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NUMERICAL APPROXIMATION OF WEAK SOLUTIONS FOR BOUNDARY VALUE PROBLEMS

By

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Introduction

There are a lot of existence theorems for boundary value problems relating to linear partial differential equations. Although formulations of boundary value problems are different according to types of differential operators such as elliptic, parabolic, hyperbolic etc., their methods of proof seem very similar. The most popular method is to use Riesz' Theorem in a Hilbert space, based on energy estimates. On the other hand, methods of numerical approximation of solutions seem strictly combined with some positive forms (e.g. [1]). Since Riesz' Theorem is highly abstract, it seems that there is a very long distance between existence of solutions and approximation of solutions. Our aim in this paper is to see that the method of approximation is just behind Riesz' Theorem.

In this paper, weak solutions are defined by using supplementary functions, which play essential role for approximation of solutions. Existence of weak solutions is proved by using Riesz' Theorem, based on weak energy estimates on the adjoint problems. The essence of this idea is found in [2], [3], etc. Concerning to approximation, the trigonometrical functions are used as basis functions. It is the most remarkable point that basis functions can be chosen without any consideration of domains or boundary conditions.

Our problem is as follows. Let Ω be a bounded open set in R^n. Let Γ be the boundary of Ω, which is a finite sum of smooth surfaces, i.e. $\Gamma = \bigcup_{1 \leqq i \leqq h} \overline{\Gamma}_i$, where Γ_i is a smooth manifold of dimension $n-1$. Let us consider a boundary value problem:

$$(\mathrm{P}) \qquad \begin{cases} Au = f \quad \text{in } \Omega, \\ B_j^{(i)} u = 0 \text{ on } \Gamma_i \ \ (j = 1, \dots, b^{(i)}, i = 1, \dots, h), \end{cases}$$

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where $\{A, B_j^{(i)}\}$ are linear partial differential operators, f is a given function of $L^2(\Omega)$, and u is an unknown function of $L^2(\Omega)$.

The following Assumption(A) is assumed throughout this paper.

ASSUMPTION(A). 1) Let $x \in \overline{\Gamma}_{i_1} \cap \cdots \cap \overline{\Gamma}_{i_\gamma}$ and $x \notin \overline{\Gamma}_{i_{\gamma+1}} \cup \cdots \cup \overline{\Gamma}_{i_h}$, then there exists $U(x)$ (: *a* neighbourhood of x) such that

$$\Phi(U(x)) = V \quad (\Phi, \Phi^{-1}\text{: smooth}),$$

$$\Phi(\Omega \cap U(x)) = \{y \in V \mid y_1 > 0, y_2 > 0, \dots, y_\gamma > 0\},$$

$$\Phi(\Gamma_{i_1} \cap U(x)) = \{y \in V \mid y_1 = 0, y_2 > 0, \dots, y_\gamma > 0\},$$

$$\Phi(\Gamma_{i_2} \cap U(x)) = \{y \in V \mid y_1 > 0, y_2 = 0, y_3 > 0, \dots, y_\gamma > 0\},$$

$$\cdots\cdots\cdots$$

$$\Phi(\Gamma_{i_\gamma} \cap U(x)) = \{y \in V \mid y_1 > 0, \dots, y_\gamma = 0\},$$

where $V = \{y \in R^n \mid |y| < 1\}$. We say that $x \in \Gamma$ is a γ-ple point of Γ $(1 \leqq \gamma \leqq n)$, if $x \in \overline{\Gamma}_{i_1} \cap \cdots \cap \overline{\Gamma}_{i_\gamma}$ and $x \notin \overline{\Gamma}_{i_{\gamma+1}} \cup \cdots \cup \overline{\Gamma}_{i_h}$.

2) $A = A(x, D_x)$ is a linear partial differential operator of order m with smooth coefficients. Let $A_0(x, D_x)$ be the principal part of A. We assume that $|A_0(x, n^{(i)}(x))| > c\ (>0)\ (x \in \Gamma_i)$, where $n^{(i)}(x)$ is the unit exterior normal vector at $x \in \Gamma_i$.

3) $B_j^{(i)} = B_j^{(i)}(x, D_x)$ is a linear partial differential operator of order $m_j^{(i)}$ $(0 \leqq m_j^{(i)} \leqq m-1)$ defined near $\overline{\Gamma}_i$ with smooth coefficients. Let $B_{j0}^{(i)}(x, D_x)$ be the principal part of $B_j^{(i)}$. We assume that $|B_{j0}^{(i)}(x, n^{(i)}(x))| > c\ (>0)\ (x \in \Gamma_i)$ and $m_j^{(i)} \neq m_k^{(i)}\ (j \neq k)$.

We use notations:

$$\|\cdot\| = \|\cdot\|_{L^2(\Omega)}, \quad (\cdot\,,\cdot) = (\cdot\,,\cdot)_{L^2(\Omega)},$$

$$\|\cdot\|_\sigma = \|\cdot\|_{H^\sigma(\Omega)}, \quad (\cdot\,,\cdot)_\sigma = (\cdot\,,\cdot)_{H^\sigma(\Omega)},$$

$$\langle\cdot\rangle_{(i)} = \|\cdot\|_{L^2(\Gamma_i)}, \quad \langle\cdot\,,\cdot\rangle_{(i)} = (\cdot\,,\cdot)_{L^2(\Gamma_i)},$$

$$\langle\cdot\rangle_{(i),\sigma} = \|\cdot\|_{H^\sigma(\Gamma_i)}, \quad \langle\cdot\,,\cdot\rangle_{(i),\sigma} = (\cdot\,,\cdot)_{H^\sigma(\Gamma_i)},$$

where σ is a non-negative integer, in general. But $\langle\cdot\rangle_{(i),\sigma}$ and $\langle\cdot\,,\cdot\rangle_{(i),\sigma}$ may be used for fractional σ, in case when Γ_i is closed.

§1. Existence of Weak Solutions

Let

$$A(x, D_x) = \sum_{|\nu| \leqq m} a_\nu(x) D_x^\nu \quad (D_{x_j} = i^{-1}\partial_{x_j}),$$

and

$$A^*(x, D_x) = \sum_{|\nu| \leqq m} D_x^\nu \overline{a_\nu(x)}.$$

Let $\{m_j^{(i)} \ (j = b^{(i)} + 1, \dots, m)\}$ be defined such as

$$\{m_j^{(i)} \ (j = 1, 2, \dots, b^{(i)})\} \cup \{m_j^{(i)} \ (j = b^{(i)} + 1, \dots, m)\} = \{0, 1, \dots, m-1\},$$

and define

$$B_j^{(i)}(x, D_x) = (d/dn^{(i)})^{m_j^{(i)}} \quad (j = b^{(i)} + 1, \dots, m).$$

Corresponding to $\{B_j^{(i)}(x, D_x) \ (j = 1, \dots, m)\}$, as is well known, there exist partial differential operators $\{B_j'^{(i)}(x, D_x)(j = 1, \dots, m)\}$ near $\overline{\Gamma}_i$ of orders $\{m_j'^{(i)} = m - 1 - m_j^{(i)}(j = 1, \dots, m)\}$ such that it follows.

Lemma 1.1. *Suppose that* $u, f \in L^2(\Omega)$ *satisfy* $Au = f$ *in* Ω *(dis.). Then* $B_j^{(i)} u \in \mathscr{D}'(\Gamma_i)$ *and it holds*

$$(f, v) - (u, A^* v) = \sum_{1 \leqq j \leqq m} \langle B_j^{(i)} u, B_j'^{(i)} v \rangle_{(i)} \tag{G}$$

for any $v \in C^\infty(\overline{\Omega})$, *satisfying* $\mathrm{supp}[v] \cap \Gamma = \mathrm{supp}[v] \cap \Gamma_i$, *where* $\langle\ ,\ \rangle_{(i)}$ *is interpreted as the duality of* $\mathscr{D}'(\Gamma_i)$ *and* $\mathscr{D}(\Gamma_i)$, where $B_j^{(i)} u$ denotes $B_j^{(i)} u|_{\Gamma_i}$.

Now we define the *adjoint problem* (P′) corresponding to (P) by

$$\begin{cases} A^* v = g & \text{in } \Omega, \\ B_j'^{(i)} v = g_j^{(i)} & \text{on } \Gamma_i \ \ (j = b^{(i)} + 1, \dots, m, i = 1, \dots, h), \end{cases} \tag{P′}$$

and we assume

$$\|v\|^2 \leqq C \left\{ \|A^* v\|^2 + \sum_{i=1}^{h} \sum_{j=b^{(i)}+1}^{m} \langle B_j'^{(i)} v \rangle_{(i), \sigma_{ij}}^2 \right\} \quad (\forall v \in H^M(\Omega)), \tag{E′}$$

where $M = \max_{i,j}(m, m_j'^{(i)} + \sigma_{ij} + 1)$.

Let $\mathscr{H}$ be a *Hilbert space* defined by the completion of $H^M(\Omega)$ with respect to the norm

$$|v|^2 = \|A^*v\|^2 + \sum_{i=1}^{h} \sum_{j=b^{(i)}+1}^{m} \langle B_j'^{(i)} v \rangle_{(i),\sigma_{ij}}^2.$$

Inner product of $\mathscr{H}$ is defined by

$$[w, v] = (A^*w, A^*v) + \sum_{i=1}^{h} \sum_{j=b^{(i)}+1}^{m} \langle B_j'^{(i)} w, B_j'^{(i)} v \rangle_{(i),\sigma_{ij}}.$$

Remark. Energy estimate (E$'$) means that

$$\|v\| \leqq C|v| \quad (\forall v \in \mathscr{H}).$$

Our reasoning depends on the following well known theorem.

Riesz' Theorem. *Let $\ell[v]$ be a continuous anti-linear functional on $\mathscr{H}$. Then there exists $w \in \mathscr{H}$ such that*

$$\ell[v] = [w, v]$$

and

$$|w| = \sup_{v \in \mathscr{H}} \frac{|\ell[v]|}{|v|}.$$

We call w a Riesz' function *of $\ell[v]$.*

Let $f \in L^2(\Omega)$, then we have

$$|(f, v)| \leqq \|f\|\,\|v\| \leqq C\|f\|\,|v| \quad (\forall v \in \mathscr{H}),$$

therefore $\ell[v] = (f, v)$ defines a continuous anti-linear functional on $\mathscr{H}$. Owing to Riesz' Theorem, there exists $w \in \mathscr{H}$ such that

$$(f, v) = [w, v] \quad (\forall v \in \mathscr{H}), \quad |w| \leqq C\|f\|.$$

It means that

$$(f, v) = (A^*w, A^*v) + \sum_{i=1}^{h} \sum_{j=b^{(i)}+1}^{m} \langle B_j'^{(i)} w, B_j'^{(i)} v \rangle_{(i),\sigma_{ij}} \quad (\forall v \in \mathscr{H}).$$

Set $u = A^*w$, then

$$(f, v) - (u, A^*v) = \sum_{i=1}^{h} \sum_{j=b^{(i)}+1}^{m} \langle B_j'^{(i)} w, B_j'^{(i)} v \rangle_{(i),\sigma_{ij}} \quad (\forall v \in \mathscr{H}).$$

LEMMA 1.2. *Suppose that*

$$(f, v) - (u, A^* v) = \sum_{i=1}^{h} \sum_{j=b^{(i)}+1}^{m} \langle B_j'^{(i)} w, B_j'^{(i)} v \rangle_{(i), \sigma_{ij}} \quad (\forall v \in \mathscr{H})$$

holds. Then

1) $(f, \phi) - (u, A^* \phi) = 0$ $(\forall \phi \in \mathscr{D}(\Omega))$, that is, $Au = f$ in Ω (dis.),
2) $\langle B_j^{(i)} u, \phi \rangle_{(i)} = 0$ $(\forall \phi \in \mathscr{D}(\Gamma_i))$, that is, $B_j^{(i)} u = 0$ on Γ_i (dis.) $(j = 1, \dots, b^{(i)})$.

PROOF. 1) is obvious. 2) Set $C_i^{\infty}(\bar{\Omega}) = \{ v \in C^{\infty}(\bar{\Omega}) \mid \mathrm{supp}[v] \cap \Gamma = \mathrm{supp}[v] \cap \Gamma_i \}$, then

$$(f, v) - (u, A^* v) = \sum_{j=b^{(i)}+1}^{m} \langle B_j'^{(i)} w, B_j'^{(i)} v \rangle_{(i), \sigma_{ij}} \quad (\forall v \in C_i^{\infty}(\bar{\Omega}))$$

holds. On the other hand, we have

$$(f, v) - (u, A^* v) = \sum_{1 \leqq j \leqq m} \langle B_j^{(i)} u, B_j'^{(i)} v \rangle_{(i)} \quad (\forall v \in C^{\infty}(\bar{\Omega}))$$

from (G). Hence we have

$$\sum_{j=1}^{m} \langle B_j^{(i)} u, B_j'^{(i)} v \rangle_{(i)} = \sum_{j=b^{(i)}+1}^{m} \langle B_j'^{(i)} w, B_j'^{(i)} v \rangle_{(i), \sigma_{ij}} \quad (\forall v \in C_i^{\infty}(\bar{\Omega})),$$

which means $B_j^{(i)} u = 0$ on Γ_i (dis.) $(j = 1, \dots, b^{(i)})$. □

We say that $u \in L^2(\Omega)$ is a weak solution of (P), if

$$\begin{cases} Au = f & \text{in } \Omega \text{ (dis.)}, \\ B_j^{(i)} u = 0 & \text{on } \Gamma_i \text{ (dis.)} \ (j = 1, \dots, b^{(i)}, i = 1, \dots, h) \end{cases}$$

is satisfied. We say that $u \in L^2(\Omega)$ is a *$\mathscr{H}$-weak solution* of (P), if $u = A^* w$, where $w \in \mathscr{H}$ satisfies

$$[w, v] = (f, v) \quad (\forall v \in \mathscr{H}).$$

We call w a *supplementary function* of $\mathscr{H}$-weak solution of (P). A supplementary function of $\mathscr{H}$-weak solution of (P) is a Riesz' function of $\ell[\cdot] = (f, \cdot)$. As is shown above, $\mathscr{H}$-weak solution of (P) is a weak solution of (P). Here we have

THEOREM I. *Assume* (E$'$). *Then for any* $f \in L^2(\Omega)$, *there exists a unique $\mathscr{H}$-weak solution* $u \in L^2(\Omega)$ *of* (P) *and it holds*

$$\|u\| \leqq C \|f\|,$$

where C *is independent of* f.

Let us consider of a generalization of Theorem I. Let μ be a natural number and let us consider a solution $u \in L^2(\Omega)$ for $f \in (H^{\mu}(\Omega))'$, where $(H^{\mu}(\Omega))'$ is the dual space of $H^{\mu}(\Omega)$ with dual norm

$$\|f\|_{-\mu} = \sup_{v \in H^{\mu}(\Omega)} \frac{|(f,v)|}{\|v\|_{\mu}},$$

where $(\cdot,\cdot)$ is interpreted as the duality of $(H^{\mu}(\Omega))'$ and $H^{\mu}(\Omega)$. If we assume

$$(\mathrm{E}')_{\mu} \qquad \|v\|_{\mu}^2 \leqq C\left\{\|A^*v\|^2 + \sum_{i=1}^{h}\sum_{j=b^{(i)}+1}^{m} \langle B_j'^{(i)} v\rangle_{(i),\sigma_{ij}}^2\right\} \quad (\forall v \in H^M(\Omega)),$$

instead of (E'), then it holds

$$\|v\|_{\mu} \leqq C|v| \quad (\forall v \in H^M(\Omega)),$$

therefore, for $f \in (H^{\mu}(\Omega))'$, we have

$$|(f,v)| \leqq \|f\|_{-\mu}\|v\|_{\mu} \leqq C\|f\|_{-\mu}|v| \quad (\forall v \in \mathscr{H}).$$

Hence, as in the above, there exists $w \in \mathscr{H}$ such that

$$(f,v) = [w,v] \quad (\forall v \in \mathscr{H}), \quad |w| \leqq C\|f\|_{-\mu},$$

owing to Riesz' Theorem. Here we have

Theorem II. *Assume* $(\mathrm{E}')_{\mu}$. *Then for any* $f \in (H^{\mu}(\Omega))'$, *there exists a unique* $\mathscr{H}$*-weak solution* $u \in L^2(\Omega)$ *of* (P) *and it holds*

$$\|u\| \leqq C\|f\|_{-\mu}.$$

Example 1. Let Ω be a bounded open set in R^2, bounded by two closed curves Γ_1 and Γ_2 without any intersection. Let us consider

$$(\mathbb{P}) \qquad \begin{cases} \Delta U = 0 & \text{in } \Omega, \\ U = \phi & \text{on } \Gamma_1, \\ (d/dn)U = 0 & \text{on } \Gamma_2, \end{cases}$$

where $\Delta = \partial_x^2 + \partial_y^2$. Choose Φ such that

$$\Phi = \phi \quad \text{on } \Gamma_1, \quad \Phi = 0 \ \text{near } \Gamma_2,$$

and set

$$u = U - \Phi, \quad f = -\Delta\Phi,$$

then (ℙ) is reduced to

$$\text{(P)} \qquad \begin{cases} \Delta u = f & \text{in } \Omega, \\ u = 0 & \text{on } \Gamma_1, \\ (d/dn)u = 0 & \text{on } \Gamma_2. \end{cases}$$

The adjoint problem of (P) is

$$\text{(P}'\text{)} \qquad \begin{cases} \Delta v = g & \text{in } \Omega, \\ v = g_1 & \text{on } \Gamma_1, \\ (d/dn)v = g_2 & \text{on } \Gamma_2. \end{cases}$$

Since we know the energy inequality for (P′):

$$\|v\|_2 \leqq C\{\|\Delta v\| + \langle v\rangle_{(1),3/2} + \langle (d/dn)v\rangle_{(2),1/2}\} \quad (\forall v \in H^2(\Omega)),$$

we may define $\mathscr{H}$ as the completion of $H^2(\Omega)$ by the norm:

$$|v|^2 = \|\Delta v\|^2 + \langle v\rangle^2_{(1),3/2} + \langle (d/dn)v\rangle^2_{(2),1/2}.$$

EXAMPLE 2. Let ω be a bounded open set in R^2 with smooth boundary $\partial\omega$. Set

$$\Omega = (0,T)\times\omega, \quad \Gamma_0 = (0,T)\times\partial\omega, \quad \Gamma_1 = \{t=0\}\times\omega, \quad \Gamma_2 = \{t=T\}\times\omega,$$

and consider

$$(\mathbb{P}) \qquad \begin{cases} (\partial_t^2 - \Delta)U = 0 & \text{in } \Omega, \\ U = 0 & \text{on } \Gamma_0, \\ U = \phi_0 \text{ and } \partial_t U = \phi_1 & \text{on } \Gamma_1, \end{cases}$$

where $\Delta = \partial_x^2 + \partial_y^2$, and $\phi_0 = \phi_1 = 0$ on $\partial\Gamma_1$. Set

$$\Phi = \phi_0 + t\phi_1, \quad u = U - \Phi, \quad f = \Delta\Phi,$$

then (ℙ) is reduced to

$$\text{(P)} \qquad \begin{cases} (\partial_t^2 - \Delta)u = f & \text{in } \Omega, \\ u = 0 & \text{on } \Gamma_0, \\ u = \partial_t u = 0 & \text{on } \Gamma_1, \end{cases}$$

The adjoint problem of (P) is

$$\text{(P}'\text{)} \qquad \begin{cases} (\partial_t^2 - \Delta)v = g & \text{in } \Omega, \\ v = g_0 & \text{on } \Gamma_0, \\ v = g_1, \partial_t v = g_2 & \text{on } \Gamma_2. \end{cases}$$

Since we know the energy inequality for (P′):

$$\|v\|_1 \leqq C\{\|(\partial_t^2-\Delta)v\| + \langle v\rangle_{(0),1} + \langle v\rangle_{(2),1} + \langle \partial_t v\rangle_{(2),0}\} \quad (\forall v \in H^2(\Omega)),$$

we may define $\mathscr{H}$ as the completion of $H^2(\Omega)$ by the norm:

$$|v|^2 = \|(\partial_t^2-\Delta)v\|^2 + \langle v\rangle_{(0),1}^2 + \langle v\rangle_{(2),1}^2 + \langle \partial_t v\rangle_{(2),0}^2.$$

§2. Approximation of Weak Solutions

Lemma 2.1. *Suppose that* $\mathrm{diam}(\Omega) < a\pi$. *Set*

$$\Omega' = \prod_{j=1}^{n} (x_j^0 - a\pi, x_j^0 + a\pi)$$

for fixed $x^0 \in \Omega$. *Then* $\bar{\Omega} \subset \Omega'$ *and there exists a continuous linear map* L_k *from* $H^k(\Omega)$ *to* $H^k(\Omega')$ *such that*

$$(L_k w)(x) = w(x) \quad \text{in } \Omega, \quad \mathrm{supp}[L_k w] \subset \Omega'.$$

Proof. Since $\bar{\Omega} \subset \Omega'$, we can choose δ (>0) such that

$$\Omega \subset \Omega_\delta \subset \Omega',$$

where Ω_δ is the δ-neighbourhood of Ω. From Assumption(A-1), there exist $x^{(1)}, \dots, x^{(J)} \in \Gamma$ and their neighbourhoods $U_1, \dots, U_J$ such that

$$\Gamma \subset \bigcup_{j=1}^{J} U_j, \quad \mathrm{diam}(U_j) < \delta,$$

$$\Phi_j(U_j) = V, \quad \Phi_j(U_j \cap \Omega) = V \cap \Sigma_j,$$

where $\Sigma_j = \{y \in R^n \mid y_1 > 0, \dots, y_{\gamma_j} > 0\}$, where $x^{(j)}$ is a γ_j-ple point of Γ. Let $\{\beta_j(x)\}$ be smooth functions such that

$$\mathrm{supp}[\beta_j(x)] \subset U_j \ \ (j = 1, \dots, J), \quad \sum_{j=1}^{J} \beta_j(x)^2 = 1 \quad \text{near } \Gamma,$$

and

$$\beta_0(x) = \begin{cases} 1 - \sum_{j=1}^{J} \beta_j(x)^2 & (x \in \Omega), \\ 0 & (x \in \Omega^c). \end{cases}$$

Let $w \in H^k(\Omega)$, then

$$w(x) = \sum_{j=1}^{J} \beta_j(x)^2 w(x) + \beta_0(x) w(x) \quad (x \in \Omega).$$

Set

$$W_j(y) = \begin{cases} (\beta_j(x)w)(\Phi_j^{-1}(y)) & (y \in V \cap \Sigma_j), \\ 0 & (y \in V^c \cap \Sigma_j), \end{cases}$$

then

$$\| W_j \|_{H^k(\Sigma_j)} \leqq C_k \| w \|_{H^k(\Omega)}.$$

Let $\{c_1, \dots, c_k\}$ be defined by

$$\sum_{s=1}^{k} c_s (-s)^r = 1 \quad (r = 0, 1, \dots, k-1).$$

Set

$$W_j^{(1)}(y) = \begin{cases} W_j(y) & (y_1 > 0), \\ \sum_{s=1}^{k} c_s W_j(-s y_1, y_2, \dots, y_n) & (y_1 < 0), \end{cases}$$

$$W_j^{(2)}(y) = \begin{cases} W_j^{(1)}(y) & (y_2 > 0), \\ \sum_{s=1}^{k} c_s W_j^{(1)}(y_1, -s y_2, y_3, \dots, y_n) & (y_2 < 0), \end{cases}$$

$$\dots\dots\dots\dots\dots,$$

and set

$$W_j^{\sim}(y) = W_j^{(\gamma_j)}(y).$$

Then we have

$$W_j^{\sim}(y) = W_j(y) \quad (y \in \Sigma_j), \quad \mathrm{supp}[W_j^{\sim}(y)] \subset V,$$

and

$$\| W_j^{\sim} \|_{H^k(R^n)} \leqq C_k \| W_j \|_{H^k(\Sigma_j)}.$$

Finally, set

$$w^{\sim}(x) = \sum_{j=1}^{J} \beta_j(x) W_j^{\sim}(\Phi_j(x)) + \beta_0(x) w(x),$$

then we have

$$\mathrm{supp}[w^{\sim}] \subset \Omega_\delta, \quad w^{\sim}(x) = w(x) \quad (x \in \Omega),$$

and

$$\|w^{\sim}\|_{H^k(\Omega_\delta)} \leqq C_k \|w\|_{H^k(\Omega)}.$$

Hence we have a continuous linear map: $w \to w^{\sim}$ from $H^k(\Omega)$ to $H^k(\Omega')$. □

The following Lemma is well known in the theory of Fourier series.

LEMMA 2.2. *Let* $w \in H^k(\Omega')$ *with* $\mathrm{supp}[w] \subset \Omega'$. *Set*

$$C_\alpha = (2a\pi)^{-n}(w, \exp(\mathrm{i}a^{-1}\, \alpha \cdot x)) \quad (\alpha = (\alpha_1, \dots, \alpha_n) \in Z^n),$$

then

$$\|w\|^2_{H^k(\Omega')} = \sum_{|\nu| \leqq k} \|D^\nu w\|^2_{L^2(\Omega')}.$$

$$= (2a\pi)^n \sum_{\alpha \in Z^n} \left\{ \sum_{|\nu| \leqq k} (a^{-1}\alpha)^{2\nu} \right\} |C_\alpha|^2$$

and

$$w(x) = \sum_{\alpha \in Z^n} C_\alpha \exp(\mathrm{i}a^{-1}\, \alpha \cdot x) \quad \textit{in } H^k(\Omega'),$$

that is,

$$\sum_{|\alpha| \leqq N} C_\alpha (a^{-1}\alpha)^\nu \exp(\mathrm{i}a^{-1}\, \alpha \cdot x) \to D_x^\nu w \quad \textit{in } L^2(\Omega')(|\nu| \leqq k)$$

as $N \to \infty$.

From Lemma 2.1 and Lemma 2.2, we have

LEMMA 2.3. *Suppose that* $\mathrm{diam}(\Omega) < a\pi$ *and* $w \in H^M(\Omega)$. *Then there exists*

$$\{C_\alpha \mid \alpha = (\alpha_1, \dots, \alpha_n) \in Z^n\}$$

such that

$$c_1 \|w\|_{H^M(\Omega)} \leqq \sum_{\alpha \in Z^n} (1 + |\alpha|)^{2M} |C_\alpha|^2 \leqq c_2 \|w\|_{H^M(\Omega)}$$

and

$$w(x) = \sum_{\alpha \in Z^n} C_\alpha \exp(\mathrm{ia}^{-1}\, \alpha \cdot x) \quad \textit{in } H^M(\Omega).$$

Therefore, set

$$w_N(x) = \sum_{|\alpha| \leqq N} C_\alpha \exp(\mathrm{ia}^{-1}\, \alpha \cdot x) \quad (x \in \Omega),$$

then it holds

$$w_N \to w \quad \textit{in } \mathscr{H} \textit{ as } N \to \infty.$$

Proof. Let $w \in H^M(\Omega)$, and set $w^\sim = L_M w \in H^M(\Omega')$, where L_M is in Lemma 2.1. Let us apply Lemma 2.2 to $w^\sim$, then there exist $\{C_\alpha\}$ such that

$$\sum_{\alpha \in Z^n} (1 + |\alpha|)^{2M} |C_\alpha|^2 \leqq C\|w^\sim\|_{H^M(\Omega')} \leqq C'\|w\|_{H^M(\Omega)}$$

and

$$w^\sim(x) = \sum_{\alpha \in Z^n} C_\alpha \exp(\mathrm{ia}^{-1}\, \alpha \cdot x) \quad \text{in } H^M(\Omega').$$

Set

$$w_N^\sim(x) = \sum_{|\alpha| \leqq N} C_\alpha \exp(\mathrm{ia}^{-1}\, \alpha \cdot x), \quad w_N = w_N^\sim|_\Omega,$$

then

$$\|w_N - w\|_{H^M(\Omega)} \leqq \|w_N^\sim - w^\sim\|_{H^M(\Omega')} \to 0 \quad (\text{as } N \to \infty),$$

therefore

$$|w_N - w| \to 0 \quad (\text{as } N \to \infty). \qquad \square$$

Now we say that $V = \{v_i \in \mathscr{H} \ (i = 1, 2, \ldots)\}$ is a *set of basis functions of* $\mathscr{H}$, if sp V(: the linear space spaned by V) is dense in $\mathscr{H}$. Since $H^M(\Omega)$ is dense in $\mathscr{H}$, we have from Lemma 2.3.

Proposition 2.1. *Let* $\mathrm{diam}(\Omega) < a\pi$. *Then*

$$V = \{\exp(\mathrm{ia}^{-1}\, \alpha \cdot x) \mid \alpha = (\alpha_1, \ldots, \alpha_n) \in Z^n\}$$

is a set of basis functions of $\mathscr{H}$.

Hereafter we assume that $V = \{v_1, v_2, \dots\}$ is a set of basis functions of $\mathscr{H}$. Let $\{\hat{v}_1, \hat{v}_2, \dots, \hat{v}_N\}$ be an ortho-normalization of $\{v_1, v_2, \dots, v_N\}$ in $\mathscr{H}$, and set

$$\begin{pmatrix} \hat{v}_1 \\ \vdots \\ \hat{v}_N \end{pmatrix} = S_N \begin{pmatrix} v_1 \\ \vdots \\ v_N \end{pmatrix}, \quad S_N = (s_{ij})_{i,j=1,\dots,N}.$$

As is well known, we have

Lemma 2.4. *For $w \in \mathscr{H}$, set*

$$w^{(N)} = \sum_{k=1}^{N} [w, \hat{v}_k]\hat{v}_k.$$

Then it holds

$$|w^{(N)} - w| \leqq |\zeta - w|$$

for any $\zeta \in \mathrm{sp}\{v_1, \dots, v_N\}$.

Lemma 2.5. *For $w \in \mathscr{H}$, set*

$$w^{(N)} = \sum_{k=1}^{N} [w, \hat{v}_k]\hat{v}_k.$$

Then it is represented by

$$w^{(N)} = ([w, v_1], \dots, [w, v_N])K_N^{-1} \begin{pmatrix} v_1 \\ \vdots \\ v_N \end{pmatrix},$$

where

$$K_N = ([v_j, v_k])_{j,k=1,\dots,N}.$$

Proof. It is clear that

$$w^{(N)} = ([w, v_1], \dots, [w, v_N])S_N^* S_N \begin{pmatrix} v_1 \\ \vdots \\ v_N \end{pmatrix}.$$

On the other hand, we have

$$[\hat{v}_i, \hat{v}_j] = \left[\sum_{p=1}^{N} s_{ip} v_p, \sum_{q=1}^{N} s_{jq} v_q\right] = \sum_{p,q=1}^{N} s_{ip}[v_p, v_q]\overline{s_{jq}},$$

that is,

$$([\![\hat{v}_i, \hat{v}_j]\!])_{i,j=1,\dots,N} = S_N K_N S_N^*,$$

that is, $S_N^* S_N = K_N^{-1}$. $\square$

PROPOSITION 2.2. *Suppose that w is a supplementary function of $\mathscr{H}$-weak solution of* (P). *Then it holds*

$$w^{(N)} = ((f, v_1), \dots, (f, v_N))K_N^{-1}\begin{pmatrix} v_1 \\ \vdots \\ v_N \end{pmatrix} \to w \quad \text{in } \mathscr{H} \quad (\text{as } N \to \infty).$$

PROOF. Let $w \in \mathscr{H}$. Since $\operatorname{sp} V$ is dense in $\mathscr{H}$, there exist $N(j)$ and $w_j \in \operatorname{sp}\{v_1, \dots, v_{N(j)}\}$ such that $|w_j - w| < 1/j$, for any positive integer j. Set

$$w^{(N)} = ([\![w, v_1]\!], \dots, [\![w, v_N]\!])K_N^{-1}\begin{pmatrix} v_1 \\ \vdots \\ v_N \end{pmatrix},$$

then we have from Lemma 2.4

$$|w^{(N)} - w| \leqq |w_j - w| \quad (\forall N \geqq N(j)).$$

Therefore we have

$$|w^{(N)} - w| \to 0 \quad (N \to \infty).$$

On the other hand, since w is a supplementary function of $\mathscr{H}$-weak solution of (P), we have

$$[\![w, v]\!] = (f, v) \quad (\forall v \in \mathscr{H})$$

therefore

$$[\![w, v_k]\!] = (f, v_k) \quad (k = 1, 2, \dots).$$

Hence we have

$$w^{(N)} = ((f, v_1), \dots, (f, v_N))K_N^{-1}\begin{pmatrix} v_1 \\ \vdots \\ v_N \end{pmatrix}. \qquad \square$$

Let $w^{(N)}$ be the one defined in Proposition 2.2 and set $u^{(N)} = A^* w^{(N)}$, then we have

$$\|u^{(N)} - u\| = \|A^* w^{(N)} - A^* w\| \leqq |w^{(N)} - w| \to 0 \quad (\text{as } N \to \infty)$$

from the definition of the norm of $\mathscr{H}$. Hence we have

Theorem III. *Suppose that u is a $\mathscr{H}$-weak solution of* (P). *Set*

$$u^{(N)} = ((f, v_1), \dots, (f, v_N)) K_N^{-1} \begin{pmatrix} A^* v_1 \\ \vdots \\ A^* v_N \end{pmatrix},$$

where $K_N = ([v_i, v_j])_{i,j=1,\dots,N}$. Then it holds that

$$\|u^{(N)} - u\| \to 0 \quad (\text{as } N \to \infty).$$

§3. Variational Problems

Let us consider of a *variational problem*, relating to a continuous anti-linear functional $\ell[v]$ on $\mathscr{H}$. Namely, define a quadratic functional on $\mathscr{H}$:

$$J[v] = |v|^2 - 2 \operatorname{Re} \ell[v],$$

and the variational problem of $J[v]$ in $\mathscr{H}$ is to seek a *stationary function* $w \in \mathscr{H}$ such that

$$J[w] = \min_{v \in \mathscr{H}} J[v].$$

Since

$$J[w+v] - J[w] = 2 \operatorname{Re}\{[w, v] - \ell[v]\} + |v|^2,$$

w satisfies

$$J[w+v] - J[w] \geqq 0 \quad (\forall v \in \mathscr{H})$$

iff

$$[w, v] - \ell[v] = 0 \quad (\forall v \in \mathscr{H}).$$

Here we have

Lemma 3.1. *$w \in \mathscr{H}$ is a Riesz' function of $\ell[v]$ in $\mathscr{H}$, i.e.*

$$[w, v] = \ell[v] \quad (\forall v \in \mathscr{H}),$$

iff w *is a stationary function of the variational problem of* $J[v]$, *i.e.*

$$J[w] = \min_{v \in \mathscr{H}} J[v].$$

Moreover,

$$|w|^2 = \left\{ \sup_{v \in \mathscr{H}} \frac{|\ell[v]|}{|v|} \right\}^2 = -\min_{v \in \mathscr{H}} J[v].$$

Set $\mathscr{H}_N = \mathrm{sp}\{v_1, \dots, v_N\}$, then we have

Lemma 3.2. $w^{(N)} \in \mathscr{H}_N$ *is a Riesz' function of* $\ell[v]$ *in* $\mathscr{H}_N$, *i.e.*

$$[w^{(N)}, v] = \ell[v] \quad (\forall v \in \mathscr{H}_N),$$

iff $w^{(N)}$ *is a stationary function of the variational problem of* $J[v]$ in $\mathscr{H}_N$, *i.e.*

$$J[w^{(N)}] = \min_{v \in \mathscr{H}_N} J[v].$$

Moreover,

$$|w^{(N)}|^2 = \left\{ \sup_{v \in \mathscr{H}_N} \frac{|\ell[v]|}{|v|} \right\}^2 = -\min_{v \in \mathscr{H}_N} J[v].$$

From Lemma 3.1 and Lemma 3.2, we have

Theorem IV. *The supplementary function* w *of* $\mathscr{H}$*-weak solution of the problem(*P*) is a stationary function of the variational problem of*

$$J[v] = |v|^2 - 2\,\mathrm{Re}(f, v) \quad \text{in } \mathscr{H},$$

and

$$w^{(N)} = ((f, v_1), \dots, (f, v_N))K_N^{-1} \begin{pmatrix} v_1 \\ \vdots \\ v_N \end{pmatrix}$$

is a stationary function of the variational problem of

$$J[v] = |v|^2 - 2\,\mathrm{Re}(f, v) \quad \text{in } \mathscr{H}_N.$$

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