)$, that is, there exists $\varphi \in \mathrm{USCC}_B(X)$ such that $e_Y(\varphi) \notin \mathrm{USCC}_B(Y)$. Then $e_Y(\varphi)(y)$ is not connected for some $y \in Y\backslash X$, whence we have $t_1 < t < t_2$ such that $t_1, t_2 \in e_Y(\varphi)(y)$ but $t \notin e_Y(\varphi)$. Since $e_Y(\varphi)$ is closed in $Y \times \mathbf{I}$ and Y is locally connected, we have a connected open neighborhood U in y in Y and $\delta > 0$ such that

$$U \times (t-\delta, t+\delta) \cap e_Y(\varphi) = \varnothing,$$

whence $t \notin \varphi(x)$ for all $x \in U \cap X$, $t_1 < t-\delta$ and $t_2 > t+\delta$. By the definition of $e_Y(\varphi)$, we have $x_i \in U \cap X$ and $s_i \in \varphi(x_i)$, $i = 1,2$, such that $|s_i - t_i| < \delta$, whence $t \notin \varphi(x_i)$ and $s_1 < t < s_2$. Since $\varphi(x_i)$ is connected, $\varphi(x_1) \subset (-\infty, t)$ and $\varphi(x_2) \subset (t, \infty)$. Since φ is u.s.c.,

$$U_1 = \{x \in U \mid \varphi(x) \subset (-\infty, t)\} \quad \text{and} \quad U_2 = \{x \in U \mid \varphi(x) \subset (t, \infty)\}$$

are open in U. It follows that $U = U_1 \cup U_2$, $U_1 \cap U_2 = \varnothing$ and $x_i \in U_i \cap X$, $i = 1,2$. Hence, $U \cap X$ is not connected, which means that $Y\backslash X$ is not locally non-separating in Y.

(b) ⇒ (a): This is observed as follows:

$$\begin{aligned} e_Y(\mathrm{USCC}(X,\mathbf{I})) &= e_Y(\mathrm{USCC}_B(X)) \cap \mathrm{USC}(Y,\mathbf{I}) \\ &\subset \mathrm{USCC}_B(Y) \cap \mathrm{USC}(Y,\mathbf{I}) = \mathrm{USCC}(Y,\mathbf{I}). \end{aligned}$$

(a) $\Rightarrow$ (c): First, note that X is dense in Y. Otherwise, $e_Y(\varphi)(y) = \varnothing$ for each $\varphi \in \mathrm{USCC}(X, \mathbf{I})$ and $y \in Y \backslash \mathrm{cl}\, X$. Now, suppose that $Y \backslash X$ is not locally non-separating in Y, that is, there exists a connected open set U in Y such that $U \cap X$ is not connected. (Note that $U \cap X \neq \varnothing$ because X is dense in Y.) Let $U \cap X = U_1 \cup U_2$, where U_1 and U_2 are disjoint non-empty open sets in X. Note that $\mathrm{cl}_X U_1 \cup \mathrm{cl}_X U_2 \supset U$. Let

$$\varphi = (X \backslash U) \times \mathbf{I} \cup U_1 \times \{0\} \cup U_2 \times \{1\} \in \mathrm{USCC}(X, \mathbf{I}).$$

Since U is connected, we have $y \in U \cap \mathrm{cl}_Y U_1 \cap \mathrm{cl}_Y U_2 \subset U \backslash X$ because X is dense in Y. It follows that $e_Y(\varphi)(y) = \{0, 1\}$. Thus $e_Y(\varphi) \notin \mathrm{USCC}(Y, \mathbf{I})$, which contradicts to $e_Y(\mathrm{USCC}(X, \mathbf{I})) \subset \mathrm{USCC}(Y, \mathbf{I})$. Therefore, $Y \backslash X$ is locally non-separating in Y. □

PROPOSITION 3. *Let X be a dense subset of a locally connected compact metric space Y with the locally non-separating complement $Y \backslash X$ in Y. Then, $e_Y(\mathrm{USCC}_B(X))$ is G_δ in $\mathrm{USCC}(Y)$ if and only if X is G_δ in Y.*

PROOF. The "only if" part follows from

$$i_Y(X) = i_Y(Y) \cap e_Y(\mathrm{USCC}_B(X)),$$

where $i_Y : Y \to \mathrm{USCC}(Y, \mathbf{I}) \subset \mathrm{USCC}_B(Y)$ is the natural closed embedding.

To see the "if" part, let $X = \bigcap_{n \in \mathbb{N}} U_n$, where each U_n is open in Y. For each $m, n \in \mathbb{N}$, let

$$G_{m,n} = \{\varphi \in \mathrm{USCC}_B(Y) \mid \rho_{\mathrm{H}}(\varphi, e_Y(\varphi | U_n)) < 1/m\}.$$

Since $e_Y(\mathrm{USCC}_B(X)) = \bigcap_{m,n \in \mathbb{N}} G_{m,n}$, it suffices to show that each $G_{m,n}$ is open in $\mathrm{USCC}_B(Y)$, or each $F_{m,n} = \mathrm{USCC}_B(Y) \backslash G_{m,n}$ is closed in $\mathrm{USCC}_B(Y)$.

Assume that a sequence $\varphi_i \in F_{m,n}$, $i \in \mathbb{N}$, converges to $\varphi \in \mathrm{USCC}_B(Y)$. Since φ is bounded, $\varphi \subset Y \times [-a, a]$ for some $a > 0$. Then, we may assume that $\varphi_i \subset Y \times [-a, a]$ for all $i \in \mathbb{N}$. Since each φ_i is compact, we can choose $(x_i, t_i) \in \varphi_i$ so that

$$\rho((x_i, t_i), e_Y(\varphi_i | U_n)) = \rho_{\mathrm{H}}(\varphi_i, e_Y(\varphi_i | U_n)) \geq 1/m.$$

Since $Y \times [-a, a]$ is compact, we may assume that (x_i, t_i) converges to $(x_0, t_0) \in Y \times [-a, a]$, whence $(x_0, t_0) \in \varphi$. We show that $\rho((x_0, t_0), e_Y(\varphi | U_n)) \geq 1/m$, which means that $\varphi \in F_{m,n}$. Then, $F_{m,n}$ would be closed in $\mathrm{USCC}(Y, [-a, a])$.

Now, assume that $\rho((x_0, t_0), e_Y(\varphi | U_n)) < 1/m$. Then, we have $(y_0, s_0) \in \varphi | U_n$ such that $\rho((x_0, t_0), (y_0, s_0)) < 1/m$. Let

$$\delta = \min\{d(y_0, Y \backslash U_n), \tfrac{1}{2}(1/m - \rho((x_0, t_0), (y_0, s_0)))\} > 0.$$

Choose i so large that $\rho_{\mathrm{H}}(\varphi_i, \varphi) < \delta$ and $\rho((x_i, t_i), (x_0, t_0)) < \delta$. Then, we have $(y_i, s_i) \in \varphi_i$ such that $\rho((y_0, s_0), (y_i, s_i)) < \delta$. Since $d(y_0, y_i) < d(y_0, Y \backslash U_n)$, it follows that $y_i \in U_n$, hence $(y_i, s_i) \in \varphi_i \mid U_n$. Therefore,

$$\rho((x_i, t_i), (y_i, s_i)) \geq \rho((x_i, t_i), e_Y(\varphi_i \mid U_n) \geq 1/m.$$

On the other hand,

$$\begin{aligned}\rho((x_i, t_i), (y_i, s_i)) &\leq \rho((x_i, t_i), (x_0, t_0)) + \rho((x_0, t_0), (y_0, s_0)) + \rho((y_0, s_0), (y_i, s_i)) \\ &< 2\delta + \rho((x_0, t_0), (y_0, s_0)) < 1/m,\end{aligned}$$

which is a contradiction. The proof is completed. □

Now, we prove Theorems 2 and 3.

PROOF OF THEOREM 2. (a) $\Rightarrow$ (b): As saw in the proof of [Fe$_2$, Proposition 2.4], $D = \mathrm{USCC}(Y, \mathbf{I}) \backslash \mathrm{USCC}(Y, (0,1))$ is a contractible Z-set in $\mathrm{USCC}(Y, \mathbf{I})$ and then

$$\mathrm{USCC}(Y, (0,1)) \approx \mathrm{USCC}(Y, \mathbf{I}) \backslash D \approx Q \times [0,1).$$

It follows from [Ch, Theorem 6.6] that

$$(\mathrm{USCC}(Y, (0,1)), e_Y(\mathrm{USCC}(X, \mathbf{I})) \backslash D) \approx (Q \times [0,1), s \times [0,1)),$$

where it should be noted that $e_Y(\mathrm{USCC}(X, \mathbf{I})) \backslash D \neq e_Y(\mathrm{USCC}(X, (0,1))$ but

$$e_Y(\mathrm{USCC}(X, \mathbf{I})) \backslash D = \{e_Y(\varphi) \mid \varphi \in \mathrm{USCC}(X, (a,b)) \text{ for some } 0 < a < b < 1\}.$$

By Theorem 1, Y is compact, whence $\mathrm{USCC}_B(Y) = \mathrm{USCC}(Y)$ and there exists a homeomorphism $h: \mathrm{USCC}(Y) \to \mathrm{USCC}(Y, (0,1))$ such that

$$h(e_Y(\mathrm{USCC}_B(X))) = \{e_Y(\varphi) \mid \varphi \in \mathrm{USCC}(X, (a,b)) \text{ for some } 0 < a < b < 1\}.$$

Consequently, we have

$$\begin{aligned}(\mathrm{USCC}_B(Y), e_Y(\mathrm{USCC}_B(X))) &\approx (\mathrm{USCC}(Y, (0,1)), e_Y(\mathrm{USCC}(X, \mathbf{I})) \backslash D) \\ &\approx (Q \times [0,1), s \times [0,1)).\end{aligned}$$

(b) $\Rightarrow$ (c): By Theorem 1, the condition (b) implies that $X \neq Y$ and Y is compact and locally connected. Moreover, $Y \backslash X$ is locally non-separating in Y by Proposition 2, and X is G_δ in Y by Proposition 3.

(c) $\Rightarrow$ (a): We first consider the case that Y is connected, hence it is a Peano continuum. In this case, $\mathrm{USCC}(Y, \mathbf{I})$ is the closure of $C(Y, \mathbf{I})$ in $\exp(Y \times \mathbf{I}) =$

$2^{Y \times \mathbf{I}}$ [Fe$_2$, Theorem 1.10]. Since $(\mathrm{USCC}(Y,\mathbf{I}), C(Y,\mathbf{I})) \approx (Q,s)$ [SU$_1$, Corollary 1'], the complement $USCC(Y,\mathbf{I}) \backslash C(Y,\mathbf{I})$ is a Z_σ-set in $\mathrm{USCC}(Y,\mathbf{I})$. By Proposition 3, $e_Y(\mathrm{USCC}_B(X))$ is G_δ in $\mathrm{USCC}_B(Y)$, whence

$$e_Y(\mathrm{USCC}(X,\mathbf{I})) = e_Y(\mathrm{USCC}_B(X)) \cap \mathrm{USCC}(Y,\mathbf{I})$$

is also G_δ in $\mathrm{USCC}(Y,\mathbf{I})$. Then, the complement

$$M = \mathrm{USCC}(Y,\mathbf{I}) \backslash e_Y(\mathrm{USCC}(X,\mathbf{I}))$$

is F_σ in $\mathrm{USCC}(Y,\mathbf{I})$ and $M \subset \mathrm{USCC}(Y,\mathbf{I}) \backslash C(Y,\mathbf{I})$, hence M is a Z_σ-set in $\mathrm{USCC}(Y,\mathbf{I})$. Let (A,B) be a pair of compacta in $\mathrm{USCC}(Y,\mathbf{I})$ such that $B \subset M$ and $\varepsilon > 0$. By all the same way as the proof of Main Theorem of [SU$_1$], but using a point $x_0 \in Y \backslash X$, we can define an embedding $h : A \to M$ such that $h|B = \mathrm{id}$ and h is ε-close to id. Applying the characterization of $B(Q) = Q \backslash s$ [An] (cf. [Ch, Lemma 8.1]), we have $(\mathrm{USCC}(Y,\mathbf{I}), M) \approx (Q, B(Q))$, hence

$$(\mathrm{USCC}(Y,\mathbf{I}), e_Y(\mathrm{USCC}(X,\mathbf{I}))) \approx (Q,s).$$

In the general case, we write $Y = \bigcup_{i=1}^n Y_i$, where each Y_i is a component of Y, which is closed and open in Y because of locally connectedness of Y. Since $Y \backslash X$ is locally non-separating in Y, each $X_i = X \cap Y_i$ is a component of X. Then

$$(\mathrm{USCC}(Y,\mathbf{I}), e_Y(\mathrm{USCC}(X,\mathbf{I}))) \approx \left(\prod_{i=1}^n \mathrm{USCC}(Y_i,\mathbf{I}), \prod_{i=1}^n e_{Y_i}(\mathrm{USCC}(X_i,\mathbf{I}))\right).$$

In case Y_i is a singleton, $X_i = Y_i$ and $\mathrm{USCC}(Y_i,\mathbf{I})$ is homeomorphic to the hyperspace of subcontinua of $\mathbf{I}$, hence $\mathrm{USCC}(Y_i,\mathbf{I}) \approx \mathbf{I}^2$ (cf. [Du, §3]). Hence the general case can be obtained the connected case. □

Proof of Theorem 3. First, assume that X is completely metrizable and has an admissible metric with Property S. Then, X has only finitely many components, which are closed and open in X. Replacing the metric, we may assume that the distance between any two components of X is positive. Thus, as in the proof of Theorem 2, it suffices to treat the case X is connected. In this case, X has a Peano compactification $\tilde{X}$ with a locally non-separating remainder $\tilde{X} \backslash X$ by [Cu, Proposition 2.4]. By complete metrizability, X is G_δ in $\tilde{X}$. Then, the "if" part follows from Theorem 2.

Conversely, assume that X has a compactification $\tilde{X}$ such that

$$(\mathrm{USCC}(\tilde{X},\mathbf{I}), e_{\tilde{X}}(\mathrm{USCC}(X,\mathbf{I}))) \approx (Q,s).$$

By Theorem 2, $X \neq \tilde{X}$, X is G_δ in $\tilde{X}$, $\tilde{X}$ is locally connected and the remainder $\tilde{X} \backslash X$ is locally non-separating in $\tilde{X}$. Then X is completely metrizable and, as is easily observed, each component of $\tilde{X}$ is a Peano compactification of a component of X with locally non-separating remainder. By [Cu, Proposition 2.4], X admits an admissible metric d with Property S. Thus we have the "only if" part. □

Supplement

As mentioned before Corollary 2, the Banach space $C_B(X)$ is not a subspace of $\mathrm{USCC}_B(X)$ in case X is non-compact (cf. [FK, Remark 3.6]). Here we show the following:

PROPOSITION 4. *In the following cases, the topology for* $C(X, \mathbf{I})$ *induced by the sup-norm is different from the one induced by the Hausdorff metric* ρ_{H}:

(1) *X has a non-complete component*;

(2) *X has a non-totally bounded component*;

(3) *X has infinitely many components X_i, $i \in \mathbf{N}$, such that* $\inf_{i\in\mathbf{N}} \operatorname{diam} X_i > 0$ *and* $\inf_{i\neq j} \operatorname{dist}(X_i, X_j) > 0$.

PROOF. (1) Let X_0 be a non-complete component of X. Then X_0 has a non-convergent Cauchy sequence $(x_i)_{i\in\mathbf{N}}$. For each $n \in \mathbf{N}$, we have $m > n$ such that $d(x_i, x_j) < (1/3)d(x_n, x_m)$ for all $i, j \geq m$. In fact, x_n is not an accumulation point of $(x_i)_{i\in\mathbf{N}}$, whence there is come $\delta > 0$ such that $d(x_n, x_i) > \delta$ for almost all $i \in \mathbf{N}$. Since $(x_i)_{i\in\mathbf{N}}$ is a Cauchy sequence, we can choose $m > n$ such that $d(x_n, x_m) > \delta$ and $d(x_i, x_j) < (1/3)\delta$ if $i, j \geq m$, whence $d(x_i, x_j) < (1/3)d(x_n, x_m)$ for all i, $j \geq m$. Therefore, by taking a subsequence, we can assume that $d(x_i, x_j) < (1/3)d(x_n, x_{n+1})$ for every $n \in \mathbf{N}$ and $i, j > n$. For each $n \in \mathbf{N}$, let $\varepsilon_n = (1/3)d(x_n, x_{n+1})$. Then, the collection $\{B(x_n, \varepsilon_n) \mid n \in \mathbf{N}\}$ is discrete in X and

$$(*) \qquad \bigcup_{i>n} B(x_i, \varepsilon_i) \subset B(x_{n+1}, 2\varepsilon_n) \subset X \setminus \bigcup_{j\leq n} B(x_j, \varepsilon_j).$$

Moreover, since X_0 is connected, it follows that

$$(\sharp_1) \qquad [0, \varepsilon_n] \subset [0, 2\varepsilon_1] \subset \{d(x_n, y) \mid y \in X_0\} \quad \text{for every } n \in \mathbf{N}.$$

We define a map $f \in C(X, \mathbf{I})$ as follows:

$$f(x) = \begin{cases} 1 - \varepsilon_i^{-1} d(x, x_i) & \text{if } x \in B(x_i, \varepsilon_i), \ i \in \mathbf{N}, \\ 0 & \text{otherwise.} \end{cases}$$

One should note that any map $g \in C(X, \mathbf{I})$ with $\sup_{x\in X}|f(x) - g(x)| = \gamma < 1/2$ is not uniformly continuous. In fact, by ($\sharp_1$), we have $y_i \in X_0$, $i \in \mathbf{N}$, such that $d(x_i, y_i) = \varepsilon_i$, whence $\lim_{i\to\infty} d(x_i, y_i) = 0$ but

$$|g(x_i) - g(y_i)| \geq |f(x_i) - f(y_i)| - |f(x_i) - g(x_i)| - |f(y_i) - g(y_i)|$$
$$\geq 1 - \gamma - \gamma = 1 - 2\gamma > 0.$$

However, for each $\varepsilon > 0$, there exists a uniformly continuous map $h \in C(X, \mathbf{I})$ with $\rho_{\mathrm{H}}(f, h) < \varepsilon$. In fact, choose $n \in \mathbb{N}$ so that $2\varepsilon_n < \varepsilon$, and define a map $h \in C(X, \mathbf{I})$ as follows:

$$h(x) = \begin{cases} 1 - 2^{-1}\varepsilon_n^{-1} d(x, x_{n+1}) & \text{if } x \in B(x_{n+1}, 2\varepsilon_n), \\ f(x) & \text{otherwise.} \end{cases}$$

It follows from ($\sharp_1$) that $f(\mathrm{cl}\, B(x_i, \varepsilon_i)) = h(\mathrm{cl}\, B(x_{n+1}, 2\varepsilon_n)) = \mathbf{I}$ for every $i > n$. Then, by ($*$), it can be easily seen that $\rho_{\mathrm{H}}(f, h) < 2\varepsilon_n < \varepsilon$.

(2) Let X_0 be a non-totally bounded component of X. Then, we have $\delta > 0$ and $x_i \in X_0$, $i \in \mathbb{N}$, such that $d(x_i, x_j) > \delta$ if $i \neq j$. Observe that

($\sharp_2$) $$[0, \delta] \subset \{d(x_i, y) \mid y \in X_0\} \quad \text{for every } i \in \mathbb{N}.$$

For each $i \in \mathbb{N}$, let $\delta_i = \min\{i^{-1}, 1/3\delta\} > 0$. Now, we define a map $f \in C(X, \mathbf{I})$ as follows:

$$f(x) = \begin{cases} 1 - \delta_i^{-1} d(x, x_i) & \text{if } x \in B(x_i, \delta_i),\ i \in \mathbb{N}, \\ 0 & \text{otherwise.} \end{cases}$$

By the same reason as the case (1), any map $g \in C(X, \mathbf{I})$ with $\sup_{x\in X}|f(x) - g(x)| < 1/2$ is not uniformly continuous. However, for each $\varepsilon > 0$, choose $n \in \mathbb{N}$ so that $n^{-1} < \varepsilon$, and define a uniformly continuous map $h \in C(X, \mathbf{I})$ defined by

$$h(x) = \begin{cases} 1 - \min\{\varepsilon, \delta\}^{-1} d(x, x_i) & \text{if } x \in B(x_i, \min\{\varepsilon, \delta\}),\ i \geq n, \\ f(x) & \text{otherwise.} \end{cases}$$

From ($\sharp_2$), it follows that

$$f(\mathrm{cl}\, B(x_i, \delta_i)) = h(\mathrm{cl}\, B(x_{n+1}, \min\{\varepsilon, \delta\})) = \mathbf{I} \quad \text{for every } i \geq n,$$

Then, we have $\rho_{\mathrm{H}}(f, h) < \varepsilon$.

(3) For each $i \in \mathbb{N}$, take $x_i \in X_i$. Choose $2\delta > 0$ so that $\delta < \inf_{i\in\mathbb{N}} \operatorname{diam} X_i$ and $\delta < \inf_{i\neq j} \operatorname{dist}(X_i, X_j)$. Since $\sup_{x\in X_i} d(x, x_i) > \delta$, it follows that

($\sharp_3$) $$[0, \delta] \subset \{d(x_i, y) \mid y \in X_i\} \quad \text{for every } i \in \mathbb{N}.$$

Then, by replacing X_0 by X_i's in the proof of the case (2), we have the proof of this case. □

Acknowledgments

The authors would like to thank the referee for his helpful comments.

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