TSUKUBA J. MATH.
Vol. 26 No. 1 (2002), 15–37

THE ALGORITHM TO CALCULATE THE PERIOD MATRIX OF THE CURVE $x^m + y^n = 1$

By

Yasuo KAMATA

Abstract. We show how to take a canonical homology basis and a basis of the space of holomorphic 1-forms on the curve $x^m + y^n = 1$, and we show how to calculate its explicit period matrix.

1. Introduction

Let R be a compact Riemann surface and g be the genus of R. Let $\{\omega_j\}$, $(j = 1, 2, \dots, g)$ be a basis of the space of holomorphic 1-forms of R and $\{\alpha_k, \beta_k\}$, $(k = 1, 2, \dots, g)$ be a basis of the first homology group of R over $\boldsymbol{Z}$, the ring of integers. Suppose that the intersection numbers of the closed paths α_k's and β_k's satisfy

$$I(\alpha_j, \beta_k) = -I(\beta_k, \alpha_j) = \delta_{jk}, \quad I(\alpha_j, \alpha_k) = I(\beta_j, \beta_k) = 0.$$

Such a basis $\{\alpha_k, \beta_k\}$, $(k = 1, 2, \dots, g)$ is called a canonical homology basis. In other words, a canonical homology basis is a set of closed paths whose intersection matrix is

$$J = \begin{pmatrix} O & I \\ -I & O \end{pmatrix},$$

where O and I are the $g \times g$ zero and identity matrices.

We denote by Ω the period matrix (or Riemann matrix) of R with respect to $\{\omega_j\}$ and $\{\alpha_k, \beta_k\}$, that is,

$$\Omega = \begin{pmatrix} \int_{\alpha_1} \omega_1 & \cdots & \int_{\alpha_g} \omega_1 & \int_{\beta_1} \omega_1 & \cdots & \int_{\beta_g} \omega_1 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \int_{\alpha_1} \omega_g & \cdots & \int_{\alpha_g} \omega_g & \int_{\beta_1} \omega_g & \cdots & \int_{\beta_g} \omega_g \end{pmatrix}.$$

Received April 3, 2000.
Revised October 31, 2001.

We write this matrix as (A, B) by two $g \times g$ matrices. Multiplying Ω by A^{-1}, we obtain the normalized period matrix $(I, T) = (I, A^{-1}B)$. The matrix T, called the theta matrix of R with respect to $\{\omega_j\}$ and $\{\alpha_k, \beta_k\}$, is non-singular symmetric and has positive definite imaginary part.

Let L be the lattice of $\boldsymbol{C}^g$ generated by the column vectors of (I, T). Then the torus $J(R) = \boldsymbol{C}^g/L$ is an algebraic variety called the Jacobian variety of R. One of the advantages of computing T lies in the theorem of Torelli which states that given Riemann surfaces R_1 and R_2 are isomorphic if and only if their Jacobian varieties $J(R_1)$ and $J(R_2)$ are isomorphic.

In general it is not so easy to give explicitly a canonical homology basis on R. Recently the combinatorial group theoretic method for general curves is presented by C. L. Tretkoff and M. D. Tretkoff [2]. Unfortunately the method needs to draw complicated figures and requires some knowledge of graph theory. If R is a hyperelliptic curve, a general method to compute its normalized period matrix is well-known (cf. [1], pp. 256–259).

If R is Fermat curve, only its period lattice can be known by David E. Rohrlich ([5], p. 79). But the representation is not so simple, and the method can not present the intersection matrix of the homology basis on R.

In the usual definition of the theta function attached to R, the period matrix is normalized. Our method can present not only the period lattice but also the intersection matrix and, hence, the normalized period matrix. Our method is an extension of the method for the case of hyperelliptic curves.

Let R be the curve $y^n = 1 - x^m$, $(n, m \in \boldsymbol{N})$, and g (≥ 1) be the genus of R. If we regard R as a n-ply branched covering of $\hat{\boldsymbol{C}} = \boldsymbol{C} \cup \{\infty\}$ and apply the Riemann-Hurwitz formula, we have

$$g = \{(n-1)(m-1) + 1 - d\}/2, \quad \text{where } d \text{ is the G.C.D. of } m \text{ and } n.$$

Let $\hat{R}$ be the projective imbedding of R into $\boldsymbol{P}^2$, the complex projective plane. That is, $\hat{R} = \{[X, Y, Z] \in \boldsymbol{P}^2 \mid Y^n = Z^n - X^m Z^{n-m}\}$. The points at infinity $(x = \infty)$ of R correspond to the set $E := \{[X, Y, 0] \in \boldsymbol{P}^2\} \cap \hat{R}$. We classify R into the following three cases:

(i) $n = m$ $(E = \{[1, \rho^j, 0] \in \boldsymbol{P}^2 \mid \rho = \exp(\pi i/n), j = 1, 2, \ldots, n\}.)$,

(ii) $n \geq m + 1$ and $d = 1$ $(E = \{[1, 0, 0] \in \boldsymbol{P}^2\}$. The point $[1, 0, 0]$ is a branch point of order n.),

(iii) $n > m + 1$ and $d > 1$ $(E = \{[1, 0, 0] \in \boldsymbol{P}^2\}$. The point $[1, 0, 0]$ is a branch point of order n/d.).

I would like to thank professor Kimio Watanabe for his instruction and helpful advice, and the referee for his careful reading and detailed suggestions.

2. Topological Model and Homology Basis

We regard R as a Riemann surface of n-ply branched covering of the Riemann sphere $\hat{\boldsymbol{C}}$. Let $X_1, X_2, \dots, X_n$ be these $\hat{\boldsymbol{C}}$'s. η^j $(j = 1, 2, \dots, m)$ are the branch points of R, where η is one of the primitive m-th roots of unity.

Since the ramification indices of these branch points are equal to n, there is no ambiguity to choose the branch points on $X_1, \dots, X_n$. Therefore any round trip path on R between two of these branch points is a closed path and belongs to $H_1(R, \boldsymbol{Z})$, the first homology group. There exist $\dim H_1(R, \boldsymbol{Z}) = 2g$ closed paths which are linearly independent over $\boldsymbol{Z}$.

It is clear that any closed path on R is homologous to the union of the round trip paths between two branch points on R. We cut open the lines between η^j and η^{j+1}, $(j = 1, 2, \dots, m-1)$ on $X_1, X_2, \dots, X_n$.

Now we consider the minimal loop that is the union of the branch cuts $[\eta^j, \eta^{j+1}]_k$ from η^j to η^{j+1} on X_k and $[\eta^{j+1}, \eta^j]_{k+1}$ from η^{j+1} to η^j on X_{k+1}. We denote such loops by

$$\alpha_{j,k} = [\eta^j, \eta^{j+1}]_k + [\eta^{j+1}, \eta^j]_{k+1}$$

for $j = 1, 2, \dots, m-1$ and $k = 1, 2, \dots, n$, where $X_{n+1} = X_1$.

Case (i). In this case, all closed paths on R are generated by the $\alpha_{j,k}$'s. Since the cut lines $[\eta, \eta^2]_1, [\eta^2, \eta^3]_1, \dots, [\eta^{m-1}, 1]_1$ belong to X_1, each of the other bank of these branch cuts belongs to another X_k. We denote the path of the opposite bank of the path $[\eta^j, \eta^{j+1}]_k$ by $[\eta^j, \eta^{j+1}]_{k'}$, $X_{k'} \in \{X_1, \dots, X_n\}$. Here $X_{k'}$ is selected by the analytic continuation of y, n-ply valued function of $x \in \hat{\boldsymbol{C}}$.

Since the union of the loops $\alpha_{1,1}, \alpha_{1,2}, \dots, \alpha_{1,n}$ is homologous to zero on R, any one of them is not required to obtain a homology basis. So we discard the loop $\alpha_{1,n}$. Because of the same reason we discard the loops $\alpha_{2,n}, \alpha_{3,n}, \dots, \alpha_{m-1,n}$.

Now consider the following path on R:

$$l : [\eta, \eta^2]_1 + [\eta^2, \eta^3]_1 + \cdots + [\eta^{m-1}, 1]_1 + [1, \eta^{m-1}]_{1'} + \cdots + [\eta^3, \eta^2]_{1'} + [\eta^2, \eta]_{1'}.$$

The path l is obviously closed and homologous to zero on X_1. Since l is a sum of the loops

$$[\eta^j, \eta^{j+1}]_1 + [\eta^{j+1}, \eta^j]_{1'}, \quad j = 1, 2, \dots, m-1,$$

in order to get independent loops, at least one of them has to be discarded. According to the same argument for $X_2, X_3, \dots, X_n$, we discard the loops

$$[\eta^{m-1}, 1]_k + [1, \eta^{m-1}]_{k'}, \quad k = 1, 2, \dots, n.$$

In other words, by the rearrangement of the branch cuts, we discard the minimal loops

$$\alpha_{m-1,k} = [\eta^{m-1}, 1]_k + [1, \eta^{m-1}]_{k+1}, \quad k = 1, 2, \dots, n.$$

Now the remaining minimal loops are $\alpha_{j,k}$, $(j = 1, 2, \dots, m-2$ and $k = 1, 2, \dots, n-1)$. The number of them is $(n-1)(m-2)$. Since $(n-1)\cdot(m-2) = 2g = \dim H_1(R, \boldsymbol{Z})$, these loops must be independent over $\boldsymbol{Z}$. Therefore they form a basis of $H_1(R, \boldsymbol{Z})$.

Case (ii). If once we add the point at $x = \infty$ to the set of the branch points, we can obtain a homology basis in the same way as in the case (i). From the minimal loops

$$\alpha_{j,k} = [\eta^j, \eta^{j+1}]_k + [\eta^{j+1}, \eta^j]_{k+1}, \quad \alpha_{m,k} = [1, \infty]_k + [\infty, 1]_{k+1}$$

$$j = 1, 2, \dots, m-1 \text{ and } k = 1, 2, \dots, n, \text{ where } X_{n+1} = X_1,$$

we discard the loops

$$\alpha_{1,n}, \alpha_{2,n}, \dots, \alpha_{m,n}, \alpha_{m,1}, \alpha_{m,2}, \dots, \alpha_{m,n-1}$$

to eliminate the linear dependence over $\boldsymbol{Z}$. Then the number of the remaining minimal loops is $mn - \{m + (n-1)\} = (n-1)(m-1) = 2g$. So the remaining minimal loops $\alpha_{j,k}$, $(j = 1, 2, \dots, m-1$ and $k = 1, 2, \dots, n-1)$ form a basis of $H_1(R, \boldsymbol{Z})$.

Case (iii). As the ramification index of $x = \infty$ is n/d, there are d points at infinity; hence, the minimal loops $\alpha_{m,k}$, $(k = 1, 2, \dots, n)$ are not always closed. Since the number of the loops corresponding to the remaining minimal loops in the case (ii) is $(n-1)(m-1) > 2g = (n-1)(m-1) + 1 - d$, they are linearly dependent over $\boldsymbol{Z}$. To remove the dependent loops, we have to construct a topological model of R by gluing $X_1, \dots, X_n$ together. To do so we investigate how the branch cuts join one another. Since each of the branch cuts $[\eta^j, \eta^{j+1}]_k$ and $[1, \infty]_k$ belongs to only one side of X_k, the paths of the opposite bank to X_k do not belong to X_k. These branch cuts are joined with the corresponding branch cut by the analytic continuation of the branch of y.

Let $\tilde{X}_k$, $(k = 1, 2, \dots, n)$ be $X_k - \{\eta, \eta^2, \dots, \eta^{m-1}, 1, \infty\}$ cut open all the branch cuts $[\eta^j, \eta^{j+1}]_k$ and $[1, \infty]_k$. Let ζ be one of the primitive n-th roots of unity and f be a branch of y on $\tilde{X}_1$. Since the interchange of the branches of y does not occur by the analytic continuation along any closed path on $\tilde{X}_k$, we can assume that the branches $\zeta^{k-1} f$, $(k = 1, 2, \dots, n)$ are assigned to $\tilde{X}_k$ by the analytic continuation.

In the first place, we consider the analytic continuation of f along a path which starts from any point on the branch cut $[\eta, \eta^2]_1$, encircles the branch point η counterclockwise, ends at the point on $[\eta, \eta^2]_{1'}$ opposite to the starting point. Then the branch turns from f to ζf. So the end point of the path should be on $[\eta, \eta^2]_2$ and the branch cut $[\eta, \eta^2]_{1'}$ is identified with the branch cut $[\eta, \eta^2]_2$. Similarly we see the branch cuts $[\eta, \eta^2]_{2'}, \dots, [\eta, \eta^2]_{n'}$ are identified with the branch cuts $[\eta, \eta^2]_3, \dots, [\eta, \eta^2]_1$, respectively.

In the next place, we take a path which starts from any point on $[\eta^2, \eta^3]_1$, encircles the branch points η^2 and η counterclockwise, ends at the point on $[\eta^2, \eta^3]_{1'}$ opposite to the starting point. Since this path encircles two branch points, the branch turns from f to $\zeta^2 f$ by the analytic continuation along this path. Therefore the branch cut $[\eta^2, \eta^3]_{1'}$ is identified with the branch cut $[\eta^2, \eta^3]_3$. Similarly we see the branch cuts $[\eta^2, \eta^3]_{2'}, \dots, [\eta^2, \eta^3]_{n'}$ are identified with the branch cuts $[\eta^2, \eta^3]_4, \dots, [\eta^2, \eta^3]_2$, respectively.

Successively using the same process, we see that the branch cuts $[\eta^j, \eta^{j+1}]_{k'}$ and $[1, \infty]_{k'}$ on each $X_{k'}$ are glued by

$$[\eta^j, \eta^{j+1}]_{k'} = [\eta^j, \eta^{j+1}]_{\langle k+j \rangle}, \quad [1, \infty]_{k'} = [1, \infty]_{\langle k+m \rangle}$$

for $j = 1, 2, \dots, m-1$ and $k = 1, 2, \dots, n$, where $\langle a \rangle$ denotes the element of $\{1, 2, \dots, n\}$ congruent to a modulo n.

$X_k, X_{\langle k+m \rangle}, \dots, X_{\langle k+(n/d-1)m \rangle}$ are glued together by the identification $[1, \infty]_{\langle k+am \rangle'} = [1, \infty]_{\langle k+(a+1)m \rangle}$, for $k = 1, 2, \dots, n$ and $a = 0, 1, \dots, n/d - 1$.

Since any two of the numbers $1, 2, \dots, d$ are not congruent each other modulo m, the set of all

$$Y_k := X_k \cup X_{\langle k+m \rangle} \cup \cdots \cup X_{\langle k+(n/d-1)m \rangle}, \quad k = 1, 2, \dots d$$

glued by the above identification is a topological model of R.

Since the sum of the loops

$$[\eta^j, \eta^{j+1}]_{\langle 1+am \rangle} + [\eta^{j+1}, \eta^j]_{\langle 1+am \rangle'}, \quad j = 1, 2, \dots, m-1 \text{ and } a = 0, 1, \dots, n/d - 1$$

is homologous to zero on Y_1, we discard one of these loops. Here we discard the loop

$$[\eta, \eta^2]_1 + [\eta^2, \eta]_{1'} = [\eta, \eta^2]_1 + [\eta^2, \eta]_2 = \alpha_{1,1}.$$

By the same reason for $Y_1, \dots, Y_{d-1}$ the loops $\alpha_{1,2}, \alpha_{1,3}, \dots, \alpha_{1,d-1}$ are discarded. We can not discard $\alpha_{1,d}$ since Y_d contains X_n and the loop $\alpha_{1,n}$ is already discarded.

Now the number of the remaining minimal loops is $(n-1)(m-1) - (d-1) = 2g$. Therefore the remaining minimal loops $\alpha_{1,d}, \alpha_{1,d+1}, \dots, \alpha_{1,n-1}, \alpha_{j,k}$, ($j = 2, 3, \dots, m-1$ and $k = 1, 2, \dots, n-1$) form a basis of $H_1(R, \boldsymbol{Z})$.

3. Holomorphic 1-Forms

We shall find the condition that the differential 1-form $\omega = x^a y^{b-n+1}\,dx$, $(a, b \in \boldsymbol{Z})$ is holomorphic. For this purpose we consider the representation of ω by a local coordinate at each point on $\hat{R}$ (cf. [6], pp. 189–192).

If $x = \alpha \in \boldsymbol{C}$ is not a branch point, then $t := x - \alpha$ is a local coordinate in a neighbourhood of α. If $\alpha \neq 0$, ω is obviously holomorphic in this neighbourhood. If $\alpha = 0$, ω is holomorphic for $a \geq 0$.

In a neighbourhood of the branch point $x = \eta^j$, $t := \sqrt[n]{x - \eta^j}$ is a local coordinate. Then $\omega = u(t)t^b\,dt$, where $u(t)$ is a non-vanishing holomorphic function on this neighbourhood. Hence ω is holomorphic for $b \geq 0$ in this neighbourhood.

To describe ω in a neighbourhood of $x = \infty$, we consider ω in the affine part $\hat{R} \cap \{[X, Y, Z] \in \boldsymbol{P}^2 \mid X \neq 0\}$. We rewrite the defining equation and ω:

$$w^n = z^n - z^{n-m}, \quad \omega = x^a y^{b-n+1}\,dx = -z^{-a-b+n-3} w^{b-n+1}\,dz,$$

$$\text{where } z = Z/X = 1/x, \ w = Y/X = y/x.$$

Case (i). The points at infinity $x = \infty$ on R correspond to the points $(z, w) = (0, \rho^j)$, $\rho = \exp(\pi i/n)$, $j = 1, 2, \ldots, n$. We choose a local coordinate $t := \sqrt[n]{w - \rho^j}$ vanishing at $(z, w) = (0, \rho^j)$. Then $\omega = t^{-a-b+n-3} u(t)\,dt$, where $u(t)$ is a non-vanishing holomorphic function on a neighbourhood of $t = 0$. Hence ω is holomorphic everywhere on $\hat{R}$ if and only if $a, b \geq 0$ and $n - 3 \geq a + b$. Because the number of the pairs of integers (a, b) which satisfy these inequalities is

$$\sum_{j=0}^{n-3} (n - 2 - j) = (n-1)(n-2)/2 = g,$$

the differentials $x^a y^{b-n+1}\,dx$ with $a, b \geq 0$, $n - 3 \geq a + b$ form a basis of the vector space of holomorphic 1-forms.

Case (ii). The unique point at infinity $x = \infty$ on R corresponds to the point $(z, w) = (0, 0)$. We choose a local coordinate $t := \sqrt[n]{z}$ vanishing at $(z, w) = (0, 0)$. Then $\omega = t^{-an-bm-m-n+mn-1} u(t)\,dt$, with $u(t)$ as above. Hence ω is holomorphic everywhere on $\hat{R}$ if and only if

$$a, b \geq 0, \quad -an - bm - m - n + mn - 1 \geq 0. \tag{1}$$

These inequalities are equivalent to being $m - 1 - (m+1)/n \geq a + bm/n \geq a \geq 0$ and $n - 1 - \{(a+1)n + 1\}/m \geq b \geq 0$. Since $0 < (m+1)/n \leq 1$, $0 \leq a \leq m - 2$.

Let N be the number of the pairs of integers (a, b) that satisfy (1). Then we have

$$2N = \sum_{j=0}^{m-2}[n - \{(j+1)n+1\}/m] + \sum_{j=0}^{m-2}[n - \{(j+1)n+1\}/m]$$

$$= \sum_{j=0}^{m-2}[n - \{(j+1)n+1\}/m] + \sum_{j=0}^{m-2}[n - \{(m-j-1)n+1\}/m],$$

where the symbol [ ] is the Gauss' symbol. Now suppose $(j+1)n = q_j m + r_j$, $q_j, r_j \in \boldsymbol{Z}$, $1 \leq r_j \leq m-1$. Then

$$[n - \{(j+1)n+1\}/m] = [n - q_j - (r_j+1)/m] = n - q_j - 1,$$

$$[n - \{(m-1-j)n+1\}/m] = [q_j + (r_j-1)/m] = q_j.$$

Therefore

$$N = \left\{\sum_{j=0}^{m-2}(n - q_j - 1) + \sum_{j=0}^{m-2} q_j\right\}/2$$

$$= (n-1)(m-1)/2$$

$$= g.$$

Hence the differentials $x^a y^{b-n+1}\, dx$ with a, b satisfying (1) form a basis of the vector space of holomorphic 1-forms.

Case (iii). The unique point at infinity $x = \infty$ on R corresponds to the point $(z, w) = (0, 0)$. Then ω is equal to $t^{(-an-bm+mn-m-n-d)/d}u(t)\, dt$ and holomorphic everywhere on $\hat{R}$ if and only if

$$a, b \geq 0, \quad -an - bm + mn - m - n - d \geq 0.$$

We rewrite this condition as follows:

$$\text{(2)} \qquad a, b \geq 0, \quad -an' - bm' + m'n'd - m' - n' - 1 \geq 0,$$

$$\text{where } n' = n/d, m' = m/d.$$

We denote by N the number of pairs (a, b) that satisfy (2). If $m' = 1$ $(d = m)$, then (2) is equivalent to the condition $0 \leq a \leq d-2$ and $0 \leq b \leq n'd - n' - 2 - an'$. So

$$N = \sum_{j=0}^{d-2}(n'd - n' - 1 - jn')$$

$$= \{(n'd - n' - 1)(d-1) - n'(d-2)(d-1)/2\}$$

$$= (m-1)(n-2)$$

$$= g.$$

If $m' \neq 1$, then the conditions are $0 \leq a \leq m-2 = m'd-2$ and $0 \leq b \leq n-1-\{(a+1)n+d\}/m = n'd-1-\{(a+1)n'+1\}/m'$. So we have

$$\begin{aligned}
2N &= 2\sum_{j=0}^{dm'-2}[n'd-\{(j+1)n'+1\}/m'] \\
&= 2\sum_{k=0}^{d-1}\sum_{j=km'}^{(k+1)m'-2}[n'd-\{(j+1)n'+1\}/m'] \\
&\quad + 2\sum_{j=1}^{d-1}[n'd-\{(jm')n'+1\}/m'] \\
&= \sum_{k=0}^{d-1}\sum_{j=km'}^{(k+1)m'-2}\{[n'd-\{(j+1)n'+1\}/m'] \\
&\quad + [n'd-\{(2km'+m'-j-1)n'+1\}/m']\} \\
&\quad + 2\sum_{j=1}^{d-1}(n'd-jn'-1).
\end{aligned}$$

Suppose $(j+1)n' = q_j m' + r_j$, $q_j, r_j \in \boldsymbol{Z}$, $1 \leq r_j \leq m'-1$. Then

$$[n'd-\{(j+1)n'+1\}/m'] = [n'd-q_j-(r_j+1)/m'] = n'd-q_j-1,$$

$$\begin{aligned}
[n'd-\{(2km'+m'-j-1)n'+1\}/m'] &= [n'd-2kn'-n'+q_j+(r_j-1)/m'] \\
&= n'd-2kn'-n'+q_j.
\end{aligned}$$

Therefore

$$\begin{aligned}
2N &= \sum_{k=0}^{d-1}\sum_{j=km'}^{(k+1)m'-2}(2n'd-2kn'-n'-1) + 2\sum_{j=1}^{d-1}(n'd-jn'-1) \\
&= (m'-1)\{(2n'd-n'-1)d-2n'(d-1)d/2\} \\
&\quad + 2\{(n'd-1)(d-1)-n'(d-1)d/2\} \\
&= nm-m-n+2-d \\
&= (n-1)(m-1)+1-d \\
&= 2g \quad (\text{So } N = g).
\end{aligned}$$

Hence the differentials $x^a y^{b-n+1}\, dx$ with a, b satisfying (2) form a basis of the vector space of holomorphic 1-forms.

4. Period Lattice

In this section we compute the period lattice relative to the differential 1-forms and the homology basis obtained in the previous Sections.

Let ζ, η be one of the primitive n-th, m-th roots of unity and f be a branch of y on $\tilde{X}_1$. Then $\zeta^{k-1} f$, a branch of y, is assigned by the analytic continuation.

The integral along $\alpha_{j,k}$ of the differential 1-form $\omega_l = x^{a_l} y^{b_l - n+1}\,dx$ is

$$\begin{aligned}\int_{\alpha_{j,k}} \omega_l &= \int_{\eta^j}^{\eta^{j+1}} x^{a_l} (\zeta^{k-1} f)^{b_l - n+1}\,dx + \int_{\eta^{j+1}}^{\eta^j} x^{a_l} (\zeta^k f)^{b_l - n+1}\,dx \\ &= \eta^{j(a_l+1)} \zeta^{(k-1)(b_l-n+1)} \int_1^{\eta} x^{a_l} f^{b_l-n+1}\,dx + \eta^{j(a_l+1)} \zeta^{k(b_l-n+1)} \int_{\eta}^{1} x^{a_l} f^{b_l-n+1}\,dx \\ &= \eta^{j(a_l+1)} \zeta^{k(b_l-n+1)} (\zeta^{-(b_l-n+1)} - 1) \int_1^{\eta} x^{a_l} f^{b_l-n+1}\,dx.\end{aligned}$$

From this equality, we find the relations

$$\int_{\alpha_{j,k+1}} \omega_l = \zeta^{b_l-n+1} \int_{\alpha_{j,k}} \omega_l = \zeta^{k(b_l-n+1)} \int_{\alpha_{j,1}} \omega_l,$$

$$\int_{\alpha_{j+1,k}} \omega_l = \eta^{a_l+1} \int_{\alpha_{j,k}} \omega_l = \eta^{j(a_l+1)} \int_{\alpha_{1,k}} \omega_l.$$

We compute the period lattice for the case (i) and (ii) at the same time. Let P_1 be the matrix whose (l,k)-component is $\int_{\alpha_{1,k}} \omega_l$ $(k = 1, 2, \ldots, n-1,$ $l = 1, 2, \ldots, g)$. Then we have

$$P_1 = \begin{pmatrix} \int_{\alpha_{1,1}} \omega_1 & \cdots & \int_{\alpha_{1,n-1}} \omega_1 \\ \vdots & \ddots & \vdots \\ \int_{\alpha_{1,1}} \omega_g & \cdots & \int_{\alpha_{1,n-1}} \omega_g \end{pmatrix}$$

$$= \begin{pmatrix} s_1 & & O \\ & \ddots & \\ O & & s_g \end{pmatrix} \begin{pmatrix} 1 & (\zeta^{b_1-n+1}) & \cdots & (\zeta^{b_1-n+1})^{n-2} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & (\zeta^{b_g-n+1}) & \cdots & (\zeta^{b_g-n+1})^{n-2} \end{pmatrix}, \quad \text{where } s_l = \int_{\alpha_{1,1}} \omega_l.$$

Similarly the matrix P_2 whose (l,k)-component is $\int_{\alpha_{2,k}} \omega_l$ $(k = 1, 2, \ldots, n-1,$ $l = 1, 2, \ldots, g)$ is given as follows:

$$P_2 = \begin{pmatrix} s_1 & & O \\ & \ddots & \\ O & & s_g \end{pmatrix} \begin{pmatrix} \eta^{(a_1+1)} & & O \\ & \ddots & \\ O & & \eta^{(a_g+1)} \end{pmatrix} \begin{pmatrix} 1 & (\zeta^{b_1-n+1}) & \cdots & (\zeta^{b_1-n+1})^{n-2} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & (\zeta^{b_g-n+1}) & \cdots & (\zeta^{b_g-n+1})^{n-2} \end{pmatrix}.$$

In the same way we can also obtain $P_3, P_4, \ldots, P_{m-2}$ for the case (i), $P_3, P_4, \ldots, P_{m-1}$ for the case (ii). Letting

$$S = \begin{pmatrix} s_1 & & O \\ & \ddots & \\ O & & s_g \end{pmatrix}, \quad A = \begin{pmatrix} \eta^{(a_1+1)} & & O \\ & \ddots & \\ O & & \eta^{(a_g+1)} \end{pmatrix}$$

and

$$B = \begin{pmatrix} 1 & (\zeta^{b_1-n+1}) & \cdots & (\zeta^{b_1-n+1})^{n-2} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & (\zeta^{b_g-n+1}) & \cdots & (\zeta^{b_g-n+1})^{n-2} \end{pmatrix},$$

we consider a $g \times 2g$ matrix

$$P' = (P_1, P_2, \dots, P_{m-2}) = S(B, AB, \dots, A^{m-3}B) \quad \text{for the case (i)},$$

$$P' = (P_1, P_2, \dots, P_{m-1}) = S(B, AB, \dots, A^{m-2}B) \quad \text{for the case (ii)}.$$

The set of the $2g$ column vectors of P' is obviously a basis of the period lattice relative to the holomorphic 1-forms selected in Section 3.

For the case (iii) we define S as above. Then the period lattice P' relative to the holomorphic 1-forms selected in Section 3 is obtained by removing the first $(d-1)$ columns from the period lattice P' of the case (ii).

In any case (i), (ii), or (iii), we change the basis of the space of the holomorphic 1-forms by multiplying S^{-1}:

$$\{\hat{\omega}_1, \hat{\omega}_2, \dots, \hat{\omega}_g\} = \{\omega_1, \omega_2, \dots, \omega_g\}S^{-1} = \{s_1^{-1}\omega_1, s_2^{-1}\omega_2, \dots, s_g^{-1}\omega_g\}.$$

Thus we can obtain the following theorem (cf. [5], p. 79).

THEOREM 1. *Let S, P', and $\{\hat{\omega}_1, \hat{\omega}_2, \dots, \hat{\omega}_g\}$ be as above. Let $P = S^{-1}P'$. The period lattice of the curve defined by $x^m + y^n = 1$ relative to the basis of the space of the holomorphic 1-forms $\{\hat{\omega}_1, \hat{\omega}_2, \dots, \hat{\omega}_g\}$ is spanned by the $2g$ column vectors of P.*

Note that each of the components of P is just a monomial of the primitive root of unity.

5. Intersection Matrix

In this section, we show how to obtain the intersection matrix.

Case (i). In order to read the intersection numbers between $\alpha_{j,k}$'s, we deform $\alpha_{j,k}$, $(j = 1, 2, \dots, m-2, k = 1, 2, \dots, n-1)$ around the branch points like the Appendix. If the minimal loops are chosen such a way, we have

$$I(\alpha_{j,k}, \alpha_{j,k+1}) = 1, \quad I(\alpha_{j,k}, \alpha_{j+1,k}) = 1, \quad I(\alpha_{j,k+1}, \alpha_{j+1,k}) = 0, \quad I(\alpha_{j,k}, \alpha_{j+1,k+1}) = -1.$$

Clearly the other intersection numbers are $I(\alpha_{j,k}, \alpha_{j',k'}) = 0$ for $|j - j'| \geq 2$ or $|k - k'| \geq 2$.

The intersection matrix L_1 of the minimal loops $\alpha_{j,1}, \alpha_{j,2}, \ldots, \alpha_{j,n-1}$, namely, the $(n-1) \times (n-1)$ matrix whose (s,t)-component is $I(\alpha_{j,s}, \alpha_{j,t})$, is given by

$$L_1 = \begin{pmatrix} 0 & 1 & 0 & \cdots & \cdots & 0 \\ -1 & 0 & 1 & \ddots & & \vdots \\ 0 & -1 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & \ddots & \ddots & 1 \\ 0 & \cdots & \cdots & 0 & -1 & 0 \end{pmatrix}.$$

The matrix L_2 of the intersection numbers of $\alpha_{j,1}, \alpha_{j,2}, \ldots, \alpha_{j,n-1}$ and $\alpha_{j+1,1}, \alpha_{j+1,2}, \ldots, \alpha_{j+1,n-1}$, namely, the $(n-1) \times (n-1)$ matrix whose (s,t)-component is $I(\alpha_{j,s}, \alpha_{j+1,t})$, is given by

$$L_2 = \begin{pmatrix} 1 & -1 & 0 & \cdots & \cdots & 0 \\ 0 & 1 & -1 & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & \ddots & 0 \\ \vdots & & & \ddots & \ddots & -1 \\ 0 & \cdots & \cdots & \cdots & 0 & 1 \end{pmatrix}.$$

Then the intersection matrix of the minimal loops $\alpha_{j,k}$, $(j = 1, 2, \ldots, m-2, k = 1, 2, \ldots, n-1)$ can be represented by

$$K = \begin{pmatrix} L_1 & L_2 & O & \cdots & \cdots & O \\ -{}^tL_2 & \ddots & \ddots & \ddots & & \vdots \\ O & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & O \\ \vdots & & \ddots & \ddots & \ddots & L_2 \\ O & \cdots & \cdots & O & -{}^tL_2 & L_1 \end{pmatrix},$$

where O is the $(n-1) \times (n-1)$ zero matrix. This is the $(m-2) \times (m-2)$ matrix of the $(n-1) \times (n-1)$ minor matrices.

Case (ii). The intersection matrix of the same representation is obtained in the same way, and the matrix is the $(m-1) \times (m-1)$ matrix of the $(n-1) \times (n-1)$ minor matrices.

Case (iii). The intersection matrix is obtained by removing the first $(d-1)$ rows and columns from the intersection matrix of the case (ii).

To get a canonical homology basis we transform the intersection matrix into J. For the case (i), for example, this transformation is equivalent to finding a $2g \times 2g$ unimodular matrix M which satisfies

$$(\alpha_1, \dots, \alpha_g, \beta_1, \dots, \beta_g) = (\alpha_{1,1}, \dots, \alpha_{1,n-1}, \alpha_{2,1}, \dots, \alpha_{2,n-1}, \dots, \alpha_{m-2,n-1})M,$$

where $\{\alpha_j, \beta_j\}$ is a canonical homology basis. It is also equivalent to finding a $2g \times 2g$ unimodular matrix M which satisfies $J = {}^t MKM$.

Finally, we obtain a normalized matrix from PM as we mentioned in Introduction. It is guaranteed by the theorem of Frobenius that we can transform the intersection matrix K into J within finitely many operations (cf. [3] p. 65, [4] p. 156). We explain the process of finding M in the next Section.

6. Examples

(1) The curve C_1 defined by $y^4 = 1 - x^4$.

This curve is of genus 3. The pairs of integers (a,b) which satisfy $a, b \geq 0$, $1 \geq a+b \geq 0$ are $(0,0), (0,1), (1,0)$. Therefore the set of

$$\omega_1 = y^{-3}\,dx, \quad \omega_2 = y^{-2}\,dx, \quad \omega_3 = xy^{-3}\,dx$$

is a basis of the space of holomorphic 1-forms on C_1.

Let η be one of the primitive fourth roots of unity. The matrices A, B and P are

$$A = \begin{pmatrix} \eta & 0 & 0 \\ 0 & \eta & 0 \\ 0 & 0 & \eta^2 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & \eta^{-3} & (\eta^{-3})^2 \\ 1 & \eta^{-2} & (\eta^{-2})^2 \\ 1 & \eta^{-3} & (\eta^{-3})^2 \end{pmatrix} = \begin{pmatrix} 1 & \eta & \eta^2 \\ 1 & \eta^2 & 1 \\ 1 & \eta & \eta^2 \end{pmatrix},$$

$$P = (B, AB) = \begin{pmatrix} 1 & \eta & \eta^2 & \eta & \eta^2 & \eta^3 \\ 1 & \eta^2 & 1 & \eta & \eta^3 & \eta \\ 1 & \eta & \eta^2 & \eta^2 & \eta^3 & 1 \end{pmatrix}.$$

The intersection matrix of the minimal loops $\alpha_{i,j}$ $(i = 1,2, j = 1,2,3)$ is

$$K = \begin{pmatrix} L_1 & L_2 \\ -{}^tL_2 & L_1 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 1 & -1 & 0 \\ -1 & 0 & 1 & 0 & 1 & -1 \\ 0 & -1 & 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 & 1 & 0 \\ 1 & -1 & 0 & -1 & 0 & 1 \\ 0 & 1 & -1 & 0 & -1 & 0 \end{pmatrix}.$$

Now we explain the method to find a unimodular matrix M which transform K into J. The i-th column of the intersection matrix is called c_i $(i = 1, \ldots, 6)$ in each step.

Step 1.

Choose a column whose the first element is non-zero, for instance, c_2. Subtract c_2 from c_4 and add c_2 to c_5 so that the first element of each column except c_2 vanishes (i.e. $c_4 \to c_4 - c_2$, $c_5 \to c_5 + c_2$). This operation substitutes $\alpha_{2,1} - \alpha_{1,2}$ for $\alpha_{2,1}, \alpha_{2,2} + \alpha_{1,2}$ for $\alpha_{2,2}$. The matrix that represents the change of the minimal loops is

$$M_1 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Then the intersection matrix changes into

$$K_1 = {}^t M_1 K M_1 = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 1 & -1 \\ 0 & -1 & 0 & 1 & -1 & 1 \\ 0 & 0 & -1 & 0 & 0 & 1 \\ 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & -1 & 0 & 0 \end{pmatrix}.$$

Step 2.

Choose a column whose the second element is non-zero and the first element is zero, for instance, c_1. Add c_1 to c_3, c_5 and subtract c_1 from c_6 so that the second element of each column except c_1 vanishes (i.e. $c_3 \to c_3 + c_1$, $c_5 \to c_5 + c_1$, $c_6 \to c_6 - c_1$). This operation substitutes $\alpha_{1,3} + \alpha_{1,1}$ for $\alpha_{1,3}, \alpha_{2,2} + \alpha_{1,1}$ for $\alpha_{2,2}$ and $\alpha_{2,3} - \alpha_{1,1}$ for $\alpha_{2,3}$. We see the matrix that represents the change of the minimal loops is

$$M_2 = \begin{pmatrix} 1 & 0 & 1 & 0 & 1 & -1 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Then the intersection matrix changes into

$$K_2 = {}^t M_2 K_1 M_2 = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & -1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & -1 & 0 & 0 \end{pmatrix}.$$

Step 3.

Choose a column whose the third element is non-zero and the first and the second elements are zero, for instance, c_4. Add c_4 to c_5 and subtract c_4 from c_6 so that the third element of each column except c_4 vanishes (i.e. $c_5 \to c_5 + c_4$, $c_6 \to c_6 - c_4$). This operation substitutes $\alpha_{2,2} + \alpha_{2,1}$ for $\alpha_{2,2}, \alpha_{2,3} - \alpha_{2,1}$ for $\alpha_{2,3}$. We see the matrix that represents the change of the minimal loops is

$$M_3 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

The intersection matrix changes into

$$K_3 = {}^t M_3 K_2 M_3 = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 & -1 & 0 \end{pmatrix}.$$

Step 4.

Choose a column whose the fourth element is non-zero and the elements from the first to the third are zero, for instance, c_3. Add c_3 to c_6 so that the fourth element of each column except c_3 vanishes (i.e. $c_6 \to c_6 + c_3$). This operation substitutes $\alpha_{2,3} + \alpha_{1,3}$ for $\alpha_{2,3}$. We see the matrix that represents the change of the minimal loops is

$$M_4 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

The intersection matrix changes into

$$K_4 = {}^t M_4 K_3 M_4 = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 \end{pmatrix}.$$

Here each column of K_4 has only one nonzero element. Letting $\gamma_1, \dots, \gamma_6$ be the loops corresponding to the columns $c_1, \dots, c_6$, we have the intersection numbers of these loops

$$I(\gamma_1, \gamma_2) = 1, \quad I(\gamma_3, \gamma_4) = 1, \quad I(\gamma_5, \gamma_6) = 1.$$

Therefore we arrange

$$(\gamma_1, \gamma_3, \gamma_5, \gamma_2, \gamma_4, \gamma_6) = (\gamma_1, \gamma_2, \gamma_3, \gamma_4, \gamma_5, \gamma_6) M_5,$$

$$\text{where } M_5 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Through out these steps, we have a canonical homology basis

$$(\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3) = (\alpha_{1,1}, \alpha_{1,2}, \alpha_{1,3}, \alpha_{2,1}, \alpha_{2,2}, \alpha_{2,3}) M$$

$$\text{where } M = M_1 M_2 M_3 M_4 M_5 = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & -1 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

If we multiply P by M, then we have

$$PM = \begin{pmatrix} 1 & 1+\eta^2 & 1+\eta+\eta^2 & \eta & 0 & \eta^2+\eta^3 \\ 1 & 2 & 1+\eta+\eta^3 & \eta^2 & \eta-\eta^2 & \eta^2+1 \\ 1 & 1+\eta^2 & 1+\eta^2+\eta^3 & \eta & \eta^2-\eta & \eta+1 \end{pmatrix}.$$

Finally using the relation $\eta^2 = -1$, we obtain a normalized period matrix

$$\Omega = \begin{pmatrix} 1 & 0 & 0 & \eta & -(\eta+1)/2 & 0 \\ 0 & 1 & 0 & -(\eta+1)/2 & (2\eta+1)/2 & (1-\eta)/2 \\ 0 & 0 & 1 & 0 & (1-\eta)/2 & \eta - 1 \end{pmatrix}.$$

The theta matrix T of Ω is certainly non-singular (Det $T = (-2\eta+1)/2 \neq 0$), symmetric, and has positive definite imaginary part (because of $\eta = i$).

(2) The curve C_2 defined by $y^5 = 1 - x^3$.

This curve is of genus 4. The pairs of integers (a, b) which satisfy $a, b \geq 0$, $6 \geq 5a + 3b \geq 0$ are $(0,0), (0,1), (0,2), (1,0)$. Therefore the set of

$$\omega_1 = y^{-4}\,dx, \quad \omega_2 = y^{-3}\,dx, \quad \omega_3 = y^{-2}\,dx, \quad \omega_4 = xy^{-4}\,dx$$

is a basis of the space of the holomorphic 1-forms on C_2.

Let η, ζ be one of the primitive cubic, fifth roots of unity. The matrices A, B and P are

$$A = \begin{pmatrix} \eta & 0 & 0 & 0 \\ 0 & \eta & 0 & 0 \\ 0 & 0 & \eta & 0 \\ 0 & 0 & 0 & \eta^2 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & \zeta^{-4} & (\zeta^{-4})^2 & (\zeta^{-4})^3 \\ 1 & \zeta^{-3} & (\zeta^{-3})^2 & (\zeta^{-3})^3 \\ 1 & \zeta^{-2} & (\zeta^{-2})^2 & (\zeta^{-2})^3 \\ 1 & \zeta^{-4} & (\zeta^{-4})^2 & (\zeta^{-4})^3 \end{pmatrix} = \begin{pmatrix} 1 & \zeta & \zeta^2 & \zeta^3 \\ 1 & \zeta^2 & \zeta^4 & \zeta \\ 1 & \zeta^3 & \zeta & \zeta^4 \\ 1 & \zeta & \zeta^2 & \zeta^3 \end{pmatrix},$$

$$P = (B, AB) = \begin{pmatrix} 1 & \zeta & \zeta^2 & \zeta^3 & \eta & \eta\zeta & \eta\zeta^2 & \eta\zeta^3 \\ 1 & \zeta^2 & \zeta^4 & \zeta & \eta & \eta\zeta^2 & \eta\zeta^4 & \eta\zeta \\ 1 & \zeta^3 & \zeta & \zeta^4 & \eta & \eta\zeta^3 & \eta\zeta & \eta\zeta^4 \\ 1 & \zeta & \zeta^2 & \zeta^3 & \eta^2 & \eta^2\zeta & \eta^2\zeta^2 & \eta^2\zeta^3 \end{pmatrix}.$$

Let σ be one of the primitive fifteenth roots of unity. Then the components of P can be expressed by the monomials of σ:

$$P = \begin{pmatrix} 1 & \sigma^3 & \sigma^6 & \sigma^9 & \sigma^5 & \sigma^8 & \sigma^{11} & \sigma^{14} \\ 1 & \sigma^6 & \sigma^{12} & \sigma^3 & \sigma^5 & \sigma^{11} & \sigma^2 & \sigma^8 \\ 1 & \sigma^9 & \sigma^3 & \sigma^{12} & \sigma^5 & \sigma^{14} & \sigma^8 & \sigma^2 \\ 1 & \sigma^3 & \sigma^6 & \sigma^9 & \sigma^{10} & \sigma^{13} & \sigma & \sigma^4 \end{pmatrix}.$$

The intersection matrix of the minimal loops $\alpha_{i,j}$ $(i = 1, 2, j = 1, 2, 3, 4)$ is

$$K = \begin{pmatrix} L_1 & L_2 \\ -{}^tL_2 & L_1 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 & 1 & -1 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 & 1 & -1 & 0 \\ 0 & -1 & 0 & 1 & 0 & 0 & 1 & -1 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 & -1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 & 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & -1 & 0 & 0 & -1 & 0 \end{pmatrix}.$$

If we execute the steps as in the first example, we can obtain two unimodular matrices

$$M' = \begin{pmatrix} 1 & 0 & 1 & 0 & 0 & 1 & -1 & 0 \\ 0 & 1 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & -1 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

and

$$M'' = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

which satisfy

$${}^tM''\,{}^tM'KM'M'' = {}^t M'' \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \end{pmatrix} M'' = J.$$

Then the matrix

$$M = M'M'' = \begin{pmatrix} 1 & 1 & 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & -1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 1 & 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

transforms K into J and gives a canonical homology basis:

$$(\alpha_1, \alpha_2, \alpha_3, \alpha_4, \beta_1, \beta_2, \beta_3, \beta_4) = (\alpha_{1,1}, \alpha_{1,2}, \alpha_{1,3}, \alpha_{1,4}, \alpha_{2,1}, \alpha_{2,2}, \alpha_{2,3}, \alpha_{2,4})M.$$

We have

$$PM = \begin{pmatrix} 1 & 1+\sigma^6 & -\sigma^3+\sigma^5-\sigma^9 & 1+\sigma^3+\sigma^8+\sigma^9 & \sigma^3 & \sigma^9 & -1-\sigma^9+\sigma^{11} & \sigma^6+\sigma^{11}+\sigma^{14} \\ 1 & 1+\sigma^{12} & -\sigma^3+\sigma^5-\sigma^6 & 1+\sigma^3+\sigma^6+\sigma^{11} & \sigma^6 & \sigma^3 & -1-\sigma^3+\sigma^2 & \sigma^2+\sigma^8+\sigma^{12} \\ 1 & 1+\sigma^3 & \sigma^5-\sigma^9-\sigma^{12} & 1+\sigma^9+\sigma^{12}+\sigma^{14} & \sigma^9 & \sigma^{12} & -1+\sigma^8-\sigma^{12} & \sigma^2+\sigma^3+\sigma^8 \\ 1 & 1+\sigma^6 & -\sigma^3-\sigma^9+\sigma^{10} & 1+\sigma^3+\sigma^9+\sigma^{13} & \sigma^3 & \sigma^9 & -1+\sigma-\sigma^9 & \sigma+\sigma^4+\sigma^6 \end{pmatrix}.$$

Using the relation $\sigma^8 = \sigma^7 - \sigma^5 + \sigma^4 - \sigma^3 + \sigma - 1$, we can represent each component of the theta matrix T by the quotient of the polynomials of degree at most seven. The result is

$$T = \frac{1}{-2\sigma^7 - \sigma^5 - 3\sigma^4 - \sigma^3 - \sigma^2 - 2\sigma - 1}(T_1 \quad T_2 \quad T_3 \quad T_4),$$

where

$$T_1 = \begin{pmatrix} -5\sigma^7 - \sigma^6 + 3\sigma^5 - 2\sigma^4 + 4 \\ 4\sigma^7 - 2\sigma^5 + 3\sigma^4 + \sigma - 2 \\ \sigma^5 + \sigma + 1 \\ 2\sigma^7 - \sigma^5 + \sigma^4 + \sigma - 1 \end{pmatrix},$$

$$T_2 = \begin{pmatrix} 4\sigma^7 - 2\sigma^5 + 3\sigma^4 + \sigma - 2 \\ -4\sigma^7 + \sigma^6 + 2\sigma^5 - 3\sigma^4 + 2\sigma^3 - \sigma + 4 \\ 2\sigma^7 - \sigma^5 + 2\sigma^4 - 1 \\ -\sigma^7 - 2\sigma^4 - 2\sigma \end{pmatrix},$$

$$T_3 = \begin{pmatrix} \sigma^5 + \sigma + 1 \\ 2\sigma^7 - \sigma^5 + 2\sigma^4 - 1 \\ -5\sigma^7 + \sigma^6 + 3\sigma^5 - 3\sigma^4 + 2\sigma^3 - \sigma^2 + 4 \\ -\sigma^7 + 2\sigma^5 + \sigma^4 + 2\sigma + 2 \end{pmatrix},$$

$$T_4 = \begin{pmatrix} 2\sigma^7 - \sigma^5 + \sigma^4 + \sigma - 1 \\ -\sigma^7 - 2\sigma^4 - 2\sigma \\ -\sigma^7 + 2\sigma^5 + \sigma^4 + 2\sigma + 2 \\ -2\sigma^7 + 2\sigma^5 + \sigma^4 + 2\sigma^3 - \sigma^2 + 2\sigma + 4 \end{pmatrix}.$$

Certainly, T is non-singular (Det $T = (\sigma^7 - 2\sigma^4 + \sigma^3 - \sigma^2 - 2)/(2\sigma^7 + \sigma^5 - \sigma^4 + 2\sigma^3 - 1) \neq 0$), symmetric with positive definite imaginary part (because of $\sigma = \cos(2\pi/15) + i\sin(2\pi/15) = (2 + 2\sqrt{5} - \sqrt{6(5+\sqrt{5})} + \sqrt{30(5+\sqrt{5})})/16 + (2\sqrt{3} + 2\sqrt{15} + \sqrt{2(5+\sqrt{5})} - \sqrt{10(5+\sqrt{5})})i/16)$.

(3) The curve C_3 defined by $y^6 = 1 - x^3$.

This curve is of genus 4. The pairs of integers (a, b) which satisfy $a, b \geq 0$, $2 \geq 2a + b \geq 0$ are $(0,0), (0,1), (0,2), (1,0)$. Therefore we choose

$$\omega_1 = y^{-4}\,dx, \quad \omega_2 = y^{-3}\,dx, \quad \omega_3 = y^{-2}\,dx, \quad \omega_4 = xy^{-4}\,dx$$

for a basis of the space of holomorphic 1-forms on C_3.

Let η, ζ be one of the primitive cubic, sixth roots of unity. The matrices A, B and P are

$$A = \begin{pmatrix} \eta & 0 & 0 & 0 \\ 0 & \eta & 0 & 0 \\ 0 & 0 & \eta & 0 \\ 0 & 0 & 0 & \eta^2 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & \zeta^{-5} & (\zeta^{-5})^2 & (\zeta^{-5})^3 & (\zeta^{-5})^4 \\ 1 & \zeta^{-4} & (\zeta^{-4})^2 & (\zeta^{-4})^3 & (\zeta^{-4})^4 \\ 1 & \zeta^{-3} & (\zeta^{-3})^2 & (\zeta^{-3})^3 & (\zeta^{-3})^4 \\ 1 & \zeta^{-5} & (\zeta^{-5})^2 & (\zeta^{-5})^3 & (\zeta^{-5})^4 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & \zeta & \zeta^2 & \zeta^3 & \zeta^4 \\ 1 & \zeta^2 & \zeta^4 & 1 & \zeta^2 \\ 1 & \zeta^3 & 1 & \zeta^3 & 1 \\ 1 & \zeta & \zeta^2 & \zeta^3 & \zeta^4 \end{pmatrix},$$

$$P = \begin{pmatrix} \zeta^2 & \zeta^3 & \zeta^4 & \eta & \eta\zeta & \eta\zeta^2 & \eta\zeta^3 & \eta\zeta^4 \\ \zeta^4 & 1 & \zeta^2 & \eta & \eta\zeta^2 & \eta\zeta^4 & \eta & \eta\zeta^2 \\ 1 & \zeta^3 & 1 & \eta & \eta\zeta^3 & \eta & \eta\zeta^3 & \eta \\ \zeta^2 & \zeta^3 & \zeta^4 & \eta^2 & \eta^2\zeta & \eta^2\zeta^2 & \eta^2\zeta^3 & \eta^2\zeta^4 \end{pmatrix}.$$

We simplify the representation of P by using the relations $\eta = \zeta^2$ and $\zeta^3 = -1$:

$$P = \begin{pmatrix} \zeta^2 & -1 & -\zeta & \zeta^2 & -1 & -\zeta & -\zeta^2 & 1 \\ -\zeta & 1 & \zeta^2 & \zeta^2 & -\zeta & 1 & \zeta^2 & -\zeta \\ 1 & -1 & 1 & \zeta^2 & -\zeta^2 & \zeta^2 & -\zeta^2 & \zeta^2 \\ \zeta^2 & -1 & -\zeta & -\zeta & -\zeta^2 & 1 & \zeta & \zeta^2 \end{pmatrix}.$$

The intersection matrix of the minimal loops $\alpha_{i,j}$ $(i = 1, 2, j = 1, 2, 3, 4, 5)$ is

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & -1 & 0 \end{pmatrix}.$$

Therefore the intersection matrix of the minimal loops $\alpha_{1,3}, \alpha_{1,4}, \alpha_{1,5}, \alpha_{2,1}, \alpha_{2,2}, \alpha_{2,3}, \alpha_{2,4}, \alpha_{2,5}$ is

$$K = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 1 & -1 & 0 \\ -1 & 0 & 1 & 0 & 0 & 0 & 1 & -1 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 & -1 & 0 & 1 & 0 \\ 1 & -1 & 0 & 0 & 0 & -1 & 0 & 1 \\ 0 & 1 & -1 & 0 & 0 & 0 & -1 & 0 \end{pmatrix}.$$

If we execute the steps as in the first example, we obtain two unimodular matrices

$$M' = \begin{pmatrix} 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

and

$$M'' = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

which satisfy

$${}^tM''\,{}^tM'KM'M'' = {}^t M'' \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \end{pmatrix} M'' = J.$$

Then the matrix

$$M = M'M'' = \begin{pmatrix} 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

transforms K into J and gives a canonical homology basis:

$$(\alpha_1, \alpha_2, \alpha_3, \alpha_4, \beta_1, \beta_2, \beta_3, \beta_4) = (\alpha_{1,3}, \alpha_{1,4}, \alpha_{1,5}, \alpha_{2,1}, \alpha_{2,2}, \alpha_{2,3}, \alpha_{2,4}, \alpha_{2,5})M.$$

Then we have

$$PM = \begin{pmatrix} \zeta^2 & \zeta^2-\zeta & \zeta^2 & \zeta^2-\zeta & -1 & 1+\zeta^2-\zeta & -1 & -\zeta^2 \\ -\zeta & \zeta^2-\zeta & \zeta^2 & 1+2\zeta^2-\zeta & 1 & \zeta^2 & -\zeta & -\zeta \\ 1 & 2 & \zeta^2 & 1+\zeta^2 & -1 & 1+2\zeta^2 & -\zeta^2 & -\zeta^2 \\ \zeta^2 & \zeta^2-\zeta & -\zeta & 1+\zeta^2 & -1 & 2-\zeta & -\zeta^2 & -2+\zeta^2 \end{pmatrix}$$

$$= \begin{pmatrix} \zeta-1 & -1 & \zeta-1 & -1 & -1 & 0 & -1 & -\zeta+1 \\ -\zeta & -1 & \zeta-1 & \zeta-1 & 1 & \zeta-1 & -\zeta & -\zeta \\ 1 & 2 & \zeta-1 & \zeta & -1 & 2\zeta-1 & -\zeta+1 & -\zeta+1 \\ \zeta-1 & -1 & -\zeta & \zeta & -1 & 2-\zeta & -\zeta+1 & \zeta-3 \end{pmatrix}.$$

$$T = (5\zeta-2)^{-1} \begin{pmatrix} \zeta-6 & 4-2\zeta & \zeta-1 & -1 \\ 4-2\zeta & -4 & 2\zeta-1 & 6\zeta+1 \\ \zeta-1 & 2\zeta-1 & -4 & 3-2\zeta \\ -1 & 6\zeta+1 & 3-2\zeta & -3\zeta-4 \end{pmatrix}.$$

We used the relation $\zeta^2 - \zeta + 1 = 0$ for simplification.

The theta matrix T is certainly non-singular ($\mathrm{Det}\, T = 20(48\zeta - 61)/(185\zeta - 416) \neq 0$), symmetric with positive definite imaginary part (because of $\zeta = (1 + \sqrt{3}i)/2$).

References

[1] H. F. Baker, Multiply Periodic Functions, Cambridge University Press (1907)
[2] C. L. Tretkoff, M. D. Tretkoff, Combinatorial group theory, Riemann surfaces and Differential equations, Contemporary Mathematics, Vol. 33 (1984), pp. 467–519
[3] C. L. Siegel, Topics in Complex Function Theory (volume 3), Wiley-Interscience, 1973
[4] A. I. Markushevich, Introduction to the Classical Theory of Abelian Functions, Translations of Mathematical Monographs Vol. 96, American Mathematical Society (1992)
[5] S. Lang, Introduction to Algebraic and Abelian Functions, Graduate Texts in Mathematics 89, Springer-Verlag (1982)
[6] T. Higuchi, Y. Tashiro, S. Yamazaki and K. Watanabe, Function theory on Riemann surface (in Japanese), Morikita Publications Inc. (1997)

Institute of Mathematics, Tsukuba University
Tsukuba, Ibaraki 305-8571, Japan
E-mail address: kamata@math.tsukuba.ac.jp

Appendix