TSUKUBA J. MATH.
Vol. 24 No. 1 (2000), 81–107

NONLINEAR WAVE EQUATION WITH POTENTIAL

By

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Abstract. We study the Cauchy problem for

$$u_{tt} - \Delta u + V(x)|u|^{p-1}u = 0$$

with $x \in \boldsymbol{R}^n$. The function $V(x)$ is positive and regular. The exponent p is subcritical or critical. By the aid of Shatah-Struwe technique (cf. [7]), we prove the existence of the global classical solution with suitable hypotheses on $V(x) : V(x) > 0,\ 3 \leq n \leq 7$ or $V(x) = |x|^2$, $n = 3$. To approach this second case we cannot follow directly the argument used in [7]: we need and prove weighted nonlinear estimates in Besov spaces.

1. Introduction

A large amount of work has been devoted to studying several questions related to the solution to the nonlinear hyperbolic Cauchy problem

$$\Box u(t,x) = \Phi(t,x,u,u_t) \quad x \in \boldsymbol{R}^n$$

$$u(0,x) = u_0(x)$$

$$u_t(0,x) = u_1(x).$$

For example, considering the case $\Phi = \Phi(u) = \pm|u|^{p-1}u$, it is possible to analyze the existence of the global solution or its eventual blow up in dependence on the size of initial data, the large time behavior of the solution, and so on. The theory began in the sixties, but in spite of the great deal of papers concerning it, some questions remain open. More precisely, fixing our attention to the nonlinear term $\Phi(u) = -|u|^{p-1}u$, with $p > (n+2)/(n-2)$ (supercritical exponent) no results clarify if there exists a global regular solution with arbitrary initial data.

Key words: Nonlinear wave equation, Large data, Besov spaces, Strichartz estimates.
AMS Classification: 35L70, 35L05, 35B65
Received December 15, 1998
Revised September 7, 1999

The positive answer to the previous question in sub-critical and critical case relies on the possibility to find a–priori estimates for the solution of the wave equation.

For example, in sub-critical case, the conservation law for the energy is crucial. In the critical case this $L^2 - L^2$ estimate is not sufficient; one has to combine $L^p - L^q$ estimate (cf. Proposition 3.6) with a multiplicative inequality for the nonlinear term (cf. Proposition 3.3).

In this work we examine carefully a remark contained in Shatah–Struwe paper [7]: in the critical case $\Phi(u) = -|u|^{4/(n-2)}u$ the authors obtain the global classical solution and they say that using the same technique this result can be generalized to suitable $\Phi(x,t,u)$. Their proof is based on the following tools:

(i) a decay lemma,

(ii) a Strichartz' inequality,

(iii) a nonlinear estimate in Besov spaces,

(iv) a contradiction argument: in a neighborhood of an eventual blow up point the solution is bounded.

In their work all the estimates are set in bounded domain, hence it is reasonable that if $\Phi(x,t,u)$ is a continuous function in t and x, having critical behavior in u, then the global existence result still holds. Here we try to classify the functions $\Phi(x,t,u)$ which give the global existence result. As will become apparent, the problem is not so simple to be solved, hence we deal only with the equation

$$u_{tt} - \Delta u = -V(x)|u|^{p-1}u \tag{1.1}$$

with positive $V(x)$.

If $V(x) > 0$, locally it behaves like a positive constant; this gives directly an extension of Shatah–Struwe result. Our approach will not distinguish the critical case from the sub–critical one and the same technique applies equally in both cases. In particular for $V(x)$ constant, we see that the Shatah-Struwe method works also in the sub–critical case. On the contrary, if $V(x)$ vanishes, the zero of the potential could compensate the blow up of the solution. On the other hand, if $V(x) = 0$ at some point $\bar{x}$, the equation at that point reduces to the linear homogeneous wave equation; hence it is natural to think that the global existence result is still valid.

Here we give the global existence result for the case $V(x)|u|^{p-1}u = |x - x_0|^2 u^5$, $n = 3$. In order to do this we find a weighted version of the nonlinear estimate in Besov spaces. The interest of this inequality relies on the fact that the multiplicative rules are not well known in Besov spaces.

The case $V(x) \neq |x - x_0|^2$ is not considered here since it is connected with the essential difficulty to establish a decay lemma. Finally we remark that in [4] other problems with vanishing potentials are studied under the strong hypothesis of small initial energy. Another simple case is the 3-dimensional subcritical case in which Jörgens' argument yields global solution without any assumption on the zeros of the potential (cf. [3]).

The plan of the work is the following. In Section 2 we collect known results and notations and we prove the decay lemma. In Section 3 we establish the weighted nonlinear inequality that enables us to consider the vanishing potential case. Here we also recall some useful properties of Besov spaces. We shall use slightly different spaces from those used in [7]: the homogeneous Slobodeckji spaces. For this reason we often treat with Hardy's inequality.

In Section 4 we prove our main theorem: the existence of a unique global solution for the Cauchy problem related to (1.1), either in the subcritical or critical case for $V(x) > 0$ or in the critical 3-dimensional case for $V(x) = |x - x_0|^2$.

Acknowledgements: I'm greatly indebted with Prof. Vladimir Georgiev for his hospitality during my visit at Academy of Science in Sofia, where we studied weighted spaces. I'd like also to thank Prof. Sergio Spagnolo who encouraged me to solve this problem.

2. Notations and Preliminary Results

We deal with the Cauchy problem

$$\Box u = -V(x)\Phi(u) \tag{2.1}$$

$$u(0,x) = f(x)$$

$$u_t(0,x) = g(x) \tag{2.2}$$

where V, Φ satisfy

(i) $V \in \mathscr{C}^2(\boldsymbol{R}^n)$, $V(x) \geq 0$;

(ii) $\Phi \in \mathscr{C}^2(\boldsymbol{R})$, $\int_0^u \Phi(t)\,dt \geq 0$, $\Phi(0) = 0$, $\Phi(u) = |u|^{p-1}u$ if $|u| \geq C > 0$;

(iii) $3 \leq n \leq 7$, $p \leq (n+2)/(n-2)$.

Since we want to prove the boundedness of u and $\Phi(u) = |u|^{p-1}u$ for large $|u|$, we can assume without restriction that, instead of (2.1), u solves the simpler equation

$$\square u = -V(x)|u|^{p-1}u. \tag{2.3}$$

We recall that for this equation there is finite speed of propagation equal to one and formal conservation law for the energy

$$E[u](t) = \frac{1}{2}\int_{\mathbf{R}^n} |u_t|^2 + |\nabla_x u|^2\,dx + \int_{\mathbf{R}^n} \frac{V(x)}{p+1}|u|^{p+1}\,dx, \tag{2.4}$$

that is $E[u](t) = E[u](0)$ for each $t \in \mathbf{R}$. We shall put $E[u](0) =: E_0$.

Using these properties one can prove the local existence result (cf. Theorem 4.3 in [6]):

THEOREM 2.1. *Let $s > n/2$; for any $(f,g) \in H^s(\mathbf{R}^n) \times H^{s-1}(\mathbf{R}^n)$ there exists $T > 0$ and a unique strong s-regular solution $u(x,t) : \mathbf{R}^n \times [0,T] \to \mathbf{R}$ for* (2.1), (2.2), *that is $u \in \mathscr{C}([0,T], H^s(\mathbf{R}^n))$, $u_t \in \mathscr{C}([0,T], H^{s-1}(\mathbf{R}^n))$ and $u_{tt} \in \mathscr{C}([0,T], H^{s-2}(\mathbf{R}^n))$. If $f \in \mathscr{C}^3(\mathbf{R}^n) \cap H^s(\mathbf{R}^n)$, $g \in \mathscr{C}^2(\mathbf{R}^n) \cap H^{s-1}(\mathbf{R}^n)$ the unique local solution of* (2.1), (2.2) *is a classical solution, i.e. $u \in \mathscr{C}^2(\mathbf{R}^n \times [0,T])$.*

Moreover, if (f,g) have compact support, then for each $t \in [0,T]$, $u(\cdot,t)$ has compact support.

Having in mind the finite speed of propagation, we consider the backward cone with vertex $z_0 = (x_0, t_0) \in \mathbf{R}^n \times \mathbf{R}$:

$$K(z_0) := \{z = (x,t) | t \le t_0, |x - x_0| \le t_0 - t\}.$$

For any $S \le T \le t_0$ we put

$$K_S^T(z_0) := \{(x,t) \in K(z_0) | S \le t \le T\}, \quad K_S(z_0) := K_S^{t_0}(z_0),$$

$$M(z_0) := \{z = (x,t) | t \le t_0, |x - x_0| = t_0 - t\},$$

$$M_S^T(z_0) := \{(x,t) \in M(z_0) | S \le t \le T\}, \quad M_S(z_0) := M_S^{t_0}(z_0),$$

$$D(t, z_0) := \{x \in \mathbf{R}^n | (x,t) \in K(z_0)\}.$$

Further we define the local energy

$$E(u, D(t,z_0)) := \int_{D(t,z_0)} \frac{1}{2}|u_t|^2 + \frac{1}{2}|\nabla_x u|^2 + \frac{V(x)}{p+1}|u|^{p+1}\,dx \tag{2.5}$$

and we state a crucial information for the proof of our theorem, that is the flux conservation law: for any $S \le T \le t_0$

$$E(u,D(S,z_0)) - E(u,D(T,z_0))$$

$$= \frac{1}{\sqrt{2}} \int_{M_S^T(z_0)} \frac{1}{2} \left| \frac{x-x_0}{|x-x_0|} u_t - \nabla_x u \right|^2 + \frac{V(x)}{p+1} |u|^{p+1} \, d\omega. \tag{2.6}$$

This is obtained integrating on $K_S^T(z_0)$ the identity

$$\partial_t \left(\frac{1}{2} |u_t|^2 + \frac{1}{2} |\nabla_x u|^2 + \frac{V(x)}{p+1} |u|^{p+1} \, dx \right) = \operatorname{div}(u_t \nabla_x u).$$

We see that for any potential $V(x)$ positive, the quantity $E(u,D(S,z_0))$ is decreasing in S and tends to zero whenever $S \to t_0$, hence

$$\lim_{S\to t_0} \int_{M_S^T(z_0)} \frac{1}{2} \left| \frac{x-x_0}{|x-x_0|} u_t - \nabla_x u \right|^2 + \frac{V(x)}{p+1} |u|^{p+1} \, d\omega = 0. \tag{2.7}$$

We conclude this section with the first crucial ingredient of Shatah-Struwe method: a Pohozaev type identity which gives the possibility to control the higher term of the local energy. This works under suitable assumptions on $V(x)$ and p.

Lemma 2.1. *Let u be a classical solution for* (2.1), (2.2) *on* $K(z_0)\backslash\{z_0\}$. *Suppose V, n, p satisfy one of the following conditions*:

(H_2) $V(x) > 0$, $3 \le n \le 7$, $\frac{n+3}{n-1} < p \le \frac{n+2}{n-2}$.

(H_2) $V(x) = |x-x_0|^2$ *with* $x_0 \in \mathbf{R}^n$, $n = 3$, $p = 5$.

Then

$$\lim_{S\to t_0} \int_{D(S,z_0)} V(x)|u|^{p+1} \, dx = 0. \tag{2.8}$$

Proof. At the begin we consider $z_0 = (0,0)$ and omit it in the notations of the domains. We multiply equation (2.3) by $tu_t + x \cdot \nabla u + ((n-1)/2)u$, obtaining the identity

$$0 = \partial_t \left(tQ_0 + \frac{n-1}{2} u_t u \right) - \operatorname{div}(tP_0) + R_0 \tag{2.9}$$

where

$$Q_0 = \frac{|u_t|^2}{2} + \frac{|\nabla_x u|^2}{2} + \frac{V(x)}{p+1}|u|^{p+1} + u_t\Big(\frac{x}{t}\cdot\nabla u\Big);$$

$$P_O = \frac{x}{t}\left(\frac{|u_t|^2}{2} - \frac{|\nabla_x u|^2}{2} - \frac{V(x)}{p+1}|u|^{p+1}\right) + \nabla u\Big(u_t + \frac{x}{t}\cdot\nabla u + \frac{n-1}{2}\frac{u}{t}\Big);$$

$$R_0 = \left(\frac{n-1}{2} - \frac{n+1}{p+1}\right)V(x)|u|^{p+1} - \frac{x\cdot\nabla V(x)}{p+1}|u|^{p+1}.$$

Integrating (2.9) on K_S^T and letting $T \to 0$, we find:

$$0 = -\int_{D(S)}\Big\{SQ_0 + \frac{n-1}{2}u_t u\Big\}\,dx + \int_{K_S} R_0\,dxdt$$
$$+\frac{1}{\sqrt{2}}\int_{M_S}\Big\{tQ_0 + \frac{n-1}{2}u_t u + x\cdot P_0\Big\}\,d\omega = \mathrm{I} + \mathrm{II} + \mathrm{III}.$$

First we observe that

$$\mathrm{I} + \mathrm{III} \geq So(1) - \int_{D(S)} S\frac{V(x)}{p+1}|u|^{p+1}\,dx$$

where $o(1)$ tends to zero when $S \to 0$. In fact, in Lemma 1.2 of [7] it is shown that

$$\mathrm{I} + \mathrm{III} \geq -S\int_{D(S)}\frac{V(x)}{p+1}|u|^{p+1}\,dx$$
$$+\frac{1}{\sqrt{2}}\int_{M_S}\frac{n+1}{2}\left|u_t - \frac{x\cdot\nabla_x u}{|x|}\right|^2 d\omega + \frac{1}{\sqrt{2}}\int_{M_S}\frac{(n-1)(n-3)}{4}\frac{u^2}{t}\,d\omega.$$

The second term is directly estimated by $-So(1)$ by virtue of (2.7). To reduce the third term to $\int_{M_S} V(x)|u|^{p+1}\,d\omega$ we apply Hölder's inequality. If $V(x) > 0$ this is a straightforward computation. In the case $V(x) = 0$ we need some additional assumptions. For example in the case $V(x) = |x|^\alpha$ we have

$$\int_{M_S}\frac{u^2}{|t|^2}\,d\omega \leq \left(\int_{M_S}|x|^{-\gamma}\,d\omega\right)^{1-2/(p+1)}\left(\int_{M_S}|x|^\alpha|u|^{p+1}\,d\omega\right)^{2/(p+1)},$$

here $\gamma = (2\alpha + p + 1)/(p-1)$; there is convergence of the first factor only if

$$\frac{n(p-1)}{2(p+1)} - \frac{1}{2} > \frac{\alpha}{p+1}.$$

In particular for $n = 3$, $p = 5$ we require $\alpha < 3$.

In order to conclude the proof it suffices to know

$$\exists \bar{S} < 0 \text{ s.t. } \forall S < 0, \quad |S| < |\bar{S}| \quad \mathrm{II} = \int_{K_S} R_0 \, \mathrm{d}x\mathrm{d}t \geq 0. \tag{2.10}$$

Hence, for $|S| < |\bar{S}|$ these relations imply

$$0 = \mathrm{I} + \mathrm{II} + \mathrm{III} \geq -\int_{D(S)} S \frac{V(x)}{p+1} |u|^{p+1} \, \mathrm{d}x + So(1).$$

Being $S < 0$, we have the desired conclusion.

It remains to check (2.10). In particular we show that R_0 is positive on K_S, that is

$$\left(\frac{n-1}{2} - \frac{n+1}{p+1}\right) V(x) \geq \frac{x \cdot \nabla V(x)}{p+1}, \quad |x| \leq |S|. \tag{2.11}$$

Suppose $V(x) > 0$ in $D(S)$; being K_S compact, we find $|\nabla V(x)| \leq C_S V(x)$ for some $C_S > 0$ and for all $x \in K_S$. Moreover C_S is a decreasing function of S in $(-\infty, 0)$. We fix $T = -(p+1)((n-1)/2 - (n+1)/(p+1))$; T is negative in force of the hypothesis (H_1). For any $S \geq \max\{T, T/C_T\}$, $x \in K_S$ we have

$$\frac{x \cdot \nabla V(x)}{p+1} \leq \left| \frac{x \cdot \nabla V(x)}{p+1} \right| \leq C_T \frac{-SV(x)}{p+1} \leq \left(\frac{n-1}{2} - \frac{n+1}{p+1}\right) V(x).$$

In the case $V(x) = 0$ at some point, in general, we cannot conclude that (2.11) holds. In the case $V(x) = |x|^\alpha$ we see that (2.11) reduces to

$$\frac{n-1}{2} - \frac{n+1}{p+1} \geq \frac{\alpha}{p+1}.$$

Being $\alpha \geq 2$ and $p \leq (n+2)/(n-2)$, this condition is verified if (H_2) holds in $D(S)$. In the case $z_0 = (x_0, t_0) \neq (0,0)$ we have

$$\lim_{S \to t_0} \int_{D(S, z_0)} V(x) |u|^{p-1} u \, \mathrm{d}x$$

$$= \lim_{S \to 0} \int_{D(S)} V(x + x_0) |u(x + x_0, t + t_0)|^{p-1} u(x + x_0, t + t_o) \, \mathrm{d}x.$$

Since $v(x,t) = u(x + x_0, t + t_0)$ solves $\Box v(x,t) = V(x + x_0) |v(x,t)|^{p-1} v(x,t)$, the assertion follows from the previous computations under the assumption either $V(x) > 0$ in $D(S, z_0)$ or $V(x) = |x - x_0|^2$ in $D(S, z_0)$. □

We emphasize that we have only used the estimate from below $p > (n+3)/(n-1)$. The previous lemma is still valid if $p = (n+3)/(n-1)$ and $V(x)$ is a positive constant, namely the classical case.

From the previous proof it follows that in the case $n=3$ $p=5$, the assumption (H_2) can be weakened: the potential can vanish of order 2 in isolated points, this means

(H_2') $V(x_0)=0$ implies $V(x)=|x-x_0|^2$ in a neighborhood of x_0;

The case $V(x)>0$ is the simplest one because Lemma 2.1 implies the following:

COROLLARY 2.1. *Let u be a classical solution for* (2.1), (2.2) *on* $K(z_0)\backslash\{z_0\}$. *Suppose* (H_1) *holds, then*

$$\lim_{S\to t_0}\int_{D(S,z_0)}|u|^{p+1}\,dx=0. \tag{2.12}$$

3. A Weighted Nonlinear Inequality

The aim of this section is to prove a generalization of a nonlinear inequality due to Ginibre-Velo (cf. [2]). We prefer to recall the necessary tools for this proof, in particular the real interpolation theory. On the contrary, we don't describe here the complex interpolation theory, though we shall use it. We shall often quote the monographs [1] and [8] in which the reader can find a general framework for these subjects.

About the notations, we omit to write $\mathbf{R}^n$ if it is a domain of a function space, denoting by $\|\cdot\|_p$ the $L^p(\mathbf{R}^n)$-norm. Finally by $\simeq$ we mean the equivalence of two positive functions A,B: we write $A\simeq B$ if there exist C_1,C_2 such that $C_1A(x)\le B(x)\le C_2A(x)$ for all x in the intersection of the domains of A,B.

We start with the abstract definition of real interpolation for a couple of Banach spaces:

DEFINITION 3.1 ([8] 1.3). *Let A_0, and A_1 be Banach spaces, both linearly and continuously embedded in a linear Hausdorff space $\mathscr{A}$.*

For each $0<t<+\infty$, one defines the K-functional related to (A_0,A_1):

$$K(t,\cdot,A_0,A_1):A_0+A_1\to\mathbf{R},$$

$$K(t,f,A_0,A_1)=\inf\{\|g\|_{A_0}+t\|h\|_{A_1}\ \text{s.t.}\ f=g+h, g\in A_0, h\in A_1\}.$$

Fixing $0<\theta<1$, $1\le q<\infty$, the intermediate space between $A_0\cap A_1$ and A_0+A_1 is

$$(A_0,A_1)_{\theta,q}:=\left\{f\in A_0+A_1\|f\|^q_{(A_0,A_1)_{\theta,q}}:=\int_0^{+\infty}t^{-sq}K(t,f,A_0,A_1)^q\frac{dt}{t}<+\infty\right\}.$$

One of the advantages of the real interpolation theory is the fact that the linear bounded operators are exact interpolation functors in the sense of the following proposition:

Proposition 3.1 ([8] 1.3.3(a)). *Let* (A_0, A_1), (B_0, B_1) *be couples of Banach spaces under the same hypotheses of the previous definition. Let* $0 < \theta < 1$, $1 \le q < +\infty$. *If*

$$T : A_0 \to B_0, \quad T : A_1 \to B_1$$

is a linear bounded operator, then

$$T : (A_0, A_1)_{\theta,q} \to (B_0, B_1)_{\theta,q}$$

is a linear bounded operator.

Now we give some examples of real interpolation spaces; we need them in what follows.

Example 3.1 ([8] 1.18.5). Let $\omega : \boldsymbol{R}^n \to \boldsymbol{R}$ be a positive continuous function; $\Omega \subset \boldsymbol{R}^n$ a measurable set. For each $1 \le p < +\infty$, $L^p(\Omega, \omega)$ is the weighted L^p space which consists of the measurable functions $f : \Omega \to \boldsymbol{R}$ such that $\omega^{1/p} f \in L^p(\Omega)$. This means

$$\|f\|^p_{L^p(\Omega,\omega)} := \int_\Omega |f|^p \omega(x)\, dx < +\infty.$$

For these spaces the following interpolation property holds ([8] 1.18.5): if $0 < \theta < 1$, $\omega_0, \omega_1 : \boldsymbol{R}^n \to \boldsymbol{R}$ are continuous positive functions, then

$$[L^{p_0}(\Omega, \omega_0^{p_0}), L^{p_1}(\Omega, \omega_1^{p_1})]_{\theta,p} = L^p(\Omega, \omega^p) \tag{3.1}$$

where

$$\frac{1}{p} = \frac{\theta}{p_0} + \frac{1-\theta}{p_1} \quad \text{and} \quad \omega(x) = \omega_0^\theta \omega_1^{1-\theta}.$$

In particular, there exists $C_\theta > 0$ such that

$$\|u\|_{L^p(\omega^p)} \le C_\theta \|u\|^\theta_{L^{p_0}(\omega_0^{p_0})} \|u\|^{1-\theta}_{L^{p_1}(\omega_1^{p_1})} \tag{3.2}$$

for all u such that the right side is finite.

Example 3.2 ([1] 5.6.2). Let A be a Banach space. One denotes by $l^s_q(A)$ the space of the sequences $\{a_j\}_{j=-\infty}^{\infty} \subset A$ having norm

$$\|a_j\|_{l_q^s(A)}^q = \sum_{j=-\infty}^{\infty} 2^{jsq} \|a_j\|_A^q$$

bounded. Let (A_0, A_1) be a couple of Banach spaces in the sense of Definition 3.1. The real interpolation gives

$$(l_{q_0}^{s_0}(A_0), l_{q_1}^{s_1}(A_1))_{\theta,q} = l_q^s((A_0, A_1)_{\theta,q}), \tag{3.3}$$

where $1 < q_0,\ q_1 < \infty$, and

$$s = (1-\theta)s_0 + \theta s_1, \quad \frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}.$$

EXAMPLE 3.3 ([8] 2.4.2(13)–(16)). Let $s > 0$, $1 < p, q < \infty$. The Besov spaces $B_{p,q}^s$ is defined in the following way:
if s is not integer

$$B_{p,q}^s = (W_p^k, W_p^h)_{\theta,q} \quad \text{with} \quad s = \theta k + (1-\theta)h, \quad h, k \in N;$$

if s is integer, by reiteration

$$B_{p,q}^s = (B_{p,q_0}^{s_0}, B_{p,q_1}^{s_1})_{\theta,q} \quad \text{with} \quad s = \theta s_0 + (1-\theta)s_1, \quad \frac{\theta}{q_0} + \frac{1-\theta}{q_1} = \frac{1}{q}.$$

In the same way, considering the homogeneous Sobolev space $\dot{W}_p^h$, after factorization out of polynomials, one obtains homogeneous Besov spaces.

In what follows we shall use also homogeneous Slobodeckji spaces:

$$\dot{W}_p^s = \begin{cases} \dot{B}_{p,p}^s & \text{if } s \text{ is not integer} \\ \dot{W}_p^s & \text{if } s \text{ is integer} \end{cases}.$$

We recall that these spaces are different from the Sobolev spaces of fractional order: suppose s is not integer, then $\dot{W}_p^s = \dot{H}_p^s$ if and only if $p = 2$.

Other relations between these spaces are given in terms of embedding theorems (cf. [1]):

$$\dot{B}_{p,q}^s \hookrightarrow L^r \quad \text{if } s \geq \frac{n}{p} - \frac{n}{r}, \quad p \leq r, q \leq r, \tag{3.4}$$

$$\dot{B}_{p,q}^s \hookrightarrow \dot{B}_{p,q_1}^s \quad \text{if } q \leq q_1. \tag{3.5}$$

Using semigroup theory one finds several equivalent norms for the spaces defined by real interpolation. For example the translation on $\boldsymbol{R}^n$ yields the next result:

Proposition 3.2 ([8] 2.5.1). *Let* $0 < s < 1$, $1 < p, q < +\infty$. *An equivalent norm for* $\dot{B}^s_{p,q}$ *is*

$$\|f\|_{\dot{B}^s_{p,q}} = \left(\int_{|h| \le 1} |h|^{-sq} \|f(x+h) - f(x)\|_p^q \frac{\mathrm{d}h}{|h|^n} \right)^{1/q}. \tag{3.6}$$

Ginibre and Velo use this norm to derive an estimate for $|u|^\lambda$ in Besov spaces:

Proposition 3.3. *Let* $0 < s < 1$, $\lambda > 1$ *and* $1 < p_1, p_2, p, q < +\infty$ *and* $\lambda > 1$, *with* $1/p = 1/p_1 + 1/p_2$. *There exists* $C > 0$ *such that*

$$\| |u|^\lambda \|_{\dot{B}^s_{p,q}} \le C \|u\|_{\dot{B}^s_{p_1,q}} \| |u|^{\lambda-1} \|_{L^{p_2}} \tag{3.7}$$

for all u *such that the norms on the right side are finite.*

We write in Slobodeckij spaces this result:

Lemma 3.1. *Under the same hypotheses of the previous proposition, there exists* $C > 0$, *depending only on* n, p, p_1, *such that*

$$\| |u|^\lambda \|_{\dot{W}^s_p} \le \frac{C}{\varepsilon} \|u\|_{\dot{W}^{s+\varepsilon}_{p_1}} \| |u|^{\lambda-1} \|_{L^{p_2}} \quad 0 < \varepsilon < 1. \tag{3.8}$$

Proof. Using Ginibre–Velo result it is clear that it suffices to prove the embedding

$$\dot{W}^{s+\varepsilon}_p \hookrightarrow \dot{B}^s_{p,q}, \quad q < p \quad \text{and} \quad \|f\|_{\dot{B}^s_{p,q}} \le \frac{C(n)}{\varepsilon} \|f\|_{\dot{W}^{s+\varepsilon}_p}$$

and after we take $p = p_1$. Since $1 \le q < p$ there exists $r \ge 1$ such that $1/q = 1/p + 1/r$. Using (3.6) and Hölder's inequality (with respect to the measure $\mathrm{d}h/|h|^n$) we find

$$\|f\|_{\dot{B}^s_{p,q}} \le \left(\int_{|h| \le 1} |h|^{-n+\varepsilon r} \right)^{1/r} \|f\|_{\dot{W}^{s+\varepsilon}_p}.$$

This gives the desired embedding. □

In order to obtain the weighted variant of this inequality, we deal with $L^p(|x|^\alpha)$ denoting with by $\|\cdot\|_{p,\alpha}$ its norm. In the same way the weighted Sobolev space $\dot{W}^1_p(|x|^\alpha)$ is the completion of $\mathscr{C}_0^\infty$ with respect to

$$\|u\|_{1,p,\alpha} := \|\nabla u\|_{p,\alpha} = \| |x|^{\alpha/p} \nabla u\|_p.$$

By real interpolation we define

$$\dot{W}^s_p(|x|^\alpha) = (L^p(|x|^\alpha), \dot{W}^1_p(|x|^\alpha))_{s,p} \quad 0 < s < 1.$$

From the general property of interpolation ([8] 1.6.2), one deduces that $\mathscr{C}_0^\infty$ is a dense subset in $\dot{W}^s_p(|x|^\alpha)$.

Now we want to represent the norms of these spaces like a sum of sequences in the classical spaces; this can be done by means of Paley–Littewood partition of unity in $\boldsymbol{R}^n$.

PROPOSITION 3.4 ([1] 6.1.17). *There exists a sequence of functions* $\{\varphi_j\}_{i\in \boldsymbol{Z}}$ *which satisfies*

(i) $\varphi_j \in \mathscr{C}_0^\infty$, $0 \le \varphi_j \le 1$;

(ii) $\operatorname{supp}\varphi_j \subset \{\xi \in \boldsymbol{R}^n | C_1 2^{-j} < |\xi| < C_2 2^{-j}\}$;

(iii) $\sum_{j\in \boldsymbol{Z}} \varphi_j(\xi) = 1$ *for* $\xi \neq 0$ *this sum contains at most two not vanishing terms*;

(iv) $\exists C > 0$ *such that* $|\nabla\varphi_j| \le \dfrac{C}{|\xi|}$ *for each* $j \in \boldsymbol{Z}$.

By the aid of this decomposition we find an equivalent norm for $L^p(|x|^\alpha)$.

LEMMA 3.2. *Let* $\{\varphi_j\}_{i\in \boldsymbol{Z}}$ *be a sequence satisfying Proposition* 3.4. *If* $u \in L^p(|x|^\alpha)$ *then*

$$\|u\|^p_{p,\alpha} \simeq \sum_{j\in \boldsymbol{Z}} 2^{-j\alpha} \|\varphi_j u\|^p_p \tag{3.9}$$

PROOF. Combining the properties (i) and (iii) for φ_j, we get $1/2^{p-1} \le \sum \varphi_j^p \le 1$. On the other hand, on the support of φ_j we have $|x|^\alpha \simeq 2^{-j\alpha}$, hence

$$\sum_{j\in \boldsymbol{Z}} 2^{-j\alpha}\|\varphi_j u\|^p_p \simeq \sum_{j\in \boldsymbol{Z}} \int_{\boldsymbol{R}^n} \varphi_j^p |x|^\alpha u^p \simeq \int_{\boldsymbol{R}^n} |x|^\alpha u^p.$$

This completes the proof. □

A similar result for $\dot{W}^1_p(|x|^\alpha)$ is obtained using of the following Hardy's inequality (cf. [5]).

PROPOSITION 3.5. *For each* $\alpha \geq 0$, $1 \leq p < n$,

$$\| |x|^{\alpha-1} f \|_p \leq \| |x|^{\alpha} \nabla f \|_p \quad \forall f \in \mathscr{C}_0^\infty(\boldsymbol{R}^n). \tag{3.10}$$

LEMMA 3.3. *Let* $1 \leq p < n$; $\{\varphi_j\}_{i \in Z}$ *satisfying Proposition* 3.4. *For any* $u \in \dot{W}_p^1(|x|^\alpha)$ *one has*

$$\|u\|_{1,p,\alpha}^p \simeq \sum_{j \in \boldsymbol{Z}} 2^{-j\alpha} \|\varphi_j u\|_{\dot{W}_p^1}^p. \tag{3.11}$$

PROOF. It suffices to prove the inequality for $\mathscr{C}_0^\infty$ functions. Having in mind the definition of $\dot{W}_p^1(|x|^\alpha)$, from Lemma 3.2 we have

$$\|u\|_{1,p,\alpha}^p := \|\nabla u\|_{p,\alpha}^p \leq C \sum_{j \in \boldsymbol{Z}} 2^{-j\alpha} \|\varphi_j \nabla u\|_p^p$$

Since $\varphi_j \nabla u = \nabla(\varphi_j u) - (\nabla \varphi_j) u$, to obtain

$$\|u\|_{1,p,\alpha}^p \leq C \sum_{j \in \boldsymbol{Z}} 2^{-j\alpha} \|\varphi_j u\|_{\dot{W}_p^1}^p$$

it suffices to find

$$\sum_{j \in \boldsymbol{Z}} 2^{-j\alpha} \|(\nabla \varphi_j) u\|_p^p \leq C \sum_{j \in \boldsymbol{Z}} 2^{-j\alpha} \|\nabla(\varphi_j u)\|_p^p.$$

The properties (iii) and (iv) of $\{\varphi_j\}$ allows us to compute

$$\|(\nabla \varphi_j) u\|_p^p \simeq \sum_{l=-1}^{1} \|(\nabla \varphi_j) \varphi_{j+l} u\|_p^p \leq C \sum_{l=-1}^{1} \left\| \frac{1}{|x|} \varphi_{j+l} u \right\|_p^p$$

Using Hardy's inequality, it follows that

$$\sum_{j \in \boldsymbol{Z}} 2^{-j\alpha} \|(\nabla \varphi_j) u\|_p^p \leq C \sum_{j \in \boldsymbol{Z}} 2^{-j\alpha} \sum_{l=-1}^{1} \|\nabla(\varphi_{j+l} u)\|_p^p \simeq \sum_{j \in \boldsymbol{Z}} 2^{-j\alpha} \|\nabla(\varphi_j u)\|_p^p.$$

Now we prove the inverse inequality. We see that

$$\sum_{j \in \boldsymbol{Z}} 2^{-j\alpha} \|\varphi_j u\|_{\dot{W}_p^1}^p \leq C \sum_{j \in \boldsymbol{Z}} 2^{-j\alpha} [\|(\nabla \varphi_j) u\|_p^p + \|\varphi_j (\nabla u)\|_p^p]$$

$$\leq C \sum_{j \in \boldsymbol{Z}} \left[\int_{\operatorname{supp} \varphi_j} |x|^\alpha |\nabla \varphi_j|^p |u|^p \, \mathrm{d}x + \int_{\operatorname{supp} \varphi_j} |x|^\alpha \varphi_j^p |\nabla u|^p \, \mathrm{d}x \right].$$

In force of the properties (ii) and (iv) of Paley–Littlewood resolution of unity, we find

$$\sum_{j\in Z} 2^{-j\alpha}\|\varphi_j u\|^p_{\dot W^1_p} \le C\left[\int_{R^n}\frac{|x|^\alpha}{|x|^p}|u|^p\,dx + \int_{R^n}|x|^\alpha|\nabla u|^p\,dx\right].$$

Using Hardy's inequality we conclude

$$\sum_{j\in Z} 2^{-j\alpha}\|\varphi_j u\|^p_{\dot W^1_p} \le C\int_{R^n}|x|^\alpha|\nabla u|^p\,dx = C\|u\|_{\dot W^1_p(|x|^\alpha)}. \qquad \square$$

Now we are in position to set the following isomorphism:

$$L^p(|x|^\alpha) \simeq l_p^{-\alpha/p}(L^p), \quad \dot W^1_p(|x|^\alpha) \simeq l_p^{-\alpha/p}(\dot W^1_p).$$

Using the property (3.3) in Example 2, we deduce the next result:

LEMMA 3.4. *Let* $1 < p < n$, $\alpha \ge 0$, $0 < s \le 1$ *and* $\{\varphi_j\}_{i\in Z}$ *satisfying Proposition* 3.4. *For any* $u \in \dot W^s_p(|x|^\alpha)$ *one has*

$$\|u\|^p_{\dot W^s_p(|x|^\alpha)} \simeq \sum_{j\in Z} 2^{-j\alpha}\|\varphi_j u\|^p_{\dot W^s_p}. \tag{3.12}$$

In this lemma an explicit relation between $\dot W^s_p(|x|^\alpha)$ and $\dot W^s_p$ appears; another important relation between these two spaces is given in the next lemma:

LEMMA 3.5. *Let* $1 < p < n$, $0 < s \le 1$, $\alpha \ge 0$. *The following inequality holds*:

$$\||x|^\alpha u\|_{\dot W^s_p} \le C\|u\|_{\dot W^s_p(|x|^{\alpha p})}. \tag{3.13}$$

PROOF. From definition of weighted L^p space we know that $\||x|^\alpha u\|_p = \|u\|_{L^p(|x|^{\alpha p})}$. On the other hand if $u \in \mathscr{C}_0^\infty$, using Hardy's inequality we get

$$\||x|^\alpha u\|_{\dot W^1_p} = \alpha\||x|^{\alpha-1}u\|_p + \||x|^\alpha\nabla u\|_p \le C\||x|^\alpha\nabla u\|_p = \|u\|_{\dot W^1_p(|x|^{\alpha p})}.$$

We see that the operator $T_\alpha : u \mapsto |x|^\alpha u$ is bounded from $L^p(|x|^{\alpha p})$ to L^p and from $\dot W^1_p(|x|^{\alpha p})$ to $\dot W^1_p$; Proposition 3.1 yields (3.13). A density argument gives the conclusion. $\square$

The same technique enable us to investigate on the behavior of cut functions in these spaces.

Lemma 3.6. *Let* $0 < s \le 1$, $1 < p < n$. *If* $\varphi \in \mathscr{C}_0^1$, $f \in \dot{B}^s_{p,q}$ *then*

$$\|\varphi f\|_{\dot{B}^s_{p,q}} \le C\|f\|_{\dot{B}^s_{p,q}}.$$

Finally we can prove the weighted version of (3.8) with a vanishing weight.

Theorem 3.1. *Let* $0 < s < 1$, $1 < p < n$, $1 < p_1, p_2 < +\infty$, $\lambda > 1$ *and* $\alpha \ge 0$. *If* $1/p = 1/p_1 + 1/p_2$ *the following inequality holds*:

$$\| |x|^\alpha |u|^{\lambda-1} u\|_{\dot{W}^s_p} \le \frac{C}{\varepsilon} \|u\|_{\dot{W}^{s+\varepsilon}_{p_1}} \| |x|^\alpha |u|^{\lambda-1}\|_{L^{p_2}} \quad 0 < \varepsilon < 1 \tag{3.14}$$

for all u *such that the norms on the right side are finite.*

Proof. From Lemma 3.5 we have

$$\| |x|^\alpha |u|^\lambda\|_{\dot{W}^s_p} \le C\| |u|^\lambda\|_{\dot{W}^s_p(|x|^{\alpha p})}. \tag{3.15}$$

On the other hand, fixed a Paley–Littlewood decomposition $\{\varphi_j\}$, Lemma 3.4 implies

$$\| |u|^\lambda\|^p_{\dot{W}^s_p(|x|^{\alpha p})} \simeq \sum_{j \in \mathbf{Z}} 2^{-j\alpha} \|\varphi_j |u|^\lambda\|^p_{\dot{W}^s_p}. \tag{3.16}$$

Since $\varphi_j |f|^\mu = \varphi_j \left|\sum_{k=-1}^1 \varphi_{j+k} f\right|^\mu$, the previous lemma yields

$$\|\varphi_j |f|^\mu\|_{\dot{W}^s_p} \le C \left\| \left| \sum_{k=-1}^1 \varphi_{j+k} f \right|^\mu \right\|_{\dot{W}^s_p}.$$

From Lemma 3.1 it follows

$$\|\varphi_j |f|^\mu\|^p_{\dot{W}^s_p} \le \frac{C}{\varepsilon^p} \sum_{k=-1}^1 \|\varphi_{j+k} f\|^p_{\dot{W}^{s+\varepsilon}_{p_1}} \| |\varphi_{j+k} f|^{\mu-1}\|^p_{L^{p_2}}.$$

Since

$$2^{-j\alpha} \| |\varphi_{j+k} f|^{\mu-1}\|^p_{L^{p_2}} \simeq \| |x|^\alpha |\varphi_{j+k} f|^{\mu-1}\|^p_{L^{p_2}},$$

we obtain

$$\sum_{j \in \mathbf{Z}} 2^{-j\alpha} \|\varphi_j |f|^\mu\|^p_{\dot{W}^s_p} \le \frac{C}{\varepsilon^p} \| |f|^{\mu-1}\|^p_{L^{p_2}(|x|^\alpha)} \sum_{j \in \mathbf{Z}} \|\varphi_j f\|^p_{\dot{W}^{s+\varepsilon}_{p_1}}.$$

Using again Lemma 3.4 with $\alpha = 0$ from (3.15) and (3.16), we get the conclusion. □

It is clear that this technique leads to other weighted estimates; for example if we choose $\{\varphi_j\}$ such that $C_1 2^{-j} \le |x - x_0| \le C_2 2^{-j}$ for all x in the support of φ_j, we check

$$\| |x - x_0|^{\alpha} |u|^{\lambda-1} u \|_{\dot{W}^s_p} \le \frac{C}{\varepsilon} \|u\|_{\dot{W}^{s+\varepsilon}_{p_1}} \| |x - x_0|^{\alpha} |u|^{\lambda-1} \|_{L^{p_2}} \quad 0 < \varepsilon < 1. \tag{3.17}$$

We conclude this section setting the local variant of Besov spaces, in which we formulate the Strichartz' estimates for the linear wave equation.

For any Ω open set in $\boldsymbol{R}^n$, $\dot{B}^s_{p,q}(\Omega)$ is the completion of $\mathscr{C}^\infty_0$ with respect to

$$\|u\|_{\dot{B}^s_{p,q}(\Omega)} = \inf\{\|v\|_{\dot{B}^s_{p,q}(\boldsymbol{R}^n)} \text{ s.t. } v_{|\Omega} = u, v \in \dot{B}^s_{p,q}(\boldsymbol{R}^n)\}.$$

These spaces on domains satisfy the same interpolation rules as the ones defined on $\boldsymbol{R}^n$. In what follows we deal with $0 < s < 1$ and $\Omega = D(t, z_0)$ for fixed $t \in \boldsymbol{R}$, $z_0 \in \boldsymbol{R}^n \times \boldsymbol{R}$. By interpolation rule we see that for any $\Phi \in \mathscr{C}^\infty_0$, $0 \le \Phi \le 1$, $\Phi = 1$ on $B_1(0)$ the function

$$\tilde{u}(x,t) = \begin{cases} u(x,t) & x \in D(t, z_0) \\ u\left(\dfrac{|t - t_0|^2 (x - x_0)}{|x - x_0|^2} + x_0, t\right) \Phi\left(\dfrac{x - x_0}{|t - t_0|}\right) & \text{otherwise} \end{cases} \tag{3.18}$$

satisfies

$$\|\tilde{u}(t)\|_{\dot{B}^s_{p,q}(\boldsymbol{R}^n)} \le C \|u\|_{\dot{B}^s_{p,q}(D(t,z_0))}. \tag{3.19}$$

In these local spaces, Strichartz' result takes the following formulation:

PROPOSITION 3.6. *Let* $(1/\bar{r}, 1/r) = ((n-1)/(2(n+1)), (n+3)/(2(n+1)))$, $f, g \in C^\infty_0(\boldsymbol{R}^n)$, $h \in L^r(\boldsymbol{R}^{n+1})$. *Let* ω *be a solution for*

$$\Box\, \omega(x,t) = h(x,t) \quad x \in \boldsymbol{R}^n, \quad n \ge 3$$

$$\omega(x,0) = f(x)$$

$$\omega_t(x,0) = g(x).$$

The following estimate holds:

$$\|\omega(x,t)\|_{L^{\tilde r}(\boldsymbol{R}^{n+1})} \le C(\|h\|_{L^r(\boldsymbol{R}^{n+1})} + \|f\|_{\dot H^{1/2}(\boldsymbol{R}^n)} + \|g\|_{\dot H^{-1/2}(\boldsymbol{R}^n)}). \tag{3.20}$$

If $z_0 = (x_0, t_0)$ *and* $s \le \tau \le t_0$, *the previous inequality gives*:

$$\|\omega\|_{L^{\tilde r}([s,\tau],\, \dot W^{1/2}_{\tilde r}(D(t,z_0)))} \le C\left(E_H(\omega, D(s,z_0))^{1/2} + \|h\|_{L^r([s,\tau],\, \dot B^{1/2}_{r,\tilde r}(D(t,z_0)))}\right) \tag{3.21}$$

where

$$E_H(\omega, D(s,z_0)) = \frac{1}{2}\int_{D(s,z_0)} |\omega_t(x,s)|^2 + |\nabla_x\omega(x,s)|^2\,\mathrm{d}x$$

To simplify the notations we put $\|\cdot\|_{L^v([s,\tau],\, \dot W^{1/2}_v(D(t,z_0)))} =: \|\cdot\|_{v,s,\tau}$.

4. Global Existence Theorem

Now we are in position to prove our main theorem:

THEOREM 4.1. *Assume that* $V(x), n, p$ *satisfy the hypothesis* (H_1) or (H_2) *given in Lemma* 2.1. *Consider* $\Phi(u) = |u|^{p-1}u$ *for large* u *a* $\mathscr{C}^2(\boldsymbol{R})$ *function.*

Let $f \in \mathscr{C}^3(\boldsymbol{R}^n) \cap H^s(\boldsymbol{R}^n)$, $g \in \mathscr{C}^2(\boldsymbol{R}^n) \cap H^{s-1}(\boldsymbol{R}^n)$, $s > n/2$. *The Cauchy Problem*

$$\square u(x,t) = -V(x)\Phi(u) \tag{4.1}$$

$$u(0,x) = f(x)$$

$$u_t(0,x) = g(x)$$

admits a unique global solution $u \in \mathscr{C}^2(\boldsymbol{R}^n \times \boldsymbol{R})$.

Moreover, if V, Φ, f, g *are* $\mathscr{C}^\infty$ *functions, then* $u \in \mathscr{C}^\infty(\boldsymbol{R}^n \times \boldsymbol{R})$.

We argue by contradiction. Let us suppose that the local solution $u(t,x)$ blows up in $T = t_0$. Our aim is to establish that for each $x_0 \in \boldsymbol{R}^n$, $u(x,t)$ belongs to $L^\infty(K_S(z_0))$ with S near t_0. Since $n \le 7$ it is sufficient to prove $u \in W^4_2(K_S(z_0))$ and then to use the embedding $W^4_2 \hookrightarrow L^\infty$.

The first step is to prove the boundedness of u in $L^{\tilde r}_{\mathrm{loc}}(\boldsymbol{R}, \dot W^{1/2}_{\tilde r})$, that is to have informations on "half derivative".

PROPOSITION 4.1. *Let* u *be a classical solution to* (4.1), (4.2) *in* $K(z_0)\backslash\{z_0\}$; *then* u *is bounded in* $L^{\tilde r}([0,t_0];\ \dot W^{1/2}_{\tilde r}(D(t,z_0)))$ *and*

$$\|u\|_{\tilde r,0,t_0} \le C(z_0, E_0). \tag{4.3}$$

PROOF. We consider the different cases (H_1) and (H_2).

(H_1) $V(x) > 0$, $3 \le n \le 7$, $(n+3)/(n-1) < p \le (n+2)/(n-2)$.

From localized Strichartz' estimate (3.21), for each $s, t \in \boldsymbol{R}$, $s \le \tau < t_0$, we have

$$\|u\|_{\bar{r},s,\tau} \le C(E_H(u, D(s, z_0)))^{1/2} + C\|V(x)|u|^{p-1}u\|_{L^r([s,\tau],\dot{B}^{1/2}_{r,\bar{r}}(D(t,z_0)))}. \tag{4.4}$$

Since $E_H(u, D(s, z_0))$ is a decreasing function in s, the first term is controlled by $E_0^{1/2}$. We want to control the last term with $\varepsilon\|u\|^{\beta}_{\bar{r},s,\tau}$ for some $\beta > 1$ and $\varepsilon > 0$.

Let $\Psi \in \mathscr{C}_0^\infty$ be a cut function $0 \le \Psi \le 1$ and $\Psi = 1$ on $D(t, z_0)$; let $\tilde{u}$ be the function defined in (3.18). Combining Lemma 3.6 and Proposition 3.3, for each $t \in [s, \tau]$ we get

$$\|V(x)|u|^{p-1}u\|_{\dot{B}^{1/2}_{r,\bar{r}}(D(t,z_0))} \le \|\Psi(x)V(x)|\tilde{u}|^{p-1}\tilde{u}\|_{\dot{B}^{1/2}_{r,\bar{r}}(\boldsymbol{R}^n)} \le C\||\tilde{u}|^{p-1}\tilde{u}\|_{\dot{B}^{1/2}_{r,\bar{r}}(\boldsymbol{R}^n)}$$

$$\le C\|\tilde{u}\|_{\dot{B}^{1/2}_{\bar{r},\bar{r}}(\boldsymbol{R}^n)}\||\tilde{u}|^{p-1}\|_{L^{p_1}(\boldsymbol{R}^n)} \le C\|u\|_{\dot{B}^{1/2}_{\bar{r},\bar{r}}(D(t,z_0))}\||u|^{p-1}\|_{L^{p_1}(D(t,z_0))}$$

with

$$\frac{1}{p_1} = \frac{1}{r} - \frac{1}{\bar{r}} = \frac{2}{n+1}. \tag{4.5}$$

After integration in t we find:

$$\|V(x)|u|^{p-1}u\|_{L^r([s,\tau],\dot{B}^{1/2}_{r,\bar{r}}(D(t,z_0)))} \le C(s,\tau)\|u\|_{\bar{r},s,\tau}\|u\|^{p-1}_{L^{p_1(p-1)}(K_s^\tau(z_0))}.$$

We observe that $C(s, \tau) = (\tau - s)^\beta$ with $\beta < 1$; it follows that there exists a constant $C(t_0) \ge C(\tau, s)$.

Now we want to estimate the factor $\|u\|^{p-1}_{L^{p_1(p-1)}(K_s^\tau(z_0))}$. If there exists $0 < \alpha < 1$ such that

$$\frac{1}{p_1(p-1)} = \alpha\left(\frac{1}{\bar{r}} - \frac{1}{2n}\right) + (1-\alpha)\frac{1}{p+1}, \tag{4.6}$$

then using interpolation rule and the embedding (3.4) we obtain

$$\|u\|_{p_1(p-1)} \le C\|u\|^{\alpha}_{(1/\bar{r}-1/2n)^{-1}}\|u\|^{1-\alpha}_{p+1} \le \|u\|^{\alpha}_{\dot{B}^{1/2}_{\bar{r},\bar{r}}}\|u\|^{1-\alpha}_{p+1}.$$

The last inequality gives

$$\|u\|^{p-1}_{L^{p_1(p-1)}(K^\tau_s(z_0))} \le C\left(\int_s^\tau \|u\|^{\alpha p_1(p-1)}_{\dot B^{1/2}_{\bar r,\bar r}(D(t,z_0))}\|u\|^{(1-\alpha)p_1(p-1)}_{L^{p+1}(D(t,z_0))}\,\mathrm{d}t\right)^{1/p_1}$$

$$\le C\sup_{s\le t\le\tau}\|u\|^{(1-\alpha)(p-1)}_{L^{p+1}(D(t,z_0))}\left(\int_s^\tau \|u\|^{\alpha p_1(p-1)}_{\dot B^{1/2}_{\bar r,\bar r}(D(t,z_0))}\,\mathrm{d}t\right)^{1/p_1}.$$

If $\alpha p_1(p-1)\le\bar r$, then by Hölder's inequality we get

$$\left(\int_s^\tau \|u\|^{\alpha p_1(p-1)}_{\dot B^{1/2}_{\bar r,\bar r}}\,\mathrm{d}t\right)^{1/p_1} \le C(s,\tau)\|u\|^{\alpha(p-1)}_{\bar r,s,\tau},$$

again we can take $C(s,\tau)\le C(t_0)$. Hence we find

$$\|u\|_{\bar r,s,\tau}\le CE_0^{1/2}+C\|u\|^{1+\alpha(p-1)}_{\bar r,s,\tau}\sup_{s\le t\le\tau}\|u\|^{(1-\alpha)(p-1)}_{L^{p+1}(D(t,z_0))}.$$

Now Corollary 2.1 implies

$$\sup_{s\le t\le\tau}\|u\|^{(1-\alpha)(p-1)}_{L^{p+1}(D(t,z_0))}<\varepsilon \quad \text{if } t\to t_0 \quad \forall\varepsilon>0.$$

For s close to t_0, we conclude

$$\|u\|_{\bar r,s,\tau}\le CE_0^{1/2}+\varepsilon\|u\|^{1+\alpha(p-1)}_{\bar r,s,\tau}. \tag{4.7}$$

Considering $\gamma=1+\alpha(p-1)$ and $X(s,\tau)=\|u\|_{\bar r,s,\tau}$, we have a continuous function such that

$$X(s,s)=0$$

$$\tau\mapsto X(s,\tau)\quad\text{is increasing for all } s\in\boldsymbol{R}$$

$$\varepsilon X^\gamma(s,\tau)-X(s,\tau)+CE_0^{1/2}\ge 0\quad\forall\varepsilon>0.$$

We choose $\varepsilon<K_\gamma^{-\gamma}(CE_0^{1/2})^{1-\gamma}$ where $K_\gamma=\min_{x>0}\{x^{\gamma-1}+x^{-1}+1\}$; this implies that $X(s,t)$ is bounded, i.e. $\|u\|_{\bar r,s,\tau}$ is bounded. Having in mind that the local solution is regular in $K_0^s(z_0)$, when $\tau\to t_0$ we have the conclusion.

It remains to prove that the exponent α in the previous computation exists, that is the following system of inequalities has a solution:

$$\begin{cases} 0<\alpha<1\\ \dfrac{1}{p_1(p-1)}=\alpha\left(\dfrac{1}{\bar r}-\dfrac{1}{2n}\right)+(1-\alpha)\dfrac{1}{p+1}\\ \alpha p_1(p-1)\le\bar r\end{cases}.$$

Having in mind the definition of p and $\bar r$, the last inequality can be written as

$$\alpha \le \frac{4}{(n-1)(p-1)}. \tag{4.8}$$

On the other hand, the condition $p > (n+3)/(n-1)$ implies $4/((n-1)(p-1)) < 1$; then the previous system is equivalent to

$$\begin{cases} 0 < \alpha \le \dfrac{4}{(n-1)(p-1)} \\ \dfrac{1}{p_1(p-1)} = \alpha\left(\dfrac{1}{\bar{r}} - \dfrac{1}{2n}\right) + (1-\alpha)\dfrac{1}{p+1} \end{cases}.$$

From the last equation we get

$$\alpha(p) = \frac{2n}{p-1}\,\frac{p(n-1)-(n+3)}{p(2n+1-n^2)+n^2+4n+1}.$$

It remains to verify

$$0 < \alpha(p) \le \frac{4}{(n-1)(p-1)}.$$

Using again the assumption $p > (n+3)/(n-1)$ we get $\alpha(p) > 0$ if and only if

$$p(2n+1-n^2)+n^2+4n+1 > 0. \tag{4.9}$$

Since $n \ge 3$ the quantity $(2n+1-n^2)$ is negative, so that we require

$$p < \frac{n^2+4n+1}{n^2-2n-1}.$$

A simple computation shows that in the case $n \ge 3$

$$\frac{n+2}{n-2} < \frac{n^2+4n+1}{n^2-2n-1}.$$

Being p subcritical or critical this implies (4.9). The same argument reduces the condition (4.8) to the discussion of the inequality

$$(n+2)[n(n-1)^2 - 2(2n+1-n^2)] \le (n-2)[n(n+3)(n-1) + 2(n^2+4n+1)]$$

and this is fulfilled.

Now we consider the case

(H_2) $V(x) = |x-x_0|^2$ with $x_0 \in \boldsymbol{R}^n$, $n=3$, $p=5$.

We fix our attention to the point $z_0 = (x_0, t_0)$ in which the potential vanishes, since otherwise there exists $s \in \boldsymbol{R}$ such that $V(x) > 0$ in $D(t, z_0)$ for all $t \in [s, t_0[$

and the assertion follows from the case (H_1). Here we use a different estimate for the last term in (4.4), in fact we have to apply Lemma 2.1 instead of its Corollary 2.1.

Combining the embedding (3.5) and the weighted estimate (3.17), we find

$$\| |x-x_0|^2 u^5 \|_{\dot{B}^{1/2}_{4/3,4}(D(t,z_0))} \leq \| |x-x_0|^2 \tilde{u}^5 \|_{\dot{W}^{1/2}_{4/3}(\boldsymbol{R}^3)}$$

$$\leq \frac{C}{\varepsilon} \|\tilde{u}\|_{\dot{W}^{1/2+\varepsilon}_4(\boldsymbol{R}^3)} \| |x-x_0|^2 \tilde{u}^4 \|_{L^2(\boldsymbol{R}^3)} \leq \frac{C}{\varepsilon} \|u\|_{\dot{W}^{1/2+\varepsilon}_4(D(t,z_0))} \|\tilde{u}\|^4_{L^8(\boldsymbol{R}^3, |x-x_0|^4)}.$$

The interpolation rule (3.2) for weighted L^p spaces gives

$$\|\tilde{u}\|_{L^8(\boldsymbol{R}^3, |x-x_0|^{4/3})} \leq \|\tilde{u}\|^{1/2}_{L^{12}(\boldsymbol{R}^3)} \|\tilde{u}\|^{1/2}_{L^6(\boldsymbol{R}^3, |x-x_0|^2)}.$$

Using the embedding (3.4), we conclude

$$\| |x-x_0|^2 u^5 \|_{\dot{B}^{1/2}_{4/3,4}(D(t,z_0))} \leq \frac{C}{\varepsilon} \|u\|^3_{\dot{W}^{1/2+\varepsilon}_4(D(t,z_0))} \| |x-x_0|^{1/3} u \|^2_{L^6(D(t,z_0))}.$$

After integration in t we get

$$\| |x-\bar{x}|^{1/6} u \|^4_{L^8(K^\tau_s(\bar{z}))} \leq \|u\|^2_{L^4([s,\tau]; \dot{W}^{1/2+\varepsilon} D(t,\bar{z}))} \sup_{s \leq t \leq \tau} \| |x-\bar{x}|^{1/3} u \|^2_{L^6(D(t,\bar{z}))}$$

$$\| |x-\bar{x}|^2 u^5 \|_{L^{4/3}([s,\tau]; \dot{B}^{1/2}_{4/3,4})} \leq C\varepsilon^{-1} \|u\|^3_{L^4([s,\tau]; \dot{W}^{1/2+\varepsilon} D(t,\bar{z}))} \sup_{s \leq t \leq \tau} \| |x-\bar{x}|^{1/3} u \|^2_{L^6(D(t,\bar{z}))}.$$

We are in position to apply Lemma 2.1, and for s close to t_0 we find

$$\sup_{s \leq t \leq \tau} \| |x-x_0|^{1/3} u \|_{L^6(D(t,z_0))} \leq C\varepsilon^2.$$

Denoting

$$X(s,\tau,\varepsilon) = \|u\|_{L^4([s,\tau]; \dot{W}^{1/2+\varepsilon} D(t,\bar{z}))},$$

we have a continuous function such that

$$X(s,s,\varepsilon) = 0 \quad \text{for all } \varepsilon > 0$$

$$X(s,\tau,\varepsilon) \quad \text{is increasing in } \varepsilon \text{ and } \tau$$

$$\varepsilon X^3(s,\tau,\varepsilon) - X(s,\tau,0) + CE_0^{1/2} \geq 0.$$

Using the fact that $X(s,\tau,\varepsilon) \to X(s,\tau,0)$ when $\varepsilon \to 0$, we see that there exists a suitable $\varepsilon_0 > 0$ such that for all $\varepsilon < \varepsilon_0$ the last condition gives

$$1 \le \frac{1 + CE_0^{1/2}}{1 + X(s,\tau,0)} + \varepsilon(1 + X(s,\tau,0))^2.$$

Taking $\varepsilon < 3^{-3}(CE_0^{1/2} + 1)^{-2}$ we get $X(s,\tau,0)$ is bounded, that is our conclusion. □

As a consequence of the embedding $\dot{W}_{\bar{r}}^{1/2}(D(t,z_0)) \hookrightarrow L^{\bar{r}}(D(t,z_0))$, using (4.3) we get

$$u \in L^{\bar{r}}(K_0(z_0)). \tag{4.10}$$

Coming back to the proof, from (4.3) we also deduce that

$$\|u\|_{L^{p_1(p-1)}(K_s^\tau(z_0))} \le Co(1) \quad \text{if } H_1 \text{ holds} \tag{4.11}$$

$$\| |x - x_0|^{1/6} u\|_{L^8(K_s^\tau(z_0))} \le Co(1) \quad \text{if } H_2 \text{ holds}, \tag{4.12}$$

where $o(1)$ tends to zero when $\tau \to t_0$.

The next step is to establish the $\bar{r}$-summability for Du. In what follows we put $D = \partial_t$ or $D = \partial_{x_i}$ for $i = 1, \dots, n$.

PROPOSITION 4.2. *Let u be a classical solution for* (4.1), (4.2) *on* $K(z_0)\backslash\{z_0\}$. *One has* $Du \in L^{\bar{r}}(K_0(z_0))$.

PROOF. By differentiating (4.1) it follows

$$\Box Du = -DV(x)|u|^{p-1}u - pV(x)|u|^{p-1}Du.$$

Having in mind the finite speed of propagation, for any $s \le \tau < t_0$, Strichartz' estimate (3.20), gives

$$\|Du\|_{L^{\bar{r}}(K_s^\tau(z_0))} \le C(E(Du, D(s,z_0)))^{1/2}$$
$$+ \|DV(x)|u|^{p-1}u\|_{L^r(K_s^\tau(z_0))} + C\|V(x)|u|^{p-1}Du\|_{L^r(K_s^\tau(z_0))}.$$

Suppose (H_1) holds.

From (4.5), using the relations (4.10) and (4.11), we have:

$$\|DV(x)|u|^{p-1}u\|_{L^r(K_s^\tau(z_0))} \le \|DV\|_{L^\infty(K_s^\tau(z_0))}\|u\|_{L^{\bar{r}}(K_s^\tau(z_0))}\|u\|^{p-1}_{L^{p_1(p-1)}(K_s^\tau(z_0))} \le Co(1),$$

$$\|V(x)|u|^{p-1}Du\|_{L^r(K_s^\tau(z_0))} \le \|V\|_{L^\infty(K_s^\tau(z_0))}\|u\|^{p-1}_{L^{p_1(p-1)}(K_s^\tau(z_0))}\|Du\|_{L^{\bar{r}}(K_0(z_0))}$$
$$\le Co(1)\|Du\|_{L^{\bar{r}}(K_0(z_0))}.$$

Summing these inequalities, we find:

$$\|Du\|_{L^{\dot{r}}(K_s^\tau(z_0))} \leq CE_0^{1/2} + Co(1)(1 + \|Du\|_{L^{\dot{r}}(K_s^\tau(z_0))}). \tag{4.13}$$

Being the local solution regular in $K_0^s(z_0)$, for $\tau \to t_0$ we get the conclusion. The situation is slightly different in the case in which (H_2) holds.

Since we deal with bounded domain, if $p \leq q$ then

$$\| |x - x_0|^{\alpha/p} u\|_{L^p(D(t,z_0))} \leq \| |x - x_0|^{\alpha/q} u\|_{L^q(D(t,z_0))}.$$

The same inequality holds when we consider $K_s^\tau(z_0)$ instead of $D(t, z_0)$. This fact enables us to estimate $\|DV(x)|u|^{p-1}u\|_{L^r(K_s^\tau(z_0))} \simeq \| |x - x_0| u^5\|_{L^{4/3}(K_s^\tau(z_0))}$. Since $9/2 \leq 6$ we have

$$\sup_{s \leq t \leq \tau} \int_{D(t,z_0)} |x - x_0|^2 |u|^{9/2} \, \mathrm{d}x = o(1).$$

Using this condition and the interpolation rule

$$L^7(|x - x_0|^{4/3}) = (L^{12}, L^{9/2}(|x - x_0|^2))_{4/7,7},$$

we obtain

$$\int_s^\tau \int_{D(t,z_0)} |x - x_0|^{4/3} u^7 \, \mathrm{d}x\mathrm{d}t = o(1)\|u\|_{4,s,\tau}^4.$$

From (4.3) we have that this quantity is bounded. Being $20/3 \leq 7$ we conclude

$$\| |x - x_0| u^5\|_{L^{4/3}(K_s^\tau(z_0))} = o(1).$$

In order to control $\|V(x)|u|^{p-1}Du\|_{L^r(K_s^\tau(z_0))}$ we can use (4.12):

$$\begin{aligned} \| |x - x_0|^2 u^4 Du\|_{L^{4/3}(K_s^\tau(z_0))} &\leq C\| |x - x_0|^{1/6} u\|_{L^8(K_s^\tau(z_0))}^4 \|Du\|_{L^4(K_0(z_0))} \\ &\leq o(1)\|Du\|_{L^4(K_0(z_0))}. \end{aligned}$$

These relations and Strichartz' estimate imply (4.13), hence the conclusion. □

COROLLARY 4.1. *Let u be a classical solution for* (4.1), (4.2) *in* $K(z_0)\backslash\{z_0\}$; *then*

$$\lim_{s \to t_0} E(u, D(s, z_0)) = 0. \tag{4.14}$$

PROOF. Being $E(u, D(s, z_0))$ decreasing in s, we have $\lim_{s\to t_0} E(u, D(s, z_0)) = l$. In force of (2.8), to obtain $l = 0$ it suffices that $\int_{D(s,z_0)} |Du|^2 \, dx \to 0$. Combining

Hölder's inequality and the previous proposition we have:

$$\|Du\|^2_{L^2(K_s^\tau(z_0))} \le [meas_{n+1}(K_s^\tau(z_0))]^{2/(n+1)}\|Du\|^2_{L^{\tilde r}(K_s^\tau(z_0))} \le C(t_0-s)^2.$$

This completes the proof. □

In order to conclude the proof of the main Theorem we need $D^2u \in L^{\tilde r}(K_0(z_0))$. We use the same trick: by differentiating two times (4.1) we get:

$$\square D^2u = -D^2V(x)|u|^{p-1}u - 2pDV(x)|u|^{p-1}Du +$$
$$-pV(x)|u|^{p-1}D^2u - p(p-1)V(x)\Phi''(u)(Du)^2.$$

Here $\Phi''(u) \simeq |u|^{p-2}$ if $n \le 5$, while $\Phi''(u)$ is bounded if $n = 6, 7$. Localized Strichartz' estimates give

$$\|D^2u\|_{L^{\tilde r}(K_s^\tau(z_0))} \le C(E(Du, D(s, z_0)))^{1/2}$$
$$+ C\|D^2V(x)|u|^{p-1}u\|_{L^r(K_s^\tau(z_0))} \tag{4.15}$$
$$+ C\|DV(x)|u|^{p-1}Du\|_{L^r(K_s^\tau(z_0))}$$
$$+ C\|V(x)|u|^{p-1}D^2u\|_{L^r(K_s^\tau(z_0))} \tag{4.16}$$
$$+ C\|V(x)\Phi''(u)(Du)^2\|_{L^r(K_s^\tau(z_0))}. \tag{4.17}$$

Suppose (H_1) holds.

We can estimate (4.15), (4.16) obtaining

$$\|D^2V|u|^p\|_{L^r(K_s^\tau(z_0))} + \|DV|u|^{p-1}Du\|_{L^r(K_s^\tau(z_0))} + \|V|u|^{p-1}D^2u\|_{L^r(K_s^\tau(z_0))}$$
$$\le C\|u\|^{p-1}_{L^{p_1(p-1)}(K_s^\tau(z_0))}(\|u\|_{L^{\tilde r}(K_s^\tau(z_0))} + \|Du\|_{L^{\tilde r}(K_s^\tau(z_0))}) + \|D^2u\|_{L^{\tilde r}(K_s^\tau(z_0))})$$
$$\le Co(1)(2 + \|D^2u\|_{L^{\tilde r}(K_s^\tau(z_0))});$$

in the last inequality we used (4.10), (4.11) and the previous proposition.

We divide the estimate for (4.17) in two cases: $n \le 5$ and $n = 6, 7$.

If $n \le 5$, by the aid of Hölder's inequality and Sobolev embedding, $W^1_{\tilde r}(K_s^\tau(\bar z)) \hookrightarrow L^\alpha(K_s^\tau(\bar z))$ with $\alpha = 2(n+1)/(n-3)$, we find:

$$\|V(x)|u|^{p-2}(Du)^2\|_{L^r(K_s^\tau(z_0))}\| \le C\|u\|^{p-2}_{L^{p+1}(K_s^\tau(z_0))}\|Du\|_{L^{\tilde r}(K_s^\tau(z_0))}\|D^2u\|_{L^{\tilde r}(K_s^\tau(z_0))}.$$

Using Corollary 2.1 and the previous proposition, we conclude that

$$\|D^2u\|_{L^{\tilde r}(K_s^\tau(z_0))} \le C + Co(1)\|D^2u\|_{L^{\tilde r}(K_s^\tau(z_0))}.$$

If $n = 6, 7$, we use the interpolation rule $L^{2r} = (L^\alpha, L^{\bar{r}})_{(n-5)/4}$, $\alpha = 2(n+1)/(n-3)$ and the embedding $W_{\bar{r}}^1(K_s^\tau(z_0)) \hookrightarrow L^\alpha(K_s^\tau(z_0))$ holds; we have

$$\|V(x)\Phi''(u)(Du)^2\|_{L^r(K_s^\tau(z_0))} \leq C\|(Du)^2\|_{L^r(K_s^\tau(z_0))} \leq C\|Du\|_{L^{\bar{r}}(K_s^\tau(z_0))}^{(9-n)/2}\|D^2u\|_{L^{\bar{r}}(K_s^\tau(z_0))}^{(n-5)/2}.$$

It follows:

$$\|D^2u\|_{L^{\bar{r}}(K_s^\tau(z_0))} \leq C + Co(1)\|D^2u\|_{L^{\bar{r}}(K_s^\tau(z_0))} + C\|D^2u\|_{L^{\bar{r}}K_s^\tau(z_0))}^{(n-5)/2}.$$

Since $(n-5)/2 \leq 1$, in both cases we conclude

$$D^2u \in L^{\bar{r}}(K_0(z_0)). \tag{4.18}$$

Suppose (H_2) holds.

The estimate for (4.15) is based on the interpolation rule $\dot{W}_p^{1/2} = (L^p, \dot{W}_p^1)_{1/2,p}$ and on embedding $\dot{W}_4^{1/2}(K_s^\tau(z_0)) \hookrightarrow L^{20/3}(K_s^\tau(z_0))$:

$$\|u^5\|_{L^{4/3}(K_s^\tau(z_0))} = \|u\|_{L^{20/3}(K_s^\tau(z_0))}^5 \leq \|u\|_{\dot{W}_4^{1/2}(K_s^\tau(z_0))}^5 \leq \|u\|_{L^4(K_s^\tau(z_0))}^{5/2}\|u\|_{\dot{W}_4^1(K_s^\tau(z_0))}^{5/2}.$$

From Proposition 4.2 and from (4.10) we get

$$\|u^5\|_{L^{4/3}(K_s^\tau(z_0))} \leq C(t_0).$$

Now we control the terms in (4.16):

$$\begin{aligned}
&\||x-x_0|u^4Du\|_{L^{4/3}(K_s^\tau(z_0))} + \||x-x_0|^2u^4D^2u\|_{L^{4/3}(K_s^\tau(z_0))}\\
&\quad \leq C(t_0)\||x-x_0|u^4\|_{L^2(K_s^\tau(z_0))}(\|Du\|_{L^4(K_s^\tau(z_0))} + \|D^2u\|_{L^4(K_s^\tau(z_0))})\\
&\quad \leq C(t_0)\||x-x_0|^{1/6}u\|_{L^8(K_s^\tau(z_0))}(\|Du\|_{L^4(K_s^\tau(z_0))} + \|D^2u\|_{L^4(K_s^\tau(z_0))}).
\end{aligned}$$

Using (4.12) we find:

$$\begin{aligned}
&\||x-x_0|u^4Du\|_{L^{4/3}(K_s^\tau(z_0))} + \||x-x_0|^2u^4D^2u\|_{L^4(K_s^\tau(z_0))}\\
&\quad \leq o(1)(\|Du\|_{L^4(K_s^\tau(z_0))} + \|D^2u\|_{L^4(K_s^\tau(z_0))}).
\end{aligned}$$

Finally we deal with (4.17):

$$\||x-x_0|^2u^3(Du)^2\|_{L^{4/3}(K_s^\tau(z_0))} \leq \||x-x_0|^2u^3\|_{L^2(K_s^\tau(z_0))}\|Du\|_{L^4(K_s^\tau(z_0))}\|D^2u\|_{L^4(K_s^\tau(z_0))}.$$

From Lemma 2.1 we obtain

$$\| |x - x_0|^2 u^3 (Du)^2 \|_{L^{4/3}(K_s^\tau(z_0))} \le o(1) \|Du\|_{L^4(K_s^\tau(z_0))} \|D^2 u\|_{L^4(K_s^\tau(z_0))}.$$

The sum of these estimates leads again to (4.18).

As a consequence of the hypotheses on $\Phi(u)$ and on p, we have

$$u \in W_r^2(K_0(z_0)) \Rightarrow |u|^{p-1} u \in W_2^2(K_0(z_0));$$

moreover being the domains bounded, we obtain the crucial information:

$$V(x)|u|^{p-1} u \in W_2^2(K_0(z_0)). \tag{4.19}$$

By energy estimate and Gronwall's lemma, the previous relation in turn implies $D^\alpha u \in L^\infty([0, t_0], L^2(D(t, z_0))$ with $|\alpha| \le 4$. More precisely we need this information for the cases $n = 6, 7$; instead if $n = 4, 5$ we have to consider only $|\alpha| \le 3$ and if $n = 3$ it suffices $|\alpha| = 2$.

The conclusion is now exactly like in [7], [9]; we repeat it to have a self-contained proof. If the local solution blows up at time $t_0 > 0$, then there exists $x_0 \in \boldsymbol{R}^n$ and $(x_n, t_n) \to (x_0, t_0)$ such that $\lim_n |u(x_n, t_n)| = +\infty$. We fix $\varepsilon > 0$ and from Corollary 4.1 we know that $E(u, D(s, z_0)) < \varepsilon$ if $\bar{S} \le s < t_0$. Since the local solution is $\mathscr{C}^2$, we can extend this inequality: there exists $\delta > 0$ such that

$$\int_{|x - x_0| \le t_0 - \bar{S} + \delta} \frac{1}{2} |u_t|^2 + \frac{1}{2} |\nabla_x u|^2 + \frac{1}{p+1} V(x) |u|^{p+1} \, dx < \varepsilon.$$

This means $E(u, D(s, \bar{z})) < \varepsilon$ where $\bar{z} = (x_0, t_0 + \delta)$; in particular in the truncate section $K_0^{t_0}(\bar{z})$, (4.3) still holds. The above argument shows that $u \in W_2^4(K_0^{t_0}(\bar{z}))$ and then $u \in L^\infty(K_0^{t_0}(\bar{z}))$.

Since $K_0(z_0) \subset K_0^{t_0}(\bar{z})$ it is impossible that the solution blows up in z_0.

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