TSUKUBA J. MATH.
Vol. 26 No. 2 (2002), 407–451

b-FUNCTIONS OF REGULAR SIMPLE PREHOMOGENEOUS VECTOR SPACES

By

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Abstract. A prehomogeneous vector space is called simple if the group under consideration is the product of a simple algebraic group and some multiplicative groups. The explicit forms of the *b*-functions of regular simple prehomogeneous vector spaces have been determined except for two spaces. In this paper, we shall calculate the *b*-functions explicitly for the remaining cases.

Introduction

Let G be a connected reductive algebraic group defined over the complex number field $\boldsymbol{C}$ and $\rho : G \to GL(V)$ its finite dimensional rational representation. A triple (G, ρ, V) is called a *prehomogeneous vector space* if there exists an open dense orbit in V. A non-zero polynomial function f on V is called a *relative invariant* corresponding to a character ϕ if $f(\rho(g)v) = \phi(g)f(v)$ for any $g \in G$ and $v \in V$. The b-functions of relative invariants were introduced by M. Sato [25, 26], and play a crucial role in the theory of prehomogeneous vector spaces. The purpose of this paper is to give a complete list of explicit forms of b-functions for a certain class of prehomogeneous vector spaces.

M. Sato and T. Kimura [23] classified the irreducible prehomogeneous vector spaces, and their b-functions are determined by A. Gyoja, T. Kimura, M. Muro, I. Ozeki and T. Yano (cf. [2, 13, 18, 20, 34]). They make use of microlocal analysis developed in [22].

As a step toward classification of non-irreducible ones, Kimura [14] classified the prehomogeneous vector spaces of simple algebraic groups with scalar multiplications. Among them, there exists a certain class of prehomogeneous vector spaces, which we call *regular simple prehomogeneous vector spaces* (see § 3). The b-functions of regular simple prehomogeneous vector spaces are investigated

Received May 2, 2001.
Revised August 17, 2001.

mainly by S. Kasai (cf. [8]), and their explicit forms have been known except for two spaces. In this paper, we determine the b-functions for these remaining cases.

Our method to determine the b-functions is due to K. Ukai [31, 32], and can be outlined as follows. First we determine the a-functions defined in §2. Then Lemma 2.8 will tell us the explicit form of the b-functions to some extent. The remaining task is to determine some positive rational numbers $\alpha_{j,r}$ in Lemma 2.8. For this work, we use

(A) the results on the b-functions of irreducible prehomogeneous vector spaces.
(B) functional equations satisfied by b-functions.

We review (B) briefly in §2.

Now we shall describe our main results. In this paper, we mainly consider the following regular simple prehomogeneous vector spaces:

(28) $(GL(1)^2 \times Sp(3), \Lambda_3 \oplus \Lambda_1, V(14) \oplus V(6))$.
(37) $(GL(1)^4 \times SL(2n+1), \Lambda_2 \oplus \Lambda_1 \oplus \Lambda_1 \oplus \Lambda_1, V(n(2n+1)) \oplus V(2n+1) \oplus V(2n+1) \oplus V(2n+1))$.

Here we follow the numbering of the spaces in Proposition 3.1.

The space (28) is investigated by S. Kasai from the view point of microlocal analysis. According to his argument, it seems that microlocal calculus does not work well for this space. Kasai [8, pp. 65], however, posed a conjecture on the b-function of this space. Now let us quote his conjecture. The space (28) has two fundamental relative invariants f_1, f_2 (see §5).

CONJECTURE 0.1 (Kasai). *The b-function of the relative invariant* $\underline{f}^{\underline{m}} = f_1^{m_1} f_2^{m_2}$ $(m_1, m_2 \in \boldsymbol{Z}_{\geq 0})$ *is given by*

$$\begin{aligned} b_{\underline{f}^{\underline{m}}}(s) = & \left\{ \prod_{\nu=0}^{m_1-1} (m_1 s + 1 + \nu)(m_1 s + 2 + \nu) \right\} \\ & \times \left\{ \prod_{\nu=0}^{m_2-1} (m_2 s + 1 + \nu)(m_2 s + 3 + \nu) \right\} \\ & \times \left\{ \prod_{\nu=0}^{m_1+m_2-1} \left((m_1 + m_2)s + \frac{5}{2} + \nu \right) \left((m_1 + m_2)s + \frac{7}{2} + \nu \right) \right\}. \end{aligned}$$

In §5, we give an affirmative answer to this conjecture (see Theorem 5.11).

On the other hand, the space (37) is one of the most complicated ones among regular simple prehomogeneous vector spaces. For example, this space has infinitely many orbits (see [17]), and hence we can not appeal to microlocal calculus to determine its b-function. The space (37) has four fundamental relative invariants f_1, f_2, f_3, f_4 (see § 7). Then the b-function of the relative invariant $\underline{f}^{\underline{m}} = \prod_{i=1}^{4} f_i^{m_i}$ $(m_i \in \boldsymbol{Z}_{\geq 0})$ is given by

$$
\begin{aligned}
b_{\underline{f}^{\underline{m}}}(s) = {} & \left\{ \prod_{\nu=0}^{m_1-1} (m_1 s + 1 + \nu) \right\} \left\{ \prod_{\nu=0}^{m_2-1} (m_2 s + 1 + \nu) \right\} \left\{ \prod_{\nu=0}^{m_3-1} (m_3 s + 1 + \nu) \right\} \\
& \times \left\{ \prod_{\nu=0}^{m_4-1} (m_4 s + 1 + \nu)(m_4 s + 2n + \nu) \right\} \\
& \times \left\{ \prod_{\nu=0}^{m_1+m_2+m_3+m_4-1} \prod_{k=2}^{n+1} ((m_1+m_2+m_3+m_4)s + 2k - 1 + \nu) \right\}.
\end{aligned}
$$

We shall prove the above in § 7 (see Theorem 7.4).

The plan of this paper is as follows. In § 1 and § 2, we review some fundamental results on prehomogeneous vector spaces. In § 3, we give a table of regular simple prehomogeneous vector spaces, and in § 4, we summarize the known results on the b-functions of these spaces. In § 5, we determine the b-function of the space (28). In § 6, we recall some properties of Pfaffians of alternating matrices, and in § 7, by using those properties, we determine the b-function of the space (37). In Appendix, we give the tables of the explicit forms of the b-functions.

Notation. We denote by I_m the identity matrix of size m and by $0_{m,n}$ the $m \times n$ zero matrix. For a matrix A, we denote by ${}^tA$ the transposed matrix. We write 0_m instead of $0_{m,m}$.

1. Preliminaries

Let G be a connected reductive algebraic group defined over the complex number field $\boldsymbol{C}$ and $\rho : G \to GL(V)$ a rational representation of G on a finite dimensional vector space V. The triplet (G, ρ, V) is called a *prehomogeneous vector space* if V has an open dense G-orbit, say $O_0 = Gv_0$. Let f be a non-zero polynomial function on V and $\phi \in \mathrm{Hom}(G, \boldsymbol{C}^{\times})$. Denote by $d := \deg f$ and $n := \dim V$. We call f a *relative invariant* (corresponding to ϕ) if $f(\rho(g)v) = \phi(g) f(v)$ for all $g \in G$ and $v \in V$. Then (G, ρ, V) and f have the following properties given in Lemmas 1.1–1.5.

LEMMA 1.1 [23, Proposition 3 in §4]. *Let f_1 and f_2 be relative invariants which correspond to the same character. Then f_2 is a constant multiple of f_1.*

LEMMA 1.2 [23, Proposition 5 in §4]. *Let $S_1, \dots, S_l$ be the irreducible components of $V \backslash O_0$ with codimension one and suppose that each S_i is the zeros of some irreducible polynomial f_i, namely $S_i = \{v \in V; f_i(v) = 0\}$. Then $f_1, \dots, f_l$ are algebraically independent relative invariants. Every relative invariant f is of the form $f = cf_1^{m_1} \cdots f_l^{m_l}$ $(c \in \boldsymbol{C}^\times, m_i \in \boldsymbol{Z})$. We call polynomials $f_1, \dots, f_l$ the fundamental relative invariants.*

LEMMA 1.3 [23, Proposition 19 in §4]. *Let $X^*(G, V)$ be the totality of the characters associated to some relative invariants of (G, ρ, V) and let G_{v_0} be the isotropy subgroup at a point v_0 of the open orbit O_0. Then*

$$X^*(G, V) = \{\phi \in \mathrm{Hom}(G, GL_1); \phi|_{G_{v_0}} \equiv 1\}.$$

Let $\{e_1, \dots, e_n\}$ be a basis of V. For $x \in V$, we define the coordinates of x by $x = x_1 e_1 + \cdots + x_n e_n$, and identify V with $\boldsymbol{C}^n$. Let $V^\vee$ be the dual space of V and $\{e_1^\vee, \dots, e_n^\vee\}$ the dual basis of $\{e_1, \dots, e_n\}$. For $y \in V^\vee$, we define the coordinates of y by $y = y_1 e_1^\vee + \cdots + y_n e_n^\vee$, and identify $V^\vee$ with $\boldsymbol{C}^n$.

LEMMA 1.4 [2, Lemma 1.5]. *Let $\rho^\vee : G \to GL(V^\vee)$ be the contragredient representation of ρ. Then the triplet $(G, \rho^\vee, V^\vee)$ is a prehomogeneous vector space. Moreover it has a relative invariant polynomial $f^\vee$ of degree d that corresponds to ϕ^{-1}.*

LEMMA 1.5 [2, Lemmas 1.6, 1.7]. *There exists a polynomial $b_f(s) = b_0 s^d + b_1 s^{d-1} + \cdots + b_d \in \boldsymbol{C}[s]$ with $b_0 \neq 0$ such that*

$$f^\vee(\mathrm{grad}_x) f(x)^{s+1} = b_f(s) f(x)^s,$$

$$f(\mathrm{grad}_y) f^\vee(y)^{s+1} = b_f(s) f^\vee(y)^s.$$

Here we put

$$\mathrm{grad}_x := \left(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}\right) \quad \text{and} \quad \mathrm{grad}_y := \left(\frac{\partial}{\partial y_1}, \dots, \frac{\partial}{\partial y_n}\right).$$

We call $b_f(s)$ the *b-function* of f. The following lemma is useful for determination of $b_f(s)$.

LEMMA 1.6. (1) *The factor* $(s+1)$ *divides* $b_f(s)$ *if* $f \notin \boldsymbol{C}^{\times}$.
(2) *Assume that*

$$f(x) = \sum c_{i_1,\dots,i_n} x_1^{i_1} \cdots x_n^{i_n},$$
$$f^{\vee}(y) = \sum c^{\vee}_{i_1,\dots,i_n} y_1^{i_1} \cdots y_n^{i_n},$$

where the sums run over $(i_1,\dots,i_n) \in \boldsymbol{Z}^n_{\geq 0}$ *such that* $i_1 + \cdots i_n = d$. *Then we have*
(i) $b_0 = f^{\vee}((\operatorname{grad}\log f)(x)) f(x)$ *for* $x \in O_0$.
(ii) $b_d = \sum c_{i_1,\dots,i_n} c^{\vee}_{i_1,\dots,i_n} i_1! \cdots i_n!$.

PROOF. Putting -1 into s, we have $f^{\vee}(\operatorname{grad}_x) f(x)^0 = b(-1) f(x)^{-1}$, and hence $b(-1) = 0$. This implies (1). (ii) follows from $b_d = f^{\vee}(\operatorname{grad}_x) f(x)$. (i) follows from

$$\frac{\partial^d}{\partial x_{i_1} \cdots \partial x_{i_d}} f(x)^{s+1} = s^d f(x)^{s-d+1} \cdot \frac{\partial f}{\partial x_{i_1}} \cdots \frac{\partial f}{\partial x_{i_d}} + (\text{terms of } \deg_s < d). \qquad \square$$

The following theorem is due to M. Kashiwara [12].

THEOREM 1.7. *Let*

$$b_f(s) = b_0 \prod_{j=1}^{d} (s + \alpha_j).$$

Then each α_j *is a positive rational number.*

Now we define regularity of prehomogeneous vector spaces. A relative invariant f is called *non-degenerate* if the Hessian

$$\det\left(\frac{\partial^2 f}{\partial x_i \partial x_j}(x)\right)$$

of $f(x)$ is not identically zero. A prehomogeneous vector space is called *regular* if there exists a non-degenerate relative invariant. For the properties of regular prehomogeneous vector spaces, see [23, §4].

2. a-Functions and b-Functions

In this section, we give the definitions of a-functions and b-functions and some properties of them. The main references of this section are [25, 26].

Let $f_1, \dots, f_l$ be the fundamental relative invariants of a prehomogeneous vector space (G, ρ, V) and $f_1^\vee, \dots, f_l^\vee$ the irreducible relative invariants of $(G, \rho^\vee, V^\vee)$ such that the characters of f_i and $f_i^\vee$ are the inverse of each other. We put $\underline{f} := (f_1, \dots, f_l)$ and $\underline{f}^\vee := (f_1^\vee, \dots, f_l^\vee)$. Let $V_{f_i} := \{v \in V; f_i(v) \neq 0\}$ and $V_{\underline{f}} := \bigcap_{i=1}^l V_{f_i}$. For l-tuple $\underline{s} = (s_1, \dots, s_l)$, we define the power $\underline{f}^{\underline{s}}$ and $\underline{f}^{\vee \underline{s}}$ formally by $\underline{f}^{\underline{s}} := \prod_{i=1}^l f_i^{s_i}$ and $\underline{f}^{\vee \underline{s}} := \prod_{i=1}^l f_i^{\vee s_i}$.

LEMMA 2.1. *For any l-tuple $\underline{m} = (m_1, \dots, m_l) \in \boldsymbol{Z}_{\geq 0}^l$, we have*

$$\underline{f}^{\underline{m}}(v)\underline{f}^{\vee \underline{m}}(\operatorname{grad} \log \underline{f}^{\underline{s}}(v)) = a_{\underline{m}}(\underline{s})$$

for all $v \in V_{\underline{f}}$ with some non-zero homogeneous polynomial $a_{\underline{m}}(\underline{s})$ which is independent of v.

We call $a_{\underline{m}}(\underline{s})$ the *a-function* of $\underline{f}$. When $\underline{m} = \varepsilon_i = (0, \dots, 0, 1, 0, \dots, 0)$, where 1 appears only at i-th place, we write $a_i(\underline{s})$ instead of $a_{\varepsilon_i}(\underline{s})$ for an abbreviation. We can easily see that $a_{\underline{m}}(\underline{s}) = \prod_{i=1}^l a_i(\underline{s})^{m_i}$ by definition. The following lemma about the structures of a-functions $a_{\underline{m}}(\underline{s})$ is stated in [25, Theorem 2] and [26, Theorem 1].

LEMMA 2.2. *The a-function $a_{\underline{m}}(\underline{s})$ is expressed as*

$$a_{\underline{m}}(\underline{s}) = \underline{A}^{\underline{m}} \prod_{j=1}^N (\gamma_j(\underline{s})^{\gamma_j(\underline{m})})^{\mu_j}$$

where $\underline{A}^{\underline{m}} = \prod_{i=1}^l A_i^{m_i}$ with $A_i \in \boldsymbol{C}^\times$, $N \in \boldsymbol{Z}_{>0}$, and $\mu_j \in \boldsymbol{Z}_{>0}$, while each $\gamma_j(\underline{s})$ is a $\boldsymbol{Z}$-linear function $\sum_{i=1}^l \gamma_{ij} s_i$ with $\gamma_{ij} \in \boldsymbol{Z}_{\geq 0}$ and $GCD(\gamma_{1j}, \dots, \gamma_{lj}) = 1$.

We have the following lemma by Lemma 2.2.

LEMMA 2.3. *The leading coefficient $a_{\underline{f}^{\underline{m}}}$ of the b-function $b_{\underline{f}^{\underline{m}}}$ of the relative invariant $\underline{f}^{\underline{m}}$ $(\underline{m} \in \boldsymbol{Z}_{\geq 0}^l)$ is given by*

$$a_{\underline{f}^{\underline{m}}} = \underline{A}^{\underline{m}} \prod_{j=1}^N (\gamma_j(\underline{m})^{\gamma_j(\underline{m})})^{\mu_j}.$$

Now we put a certain assumption on the a-function $a_{\underline{m}}(\underline{s})$.

ASSUMPTION 2.4. *For all $1 \leq j \leq N$, there exists $\underline{m} \in \boldsymbol{Z}_{\geq 0}^l$ such that $\gamma_j(\underline{m}) = 1$.*

Lemma 2.5. *For any l-tuple $\underline{m} = (m_1, \dots, m_l) \in \boldsymbol{Z}^l_{\geq 0}$, we have*

$$\underline{f}^{\vee \underline{m}}(\mathrm{grad})\underline{f}^{\underline{s}+\underline{m}} = b_{\underline{m}}(\underline{s})\underline{f}^{\underline{s}}$$

with some non-zero polynomial $b_{\underline{m}}(\underline{s})$.

This polynomial $b_{\underline{m}}(\underline{s})$ is called the *b-function* of $\underline{f}$. We write $b_i(\underline{s})$ instead of $b_{\varepsilon_i}(\underline{s})$ for an abbreviation. Let a-function $a_{\underline{m}}(\underline{s})$ be as in Lemma 2.2. The following lemmas about the structures of $b_i(\underline{s})$ and $b_{\underline{m}}(\underline{s})$ are stated in [25, Theorem 3], [26, Theorem 2]. In the following, we always assume that the a-function $a_{\underline{m}}(\underline{s})$ satisfies Assumption 2.4.

Lemma 2.6. *The b-function $b_i(\underline{s})$ is expressed as*

$$b_i(\underline{s}) = A_i \prod_{j=1}^{N} \prod_{\nu=0}^{\gamma_j(\varepsilon_i)-1} \prod_{r=1}^{\mu_j} (\gamma_j(\underline{s}) + \alpha_{j,r} + \nu)$$

with some $\alpha_{j,r} \in \boldsymbol{Q}_{>0}$.

Lemma 2.7. *The b-function $b_{\underline{m}}(\underline{s})$ is expressed as*

$$b_{\underline{m}}(\underline{s}) = \underline{A}^{\underline{m}} \prod_{j=1}^{N} \prod_{\nu=0}^{\gamma_j(\underline{m})-1} \prod_{r=1}^{\mu_j} (\gamma_j(\underline{s}) + \alpha_{j,r} + \nu)$$

with the same $\alpha_{j,r} \in \boldsymbol{Q}_{>0}$ as in Lemma 2.6.

Lemma 2.8. *Let the b-function $b_{\underline{m}}(\underline{s})$ be as in Lemma 2.7. Then the b-function $b_{\underline{f}^{\underline{m}}}(s)$ of the relative invariant $\underline{f}^{\underline{m}}$ $(\underline{m} \in \boldsymbol{Z}^l_{\geq 0})$ is given by*

$$b_{\underline{f}^{\underline{m}}}(s) = \underline{A}^{\underline{m}} \prod_{j=1}^{N} \prod_{\nu=0}^{\gamma_j(\underline{m})-1} \prod_{r=0}^{\mu_j} (\gamma_j(\underline{m})s + \alpha_{j,r} + \nu).$$

Remark 2.9. In the expression in Lemmas 2.6–2.8, we interpret the symbol of the product $\prod_{\nu=0}^{\gamma_j(\underline{m})-1}(\bullet)$ as follows:

$$\prod_{\nu=0}^{\gamma_j(\underline{m})-1}(\bullet) = \begin{cases} \prod_{\nu=0,1,\dots,\gamma_j(\underline{m})-1}(\bullet) & \text{if } \gamma_j(\underline{m}) > 0, \\ 1 & \text{if } \gamma_j(\underline{m}) = 0, \\ \prod_{\nu=-1,-2,\dots,\gamma_j(\underline{m})}(\bullet)^{-1} & \text{if } \gamma_j(\underline{m}) < 0. \end{cases}$$

When (G, ρ, V) is a regular prehomogeneous vector space, the b-function $b_{\underline{m}}(\underline{s})$ satisfies a certain functional equation, which comes from the fundamental theorem of prehomogeneous vector spaces.

LEMMA 2.10. *Let (G, ρ, V) be a regular prehomogeneous vector space. Then it has a relative invariant $\underline{f}$ corresponds to the character $\det \rho(g)^2$ where $\det \rho(g)$ denotes the determinant of $\rho(g)$ in V. We define $2\underline{\kappa} \in \boldsymbol{Z}^l$ by the condition*

$$\underline{f}^{2\underline{\kappa}}(\rho(g)v) = \det \rho(g)^2 \underline{f}^{2\underline{\kappa}}(v).$$

For the proof of the above lemma, see [23, Proposition 8 in §4]. The proof of the following proposition can be found in [25, Theorem 4].

PROPOSITION 2.11. *Let (G, ρ, V) be a regular prehomogeneous vector space. Then we have the following functional equations.*

$$b_{\underline{m}}(\underline{s}) = (-1)^{\deg \underline{f}^{\underline{m}}} b_{\underline{m}}(-\underline{s} - \underline{m} - \underline{\kappa})$$

where $\deg \underline{f}^{\underline{m}} = \sum_{i=1}^{l} m_i \deg f_i$.

Combining Lemma 2.6 and Proposition 2.11, we have the following theorem.

THEOREM 2.12. *Put*

$$\beta_{\gamma_j}(u) := \prod_{r=1}^{\mu_j} (u + \alpha_{j,r})$$

and let $\underline{\kappa}$ be as in Lemma 2.10. Then for each j, we have the following functional equation

$$\beta_{\gamma_j}(u) = (-1)^{\mu_j} \beta_{\gamma_j}(-u - \gamma_j(\underline{\kappa}) - 1)$$

for any u.

In the case of $l = 1$, we have the following proposition (cf. [24]).

PROPOSITION 2.13. *Let (G, ρ, V) be a regular prehomogeneous vector space with the irreducible relative invariant f. Denote by $d := \deg f$ and $n := \dim V$. Then we have the following functional equation:*

$$b_f(s) = (-1)^d b_f\left(-s - \frac{n}{d} - 1\right).$$

3. A Classification of Regular Simple Prehomogeneous Vector Spaces

Let G_s be a simple algebraic group defined over $\boldsymbol{C}$ and $\rho_s : G_s \to GL(V)$ be a rational representation of G_s on a finite dimensional vector space V. Assume that ρ_s and V are of the form $\rho_s = \rho_1 \oplus \cdots \oplus \rho_v$ and $V = V_1 \oplus \cdots \oplus V_v$ where $\rho_i : G_s \to GL(V_i)$ is an irreducible representation of G_s on V_i. We put $G = GL(1)^v \times G_s$ and define a representation ρ of G on V by

$$\rho(\alpha_1, \dots, \alpha_v; g)v = (\alpha_1 \rho_1(g)v_1, \dots, \alpha_v \rho_v(g)v_v)$$

for $v = (v_1, \dots, v_v) \in V$ and $(\alpha_1, \dots, \alpha_v; g) \in G$. By abuse of notation, we write $\rho = \rho_1 \oplus \cdots \oplus \rho_v$. If the triplet $(G, \rho, V) = (GL(1)^v \times G_s, \rho_1 \oplus \cdots \oplus \rho_v, V_1 \oplus \cdots \oplus V_v)$ is a prehomogeneous vector space, we call it a *simple prehomogeneous vector space*.

Simple prehomogeneous vector spaces are classified by M. Sato, T. Shintani and T. Kimura ([23], [24] and [14] with a correction [16]). For $v = 1$ (resp. $v \geq 2$), regular simple prehomogeneous vector spaces are classified in [23] (resp. [14]), and the results can be summarized as follows:

Proposition 3.1. *Let Λ' be a (odd or even) half-spin representation of $Spin(2n)$. When there is a possible ambiguity, we write Λ_e (resp. Λ_o) for the even (resp. odd) half-spin representation of $Spin(2n)$.*

Let Λ be the spin representation of $Spin(2n+1)$, and χ the vector representation of $Spin(m)$ so that $(Spin(m), \chi) \cong (SO(m), \Lambda_1)$.

We denote by l the number of fundamental relative invariants. Then all regular simple prehomogeneous vector spaces are given as follows:

(I) *Irreducible case* $(v = 1)$
(1) $\left(GL(1) \times SL(n), 2\Lambda_1, V\left(\frac{1}{2}n(n+1)\right)\right)$.
(2) $(GL(1) \times SL(2n), \Lambda_2, V(n(2n-1)))$.
(3) $(GL(1) \times SL(2), 3\Lambda_1, V(4))$.
(4) $(GL(1) \times SL(6), \Lambda_3, V(20))$.
(5) $(GL(1) \times SL(7), \Lambda_3, V(35))$.
(6) $(GL(1) \times SL(8), \Lambda_3, V(56))$.
(7) $(GL(1) \times SO(n), \Lambda_1, V(n))$.
(8) $(GL(1) \times Sp(3), \Lambda_3, V(14))$.
(9) $(GL(1) \times Spin(7), \Lambda, V(8))$.
(10) $(GL(1) \times Spin(9), \Lambda, V(16))$.
(11) $(GL(1) \times Spin(11), \Lambda, V(32))$.
(12) $(GL(1) \times Spin(12), \Lambda', V(32))$.

(13) $(GL(1) \times Spin(14), \Lambda', V(64))$.
(14) $(GL(1) \times G_2, \Lambda_2, V(7))$.
(15) $(GL(1) \times E_6, \Lambda_1, V(27))$.
(16) $(GL(1) \times E_7, \Lambda_6, V(56))$.

(II) *Non-irreducible case* $(v \geq 2)$.

(II-1) $l = 1$.
(17) $(GL(1)^2 \times SL(n), \Lambda_1 \oplus \Lambda_1^*, V(n) \oplus V(n)^*)$.
(18) $(GL(1)^n \times SL(n), \Lambda_1 \overbrace{\oplus \cdots \oplus}^{n} \Lambda_1, V(n) \overbrace{\oplus \cdots \oplus}^{n} V(n))$.
(19) $(GL(1)^2 \times SL(2n+1), \Lambda_2 \oplus \Lambda_1, V(n(2n+1)) \oplus V(2n+1))$.
(20) $(GL(1)^2 \times Sp(n), \Lambda_1 \oplus \Lambda_1, V(2n) \oplus V(2n))$.
(21) $(GL(1)^2 \times Spin(10), \Lambda_e \oplus \Lambda_e, V(16) \oplus V(16))$ (*or equivalently*, $(GL(1)^2 \times Spin(10), \Lambda_o \oplus \Lambda_o, V(16) \oplus V(16))$).

(II-2) $l = 2$
(22) $(GL(1)^3 \times SL(2n), \Lambda_2 \oplus \Lambda_1 \oplus \Lambda_1, V(n(2n-1)) \oplus V(2n) \oplus V(2n))$.
(23) $(GL(1)^3 \times SL(2n), \Lambda_2 \oplus \Lambda_1^* \oplus \Lambda_1^*, V(n(2n-1)) \oplus V(2n)^* \oplus V(2n)^*)$.
(24) $\left(GL(1)^2 \times SL(n), 2\Lambda_1 \oplus \Lambda_1, V\left(\frac{1}{2}n(n+1)\right) \oplus V(n)\right)$.
(25) $\left(GL(1)^2 \times SL(n), 2\Lambda_1 \oplus \Lambda_1^*, V\left(\frac{1}{2}n(n+1)\right) \oplus V(n)^*\right)$.
(26) $(GL(1)^2 \times SL(7), \Lambda_3 \oplus \Lambda_1, V(35) \oplus V(7))$.
(27) $(GL(1)^2 \times SL(7), \Lambda_3 \oplus \Lambda_1^*, V(35) \oplus V(7)^*)$.
(28) $(GL(1)^2 \times Sp(3), \Lambda_3 \oplus \Lambda_1, V(14) \oplus V(6))$.
(29) $(GL(1)^2 \times Spin(10), \chi \oplus \Lambda', V(10) \oplus V(16))$.
(30) $(GL(1)^3 \times SL(2n), \Lambda_2 \oplus \Lambda_1 \oplus \Lambda_1^*, V(n(2n-1)) \oplus V(2n) \oplus V(2n)^*)$.
(31) $(GL(1)^2 \times Spin(7), \chi \oplus \Lambda, V(7) \oplus V(8))$.
(32) $(GL(1)^2 \times Spin(8), \chi \oplus \Lambda', V(8) \oplus V(8))$.
(33) $(GL(1)^2 \times Spin(12), \chi \oplus \Lambda', V(12) \oplus V(32))$.

(II-3) $l = 3$.
(34) $(GL(1)^3 \times SL(2), \Lambda_1 \oplus \Lambda_1 \oplus \Lambda_1, V(2) \oplus V(2) \oplus V(2))$.

(II-4) $l \geq 4$.
(35) $(GL(1)^{n+1} \times SL(n), \Lambda_1 \overbrace{\oplus \cdots \oplus}^{n} \Lambda_1 \oplus \Lambda_1, V(n) \overbrace{\oplus \cdots \oplus}^{n} V(n) \oplus V(n))$.
(36) $(GL(1)^{n+1} \times SL(n), \Lambda_1 \overbrace{\oplus \cdots \oplus}^{n} \Lambda_1 \oplus \Lambda_1^*, V(n) \overbrace{\oplus \cdots \oplus}^{n} V(n) \oplus V(n)^*)$.
(37) $(GL(1)^4 \times SL(2n+1), \Lambda_2 \oplus \Lambda_1 \oplus \Lambda_1 \oplus \Lambda_1, V(n(2n+1)) \oplus V(2n+1) \oplus V(2n+1) \oplus V(2n+1))$.
(38) $(GL(1)^4 \times SL(2n+1), \Lambda_2 \oplus \Lambda_1 \oplus \Lambda_1^* \oplus \Lambda_1^*, V(n(2n+1)) \oplus V(2n+1) \oplus V(2n+1)^* \oplus V(2n+1)^*)$.

For the spaces (35) *and* (36), *we assume that* $n \geq 3$. *We have* $l = n + 1$ *for the spaces* (35) *and* (36), *and* $l = 4$ *for the spaces* (37) *and* (38).

Remark 3.2. Although the space (34) is a special case of (35), we treat them separately because the space (34) has finitely many orbits and its microlocal structure is determined by S. Kasai, while the space (35) with $n \geq 3$ has infinitely many orbits.

4. Known Results

In this section, we summarize the known results on the b-functions of regular simple prehomogeneous vector spaces. First Kimura [13] and Ozeki [20] determined the b-functions for all the regular simple prehomogeneous vector spaces with $v = 1$. When $v \geq 2$ and $l = 1$, the calculation of the b-functions can be reduced to the case of $v = 1$. In view of Proposition 3.1, there are twelve regular simple prehomogeneous vector spaces with $l = 2$. We can easily determine the b-functions of four of the twelve spaces, since these four spaces are direct sums of some regular prehomogeneous vector spaces whose b-functions are known. These are the cases of the spaces (30)–(33). Kasai investigated the remaining eight spaces by using the method of microlocal analysis, and Kasai determined their b-functions except for the space (28). A brief summary of his results is included in [8].

Now we describe a different aspect of the study on the determination of the b-functions. By using the results of [24] and [27], *one can derive the explicit form of the b-function from the explicit formula for the functional equation satisfied by the* (*local*) *zeta functions*. Thus it is enough to know the explicit formulae for the (local) functional equations for our purpose. In [30], T. Suzuki gave the explicit formulae for the functional equations of the spaces (24) and (25). The space (34), which has three relative invariants, is investigated by F. Sato [27], and we know the explicit formula for the functional equation satisfied by the zeta functions.

As for the spaces with $l \geq 4$, we found that each of the spaces (35)–(38) has infinitely many orbits ([17]). Hence it seems that the study from the view point of microlocal analysis is not so effective. However, T. Kimura [15] and H. Hosokawa [5] investigated these prehomogeneous vector spaces by using a different method. Kimura developed a method for explicit calculation of the functional equation satisfied by the local zeta functions attached to a prehomogeneous vector space. This method works without the finitely many orbits condition. On the other hand, in order to use his method, one needs to know the explicit form of the Igusa local zeta function associated with the prehomogeneous vector space in question. Hosokawa, however, determined the Igusa local zeta function of the spaces (35), (36) and (38). As its application, Kimura determined the explicit formulae for the functional equations (cf. [15, §4]), and hence we obtain the explicit forms of the b-functions of the spaces (35), (36) and (38). We note that Kasai [8] determined the b-functions of the spaces (35) and (36) by using the Capelli identity.

Hence the b-functions of the spaces (28) and (37) have been unknown before the present paper. We determine the b-function of the space (28) in §5 and that of the space (37) in §7.

Unfortunately, the article [8] is written in Japanese, and the results in [8] with detailed proofs have been published in separate papers [9, 10, 11]. For the convenience of readers, we shall give the tables of the explicit forms of the b-functions of non-irreducible regular simple prehomogeneous vector spaces in Appendix.

5. The Space (28)

In this section, we shall determine the b-function of the regular simple prehomogeneous vector space

$$(GL(1)^2 \times Sp(3), \Lambda_3 \oplus \Lambda_1, V(14) \oplus V(6)). \tag{28}$$

In [8], Kasai gave some comments on the microlocal structure of this space and posed a conjecture on the b-function. At first, following the Kasai's argument, we explain the reason why it is difficult to calculate the b-function by using microlocal calculus.

When we appeal to microlocal calculus, we first determine the orbital decomposition of a given prehomogeneous vector space and investigate the conormal bundles of the orbits. Then we construct the holonomy diagram which expresses how each conormal bundle intersects the others. A conormal bundle Λ that satisfies certain conditions is called a *good Lagrangian*, and we can associate with a good Lagrangian Λ, its local b-function $b_\Lambda(s)$. If Λ and Λ' are good Lagrangians whose intersection is of codimension one, we can determine the ratio $b_\Lambda(s)/b_{\Lambda'}(s)$ of the local b-functions. Finally we multiply all the ratios in the path that connects the conormal bundle of the origin and that of V, i.e., the zero section. As a result, we obtain the b-function, because the local b-function of the zero section is nothing but the original b-function. For the detail of the above, see [22].

The aspect of a holonomy diagram, however, is not so simple in some cases. For example, three good Lagrangians (say Λ, Λ' and Λ'') may have a common intersection of codimension one—it may happen that $\dim \Lambda \cap \Lambda' = \dim V - 1$ and $\Lambda \cap \Lambda' \subset \Lambda''$ (in the holonomy diagram, it is expressed as Figure 1). In such cases, we have to appeal to another formula in the determination of the ratios of the local b-functions. We call it the *three Lagrangians formula*. The whole theory of the formula, however, is not published yet and the only reference to the author's knowledge is Ozeki [19], in which Ozeki gave the three Lagrangians formula without the proof. Moreover, the description in [19] is applicable only to irreducible prehomogeneous vector spaces. For the three Lagrangians formula, see also [13, Remark 1.10], [18] and [34, Remark 2.4].

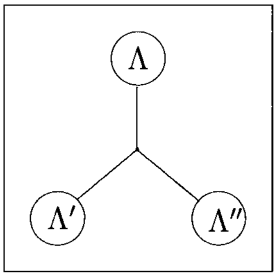

Figure 1: Three good Lagrangians

Let us return to our case. By looking at the holonomy diagram in [8, pp. 34–35], we observe that we *can not avoid* the three Lagrangians formula. We note that the same phenomenon does not occur for the other regular simple prehomogeneous vector spaces given in Proposition 3.1. Thus we can say that the prehomogeneous vector space in question is conspicuous for its microlocal structure. Our approach in the present paper does not rely on such advanced formulae which are developed with the aid of microlocal analysis.

Now we shall investigate the prehomogeneous vector space (28) in detail. First we define the irreducible regular prehomogeneous vector space $\left(GL(6), \Lambda_3, \bigwedge^3 V_1\right)$. Let $V_1 = \sum_{i=1}^{6} \boldsymbol{C} e_i$ be a 6-dimensional vector space with a basis $\{e_1, \dots, e_6\}$. The group $GL(6)$ acts on V_1 by $\Lambda_1(g)(e_1, \dots, e_6) = (e_1, \dots, e_6)g$ for $g \in GL(6)$. Denote by $\bigwedge^3 V_1$ the space of skew-symmetric tensors of rank 3, that is, $\bigwedge^3 V_1 = \sum_{1 \le i < j < k \le 6} \boldsymbol{C}(e_i \wedge e_j \wedge e_k)$. Then Λ_1 gives rise to the action Λ_3 of $GL(6)$ on $\bigwedge^3 V_1$ as follows: $\Lambda_3(g)(e_i \wedge e_j \wedge e_k) = (\Lambda_1(g)e_i) \wedge (\Lambda_1(g)e_j) \wedge (\Lambda_1(g)e_k)$ for $g \in GL(6)$. It is known that the triplet $(GL(6), \Lambda_3, \bigwedge^3 V_1)$ is an irreducible regular prehomogeneous vector space (see [23, Proposition 7 in §5]).

We shall give the irreducible relative invariant of this space explicitly. For $1 \le l \le 6$, we define the operator $\partial/\partial e_l$ on $\bigwedge^3 V_1$ by

$$\frac{\partial}{\partial e_l}(e_i \wedge e_j \wedge e_k) = \delta_{il} e_j \wedge e_k - \delta_{jl} e_i \wedge e_k + \delta_{kl} e_i \wedge e_j$$

where $\delta_{\mu\nu}$ is the Kronecker delta. Put $\tau = e_1 \wedge \cdots \wedge e_6$ and let $\tilde{x} = \sum x_{ijk} e_i \wedge e_j \wedge e_k$ be an element of $\bigwedge^3 V_1$. For $1 \le i, j \le 6$, we define the homogeneous polynomial $\varphi_{ij}(\tilde{x})$ of degree 2 on $\bigwedge^3 V_1$ by the relation

$$\tilde{x} \wedge \frac{\partial \tilde{x}}{\partial e_i} \wedge e_j = \varphi_{ij}(\tilde{x}) \tau.$$

Let $\varphi(\tilde{x}) = (\varphi_{ij}(\tilde{x}))_{1 \le i, j \le 6}$. Again by [23, Proposition 7 in §5], we have the following lemma.

LEMMA 5.1. (1) *We have* $\varphi(\Lambda_3(g)\tilde{x}) = (\det g) \cdot g\varphi(\tilde{x})g^{-1}$ *for* $g \in GL(6)$.
(2) *We have* $\varphi(\tilde{x})^2 = f(\tilde{x}) \cdot I_6$ *with some irreducible polynomial* $f(\tilde{x})$ *of degree* 4. *Moreover*, $f(\tilde{x})$ *is a relative invariant on* $\bigwedge^3 V_1$ *corresponds to the character* $(\det g)^2$.

Now we restrict Λ_3 to the subgroup

$$Sp(3) = \{g \in GL(6);\, {}^t g J g = J\}, \quad J = \left(\begin{array}{c|c} 0 & I_3 \\ \hline -I_3 & 0 \end{array}\right).$$

Then we have the decomposition $\bigwedge^3 V_1 = V(14) \oplus V(6)$ as the representation space of $Sp(3)$, since the restriction of weights $\{\lambda_i + \lambda_j + \lambda_k; 1 \le i < j < k \le 6\}$ of $GL(6)$ decomposes into $\{\pm\lambda_1 \pm \lambda_2 \pm \lambda_3, \pm\lambda_1, \pm\lambda_2, \pm\lambda_3\} \cup \{\pm\lambda_i; i = 1, 2, 3\}$. The action ρ_1 of $Sp(3)$ on $V(14)$ remains Λ_3, however, the action ρ_2 of $Sp(3)$ on $V(6)$ becomes Λ_1. Moreover, an element $\tilde{x} = \sum x_{ijk} e_i \wedge e_j \wedge e_k$ of $\bigwedge^3 V_1$ belongs to $V(14)$ if and only if $x_{i14} + x_{i25} + x_{i36} = 0$ for $1 \le i \le 6$. For $\tilde{x} \in \bigwedge^3 V_1$, we denote by (x, v) the element of $V(14) \oplus V(6)$ corresponding to $\tilde{x}$. We also denote by x an arbitrary element of $V(14)$ and let

$$V(14) = \left\{ x = \sum_{1 \le i < j < k \le 6} x_{ijk} e_i \wedge e_j \wedge e_k;\, x_{i14} + x_{i25} + x_{i36} = 0 \ \ (1 \le i \le 6) \right\}.$$

Let $G' = GL(1) \times Sp(3)$, $G = GL(1)^2 \times Sp(3)$, $V' = V(14)$ and $V = V(14) \oplus V(6)$. The group G' acts on V' by $\rho'(\alpha; g)x = \alpha\rho_1(g)x$ for $(\alpha; g) \in G'$ and $x \in V'$. Let $\tilde{g} = (\alpha, \beta; g) \in G$ and $\tilde{x} = (x, v) \in V$. Then we define the action ρ of G on V by $\rho(\tilde{g})\tilde{x} = (\alpha\rho_1(g)x, \beta\rho_2(g)v)$. By [23, Proposition 22 in §5], the triplet $(G', \rho', V') = (GL(1) \times Sp(3), \Lambda_3, V(14))$ is an irreducible regular prehomogeneous vector space, and by [14, Proposition 2.38], the triplet $(G, \rho, V) = (GL(1)^2 \times Sp(3), \Lambda_3 \oplus \Lambda_1, V(14) \oplus V(6))$ is a regular simple prehomogeneous vector space.

We shall consider the relative invariants of (G', ρ', V') and (G, ρ, V). Let $f(\tilde{x})$ be the irreducible relative invariant of $(GL(6), \Lambda_3, \bigwedge^3 V_1)$ given in Lemma 5.1. Denote by $f_1(x)$ the restriction of $f(\tilde{x})$ to the subspace $V(14)$:

$$f_1(x) = f|_{V(14)}. \tag{5.1}$$

By [23, pp. 108], we see that $f_1(x)$ remains irreducible as a polynomial on $V(14)$, and hence $f_1(x)$ is the irreducible relative invariant of (G', ρ', V'). Since the generic isotropy subalgebra of (G, ρ, V) is isomorphic to $\mathfrak{sl}_2$, we observe that (G, ρ, V) has two fundamental relative invariants by Lemma 1.3. Clearly the above $f_1(x)$ is one of the irreducible relative invariants of (G, ρ, V). Hence it remains to construct another relative invariant. We define the polynomial f_2 on V by

$$f_2(\tilde{x}) = \left(-\frac{1}{2}\right) \cdot {}^t vJ\varphi(x)v \tag{5.2}$$

for $\tilde{x} = (x, v) \in V$. By Lemma 5.1, we easily see that f_2 is a relative invariant of (G, ρ, V) corresponding to the character $\alpha^2\beta^2$. We can check the irreducibility of f_2 by using Lemma 1.3. Hence we have the following proposition.

Proposition 5.2. (1) *The triplet* $(G', \rho', V') = (GL(1) \times Sp(3), \Lambda_3, V(14))$ *is an irreducible regular prehomogeneous vector space. This space has the irreducible relative invariant* f_1 *of degree* 4, *which corresponds to the character* $\phi_1 = \alpha^4$.

(2) *The triplet* $(G, \rho, V) = (GL(1)^2 \times Sp(3), \Lambda_3 \oplus \Lambda_1, V(14) \oplus V(6))$ *is a regular simple prehomogeneous vector space. This space has two irreducible relative invariants* f_1 *of degree* 4 *and* f_2 *of degree* 4. *Let* ϕ_i *be the corresponding character of* f_i *for* $i = 1, 2$. *Then we have* $\phi_1 = \alpha^4$ *and* $\phi_2 = \alpha^2\beta^2$.

Remark 5.3. In [14, pp. 97], Kimura asserts that $f(\tilde{x})$ is again a relative invariant of (G, ρ, V). This is not correct, because the subgroup $\rho(G)$ of $GL(V)$ is *not* contained in $\Lambda_3(GL(6))$. This correction is already pointed out by Kasai ([8, pp. 63]). As we will see in Proposition 5.5, $f_2(\tilde{x})$ is of integral coefficients. The constant factor $(-1/2)$ in (5.2) is necessary in order that the common factor of the coefficients of $f_2(\tilde{x})$ is 1.

To simplify the notation, we write (ijk) instead of $e_i \wedge e_j \wedge e_k$. Since we have $x_{i14} + x_{i25} + x_{i36} = 0$ $(1 \leq i \leq 6)$, the set of vectors

$$\left\{ \begin{array}{l} (123), (124+136), (125-136), (126), (134-235), (135), \\ (145+356), (146-256), (156), (234), (245-346), \\ (246), (345), (456) \end{array} \right\} \tag{5.3}$$

is a basis of $V(14)$. We define the coordinates with respect to this basis as follows:

$$\begin{aligned} x = {} & x_{123}(123) + x_{124}(124+236) + x_{125}(125-136) + x_{126}(126) \\ & + x_{134}(134-235) + x_{135}(135) + x_{145}(145+356) + x_{146}(146-256) \\ & + x_{156}(156) + x_{234}(234) + x_{245}(245-346) + x_{246}(246) \\ & + x_{345}(345) + x_{456}(456). \end{aligned} \tag{5.4}$$

Expanding the relative invariants f_1 along the definition (5.1), we obtain the following result.

PROPOSITION 5.4.

$$\begin{aligned} f_1(x) = {} & 4x_{126}x_{145}^2x_{234} - 8x_{125}x_{145}x_{146}x_{234} - 4x_{135}x_{146}^2x_{234} \\ & + 4x_{124}x_{145}x_{156}x_{234} + 4x_{134}x_{146}x_{156}x_{234} + x_{156}^2x_{234}^2 \\ & - 8x_{126}x_{134}x_{145}x_{245} + 8x_{124}x_{135}x_{146}x_{245} - 8x_{123}x_{145}x_{146}x_{245} \\ & - 8x_{124}x_{134}x_{156}x_{245} - 4x_{125}x_{156}x_{234}x_{245} - 4x_{126}x_{135}x_{245}^2 \\ & - 4x_{123}x_{156}x_{245}^2 + 8x_{125}x_{134}x_{145}x_{246} - 4x_{124}x_{135}x_{145}x_{246} \\ & + 4x_{123}x_{145}^2x_{246} + 4x_{134}x_{135}x_{146}x_{246} - 4x_{134}^2x_{156}x_{246} \\ & - 2x_{135}x_{156}x_{234}x_{246} + 4x_{125}x_{135}x_{245}x_{246} + x_{135}^2x_{246}^2 \\ & + 4x_{124}x_{126}x_{145}x_{345} - 8x_{124}x_{125}x_{146}x_{345} - 4x_{126}x_{134}x_{146}x_{345} \\ & - 4x_{123}x_{146}^2x_{345} + 4x_{124}^2x_{156}x_{345} - 2x_{126}x_{156}x_{234}x_{345} \\ & + 4x_{125}x_{126}x_{245}x_{345} - 4x_{125}^2x_{246}x_{345} - 2x_{126}x_{135}x_{246}x_{345} \\ & - 4x_{123}x_{156}x_{246}x_{345} + x_{126}^2x_{345}^2 - 8x_{124}x_{125}x_{134}x_{456} \end{aligned}$$

$$- 4x_{126}x_{134}^2x_{456} + 4x_{124}^2x_{135}x_{456} - 4x_{123}x_{124}x_{145}x_{456}$$
$$- 4x_{123}x_{134}x_{146}x_{456} - 4x_{125}^2x_{234}x_{456} - 4x_{126}x_{135}x_{234}x_{456}$$
$$- 2x_{123}x_{156}x_{234}x_{456} - 4x_{123}x_{125}x_{245}x_{456} - 2x_{123}x_{135}x_{246}x_{456}$$
$$- 2x_{123}x_{126}x_{345}x_{456} + x_{123}^2x_{456}^2.$$

By abuse of notation, we denote by $\{e_1, \dots, e_6\}$ the basis of $V(6)$ such that the action ρ_2 of $Sp(3)$ is given by $(e_1, \dots, e_6) \mapsto (e_1, \dots, e_6)g$ for $g \in Sp(6)$. For $v \in V(6)$, we take the coordinates with respect to this basis as follows: $v = v_1e_1 + \cdots + v_6e_6$. Combining with (5.4), we then have the coordinates of $\tilde{x} = (x, v) \in V(14) \oplus V(6)$. We expand f_2 along the definition (5.2) and obtain the following proposition.

Proposition 5.5.

$$\begin{aligned}
f_2(\tilde{x}) = {} & v_1^2(x_{234}x_{456} + x_{245}^2 + x_{246}x_{345}) + 2v_1v_2(-x_{134}x_{456} - x_{145}x_{245} - x_{146}x_{345}) \\
& + 2v_1v_3(x_{124}x_{456} - x_{145}x_{246} + x_{146}x_{245}) \\
& + v_1v_4(-x_{123}x_{456} + 2x_{125}x_{245} + x_{135}x_{246} + x_{126}x_{345} - x_{156}x_{234}) \\
& + 2v_1v_5(-x_{124}x_{245} - x_{134}x_{246} + x_{146}x_{234}) \\
& + 2v_1v_6(-x_{124}x_{345} + x_{134}x_{245} - x_{145}x_{234}) \\
& + v_2^2(-x_{135}x_{456} + x_{145}^2 - x_{156}x_{345}) + 2v_2v_3(x_{125}x_{456} + x_{145}x_{146} + x_{156}x_{245}) \\
& + 2v_2v_4(-x_{125}x_{145} + x_{134}x_{156} - x_{135}x_{146}) \\
& + v_2v_5(-x_{123}x_{456} + 2x_{124}x_{145} - x_{135}x_{246} + x_{156}x_{234} + x_{126}x_{345}) \\
& + 2v_2v_6(-x_{125}x_{345} + x_{134}x_{145} + x_{135}x_{245}) + v_3^2(x_{126}x_{456} + x_{146}^2 + x_{156}x_{246}) \\
& + 2v_3v_4(-x_{124}x_{156} + x_{125}x_{146} - x_{126}x_{145}) \\
& + 2v_3v_5(x_{124}x_{146} + x_{125}x_{246} - x_{126}x_{245}) \\
& + v_3v_6(-x_{123}x_{456} + 2x_{134}x_{146} - x_{126}x_{345} + x_{156}x_{234} + x_{135}x_{246}) \\
& + v_4^2(x_{123}x_{156} + x_{125}^2 + x_{126}x_{135}) + 2v_4v_5(-x_{123}x_{146} - x_{124}x_{125} - x_{126}x_{134}) \\
& + 2v_4v_6(x_{123}x_{145} - x_{124}x_{135} + x_{125}x_{134}) + v_5^2(-x_{123}x_{246} + x_{124}^2 - x_{126}x_{234}) \\
& + 2v_5v_6(x_{123}x_{245} + x_{124}x_{134} + x_{125}x_{234}) + v_6^2(x_{134}^2 + x_{123}x_{345} + x_{135}x_{234}).
\end{aligned}$$

Now we consider the dual triplets of (G, ρ, V) and (G', ρ', V'). First we introduce a skew-symmetric bilinear form on $\bigwedge^3 V_1$ by

$$\langle (ijk), (lmn) \rangle = \operatorname{sgn}\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ i & j & k & l & m & n \end{pmatrix} \tag{5.5}$$

where sgn is the signature on the symmetric group S_6 which is extended by

$$\operatorname{sgn}\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ i & j & k & l & m & n \end{pmatrix} = 0 \quad \text{if } \{i, j, k, l, m, n\} \neq \{1, 2, 3, 4, 5, 6\}.$$

By [3, §2.9], the bilinear form (5.5) is $Sp(3)$-invariant.

Let $(V')^\vee = V(14)^\vee$ be the dual space of $V' = V(14)$ and $(\rho')^\vee : G' \to GL((V')^\vee)$ the contragredient representation of ρ'. We identify $(V')^\vee$ with V' by the bilinear form (5.5). Then the action $(\rho')^\vee$ of G' on $(V')^\vee$ is given by $(\rho')^\vee(\alpha; g)y = \alpha^{-1}\rho_1(g)y$ for $(\alpha; g) \in G'$ and $y \in (V')^\vee$. The dual basis of (5.3) is given as follows:

LEMMA 5.6.

$$(123)^\vee = (456), \quad (124 + 236)^\vee = -\frac{1}{2}(145 + 356),$$

$$(125 - 136)^\vee = -\frac{1}{2}(245 - 346), \quad (126)^\vee = -(345),$$

$$(134 - 235)^\vee = -\frac{1}{2}(146 - 256), \quad (135)^\vee = -(246),$$

$$(145 + 356)^\vee = \frac{1}{2}(124 + 236), \quad (146 - 256)^\vee = \frac{1}{2}(134 - 235),$$

$$(156)^\vee = (234), \quad (234)^\vee = -(156), \quad (245 - 346)^\vee = \frac{1}{2}(125 - 136),$$

$$(246)^\vee = (135), \quad (345)^\vee = (126), \quad (456)^\vee = -(123).$$

We define the coordinates of $(V')^\vee$ with respect to the above basis by

$$\begin{aligned} y = {} & y_{123}(123)^\vee + y_{124}(124 + 236)^\vee + y_{125}(125 - 136)^\vee + y_{126}(126)^\vee \\ & + y_{134}(134 - 235)^\vee + y_{135}(135)^\vee + y_{145}(145 + 356)^\vee + y_{146}(146 - 256)^\vee \\ & + y_{156}(156)^\vee + y_{234}(234)^\vee + y_{245}(245 - 346)^\vee + y_{246}(246)^\vee \\ & + y_{345}(345)^\vee + y_{456}(456)^\vee \end{aligned}$$

$$= -y_{456}(123) + \frac{1}{2}y_{145}(124+236) + \frac{1}{2}y_{245}(125-136) + y_{345}(126)$$
$$+ \frac{1}{2}y_{146}(134-235) + y_{246}(135) - \frac{1}{2}y_{124}(145+356) - \frac{1}{2}y_{134}(146-256)$$
$$- y_{234}(156) + y_{156}(234) - \frac{1}{2}y_{125}(245-346) - y_{135}(246)$$
$$- y_{126}(345) + y_{123}(456). \tag{5.6}$$

Let $f_1^\vee$ be a polynomial function on $(V')^\vee$ defined by

$$f_1^\vee(y) = f_1(y) \tag{5.7}$$

for any $y \in (V')^\vee \cong V'$. We observe that $f_1^\vee$ is a relative invariant of the prehomogeneous vector space $(G', (\rho')^\vee, (V')^\vee)$. Moreover, $f_1^\vee$ corresponds to the character $\phi_1^\vee = \alpha^{-4}$. We expand $f_1^\vee$ with respect to the coordinates (5.6) and obtain the following proposition.

Proposition 5.7.

$$\begin{aligned} f_1^\vee(y) = {} & y_{124}^2 y_{156} y_{345} - y_{124} y_{134} y_{156} y_{245} - y_{134}^2 y_{156} y_{246} \\ & + y_{124} y_{145} y_{156} y_{234} + y_{134} y_{146} y_{156} y_{234} + y_{156}^2 y_{234}^2 \\ & - y_{124} y_{125} y_{146} y_{345} + y_{125} y_{134} y_{145} y_{246} - y_{124} y_{125} y_{134} y_{456} \\ & - y_{125} y_{145} y_{146} y_{234} - y_{125} y_{156} y_{234} y_{245} - y_{125}^2 y_{246} y_{345} \\ & - y_{125}^2 y_{234} y_{456} + y_{124} y_{135} y_{146} y_{245} - y_{124} y_{135} y_{145} y_{246} \\ & + y_{124}^2 y_{135} y_{456} + y_{134} y_{135} y_{146} y_{246} - y_{135} y_{146}^2 y_{234} \\ & - 2 y_{135} y_{156} y_{234} y_{246} + y_{125} y_{135} y_{245} y_{246} + y_{135}^2 y_{246}^2 \\ & + y_{124} y_{126} y_{145} y_{345} - y_{126} y_{134} y_{145} y_{245} - y_{126} y_{134} y_{146} y_{345} \\ & - y_{126} y_{134}^2 y_{456} + y_{126} y_{145}^2 y_{234} - 2 y_{126} y_{156} y_{234} y_{345} \\ & + y_{125} y_{126} y_{245} y_{345} - y_{126} y_{135} y_{245}^2 - 2 y_{126} y_{135} y_{246} y_{345} \\ & - 4 y_{126} y_{135} y_{234} y_{456} + y_{126}^2 y_{345}^2 - y_{123} y_{145} y_{146} y_{245} \\ & - y_{123} y_{146}^2 y_{345} + y_{123} y_{145}^2 y_{246} - y_{123} y_{124} y_{145} y_{456} \end{aligned}$$

$$- y_{123}y_{134}y_{146}y_{456} - y_{123}y_{156}y_{245}^2 - 4y_{123}y_{156}y_{246}y_{345}$$
$$- 2y_{123}y_{156}y_{234}y_{456} - y_{123}y_{125}y_{245}y_{456} - 2y_{123}y_{135}y_{246}y_{456}$$
$$- 2y_{123}y_{126}y_{345}y_{456} + y_{123}^2y_{456}^2.$$

Now let us define a skew-symmetric bilinear form on $V(6)$ by

$$\langle v, w\rangle = {}^t vJw \tag{5.8}$$

for $v, w \in V(6)$. Clearly this bilinear form is $Sp(3)$-invariant. Let $V(6)^\vee$ be the dual space of $V(6)$. We identify $V(6)^\vee$ with $V(6)$ by the bilinear form (5.8). The dual basis $\{e_1^\vee, \dots, e_6^\vee\}$ of $\{e_1, \dots, e_6\}$ is given by

$$e_1^\vee = e_4, \quad e_2^\vee = e_5, \quad e_3^\vee = e_6, \quad e_4^\vee = -e_1, \quad e_5^\vee = -e_2, \quad e_6^\vee = -e_3.$$

We define the coordinates of $w \in V(6)^\vee$ with respect to the above basis as follows:

$$\begin{aligned} w &= w_1e_1^\vee + w_2e_2^\vee + w_3e_3^\vee + w_4e_4^\vee + w_5e_5^\vee + w_6e_6^\vee \\ &= -w_4e_1 - w_5e_2 - w_6e_3 + w_1e_4 + w_2e_5 + w_3e_6. \end{aligned} \tag{5.9}$$

Let $V^\vee = (V(14) \oplus V(6))^\vee = V(14)^\vee \oplus V(6)^\vee$ be the dual space of V, and $\rho^\vee : G \to GL(V^\vee)$ the contragredient representation of ρ. We identify $V^\vee$ with V by the bilinear forms (5.5), (5.8). Then the action $\rho^\vee$ of G on $V^\vee$ is given by $\rho^\vee(\tilde{g})\tilde{y} = (\alpha^{-1}\rho_1(g)y, \beta^{-1}\rho_2(g)w)$ for $\tilde{g} = (\alpha, \beta; g) \in G$ and $\tilde{y} = (y, w) \in V^\vee$.

We define the polynomial function $f_2^\vee$ on $V^\vee$ by

$$f_2^\vee(\tilde{y}) = 4f_2(\tilde{y}) \tag{5.10}$$

for any $\tilde{y} \in V^\vee \cong V$. Then, $f_2^\vee$ is a relative invariant of the prehomogeneous vector space $(G, \rho^\vee, V^\vee)$. Moreover, $f_2^\vee$ corresponds to the character $\phi_2^\vee = \alpha^{-2}\beta^{-2}$. We expand $f_2^\vee$ with respect to the coordinates (5.6), (5.9), and obtain the following result.

Proposition 5.8.

$$\begin{aligned} f_2^\vee(\tilde{y}) = {} & w_4^2(4y_{123}y_{156} + y_{125}^2 + 4y_{126}y_{135}) \\ & + 2w_4w_5(-2y_{123}y_{146} - y_{124}y_{125} - 2y_{126}y_{134}) \\ & + 2w_4w_6(2y_{123}y_{145} - 2y_{124}y_{135} + y_{125}y_{134}) \\ & - w_1w_4(4y_{123}y_{456} - 2y_{125}y_{245} - 4y_{135}y_{146} - 4y_{126}y_{345} + 4y_{156}y_{234}) \\ & - 2w_2w_4(y_{125}y_{145} + 2y_{135}y_{146} - 2y_{134}y_{156}) \end{aligned}$$

$$
\begin{aligned}
&- 2w_3w_4(2y_{126}y_{145} - y_{125}y_{146} + 2y_{124}y_{156}) \\
&+ w_5^2(-4y_{123}y_{246} + y_{124}^2 - 4y_{126}y_{234}) \\
&+ 2w_5w_6(2y_{123}y_{245} + y_{124}y_{134} + 2y_{125}y_{234}) \\
&- 2w_1w_5(y_{124}y_{245} - 2y_{146}y_{234} + 2y_{134}y_{246}) \\
&- w_2w_5(4y_{123}y_{456} - 2y_{124}y_{145} + 4y_{135}y_{246} - 4y_{156}y_{234} - 4y_{126}y_{345}) \\
&- 2w_3w_5(2y_{126}y_{245} - y_{124}y_{146} - 2y_{125}y_{246}) \\
&+ w_6^2(4y_{123}y_{345} + y_{134}^2 + 4y_{135}y_{234}) \\
&- 2w_1w_6(2y_{145}y_{234} - y_{134}y_{245} + 2y_{124}y_{345}) \\
&- 2w_2w_6(-y_{134}y_{145} - 2y_{135}y_{245} + 2y_{125}y_{345}) \\
&- w_3w_6(4y_{123}y_{456} - 2y_{134}y_{146} + 4y_{126}y_{345} - 4y_{156}y_{234} - 4y_{135}y_{246}) \\
&+ w_1^2(4y_{234}y_{456} + y_{245}^2 + 4y_{246}y_{345}) \\
&+ 2w_1w_2(-2y_{134}y_{456} - y_{145}y_{245} - 2y_{246}y_{345}) \\
&+ 2w_1w_3(2y_{124}y_{456} - 2y_{145}y_{246} + y_{146}y_{245}) \\
&+ w_2^2(-4y_{135}y_{456} + y_{145}^2 - 4y_{156}y_{345}) \\
&+ 2w_2w_3(2y_{125}y_{456} + y_{145}y_{146} + 2y_{156}y_{245}) \\
&+ w_3^2(y_{146}^2 + 4y_{126}y_{456} + 4y_{156}y_{246}).
\end{aligned}
$$

We shall determine the b-function $b_{f_1}(s)$ of f_1. Although this b-function is already calculated by T. Kimura [13], our approach is far different from his one, namely, microlocal calculus. It may be of some interests to introduce our method, which is initiated by K. Ukai [31].

Let $b_{f_1}(s) = \sum_{i=0}^{4} b_i^{(1)} s^{4-i} \in \boldsymbol{C}[s]$. By Lemma 1.6 (i), we have

$$b_0^{(1)} = f_1^{\vee}(\operatorname{grad}\log f_1(x)) f_1(x)$$

for generic x. Recall that the rational map $\operatorname{grad}\log f_1$ is defined by

$$\operatorname{grad}\log f_1(x) = \left(\frac{1}{f_1(x)}\cdot\frac{\partial f_1}{\partial x_{123}}(x), \ldots, \frac{1}{f_1(x)}\cdot\frac{\partial f_1}{\partial x_{456}}(x)\right).$$

We see that $x_0 = (123) + (456) \in V' = V(14)$ is a generic point of (G', ρ', V') by calculating the isotropy subalgebra at x_0. Then, by Proposition 5.4, we have

$$\frac{\partial f_1}{\partial x_{123}}(x_0) = \frac{\partial f_1}{\partial x_{456}}(x_0) = 2,$$

$$\frac{\partial f_1}{\partial x_{ijk}}(x_0) = 0 \quad \text{otherwise}.$$

Hence it follows from Proposition 5.7 that $b_0^{(1)} = 2^4$. By using Lemma 1.6 (ii), we can determine the constant term $b_4^{(1)}$ of $b_{f_1}(s)$ and the result is $b_4^{(1)} = 280 = 2^3 \cdot 5 \cdot 7$.

Now we write $b_{f_1}(s) = b_0^{(1)} \prod_{i=1}^4 (s + \alpha_i)$. We have that $\alpha_i \in \boldsymbol{Q}_{>0}$ by Theorem 1.7. Since $\alpha_1\alpha_2\alpha_3\alpha_4 = b_4^{(1)}/b_0^{(1)}$, we obtain

$$\alpha_1\alpha_2\alpha_3\alpha_4 = \frac{5 \cdot 7}{2}. \tag{5.11}$$

Moreover, by Proposition 2.13, the following functional equation holds:

$$b_{f_1}(s) = b_{f_1}\left(-s - \frac{9}{2}\right). \tag{5.12}$$

In view of Lemma 1.6 (1), we may assume that $\alpha_1 = 1$. Hence it follows from (5.12) that $(s + 7/2)$ divides $b_{f_1}(s)$. Letting $\alpha_4 = 7/2$, we obtain $\alpha_2 + \alpha_3 = 9/2$ and $\alpha_2\alpha_3 = 5$ by (5.11) and (5.12). This immediately implies the following.

PROPOSITION 5.9.

$$b_{f_1}(s) = 2^4(s+1)(s+2)\left(s + \frac{5}{2}\right)\left(s + \frac{7}{2}\right).$$

Next we shall calculate the a-functions $a_{\underline{m}}(\underline{s})$. Let $v_0 = e_1 + e_4 \in V(6)$ and $\tilde{x}_0 = (x_0, v_0) \in V = V(14) \oplus V(6)$. Then we observe that $\tilde{x}_0$ is a generic point of (G, ρ, V). Moreover, by using Proposition 5.8, we obtain

$$\operatorname{grad} \log f_2(\tilde{x}_0) = ((123)^\vee - (156)^\vee - (234)^\vee + (456)^\vee, e_1^\vee + e_4^\vee).$$

Hence it follows from Lemma 2.1 and Proposition 5.7 that

$$\begin{aligned} a_1(\underline{s}) &= f_1^\vee(\operatorname{grad} \log \underline{f}^{\underline{s}}(\tilde{x}_0)) f_1(\tilde{x}_0) \\ &= f_1^\vee(s_1 \operatorname{grad} \log f_1(x_0) + s_2 \operatorname{grad} \log f_2(\tilde{x}_0)) f_1(x_0) \\ &= (-s_2)^4 - 2(-s_2)^2(2s_1 + s_2)^2 + (2s_1 + s_2)^4 \\ &= 2^4 s_1^2 (s_1 + s_2)^2. \end{aligned}$$

Let

$$a_{\underline{m}}(\underline{s}) = \underline{A}^{\underline{m}} \prod_{j=1}^{N} (\gamma_j(\underline{s})^{\gamma_j(\underline{m})})^{\mu_j}$$

be the *a*-function as in Lemma 2.2. Since $a_1(\underline{s})$ is as above, we have $N = 3$ and may assume that

$$\gamma_1(\underline{s}) = s_1, \quad \gamma_2(\underline{s}) = s_2, \quad \gamma_3(\underline{s}) = s_1 + s_2.$$

Moreover we see that $\mu_1 = 2$, $\mu_3 = 2$ and $A_1 = b_0^{(1)} = 2^4$. Hence it remains to determine μ_2 and A_2. Since $a_2(\underline{s}) = A_2 s_2^{\mu_2}(s_1 + s_2)^2$ and $\deg f_2 = 4$, we get $\mu_2 = 2$. We can calculate the leading coefficient $b_0^{(2)}$ of $b_{f_2}(s)$ by using Lemma 1.6, and consequently $A_2 = b_0^{(2)} = 2^4$. We therefore obtain the following proposition.

Proposition 5.10.

$$a_{\underline{m}}(\underline{s}) = 2^{4(m_1+m_2)} s_1^{2m_1} s_2^{2m_2} (s_1 + s_2)^{2(m_1+m_2)}.$$

Now we shall consider the *b*-function $b_{\underline{f}^{\underline{m}}}(s)$ of $\underline{f}^{\underline{m}}$. Note that the above *a*-function satisfies Assumption 2.4 and consequently we can apply Lemmas 2.6–2.8 to the *b*-functions. By Lemma 2.6, the *b*-functions $b_1(\underline{s})$ and $b_2(\underline{s})$ are expressed as

$$b_1(\underline{s}) = 2^4(s_1 + \alpha_{1,1})(s_1 + \alpha_{1,2})(s_1 + s_2 + \alpha_{3,1})(s_1 + s_2 + \alpha_{3,2}),$$

$$b_2(\underline{s}) = 2^4(s_2 + \alpha_{2,1})(s_2 + \alpha_{2,2})(s_1 + s_2 + \alpha_{3,1})(s_1 + s_2 + \alpha_{3,2})$$

with some $\alpha_{j,r} \in \boldsymbol{Q}_{>0}$. Let us describe the functional equations explicitly. Since $\det \rho(g)^2 = \alpha^{28}\beta^{12}$, we see that $\underline{\kappa}$ in Lemma 2.10 is given by $\underline{\kappa} = (2, 3)$. Put

$$\beta_{\gamma_1}(u) := \prod_{r=1}^{2}(u + \alpha_{1,r}), \quad \beta_{\gamma_2}(u) := \prod_{r=1}^{2}(u + \alpha_{2,r}), \quad \beta_{\gamma_3}(u) := \prod_{r=1}^{2}(u + \alpha_{3,r}).$$

Then, by Theorem 2.12, we have

$$\beta_{\gamma_1}(u) = \beta_{\gamma_1}(-u-3), \quad \beta_{\gamma_2}(u) = \beta_{\gamma_2}(-u-4), \quad \beta_{\gamma_3}(u) = \beta_{\gamma_3}(-u-6),$$

and hence

$$\begin{aligned} \{\alpha_{1,1}, \alpha_{1,2}\} &= \{3 - \alpha_{1,1}, 3 - \alpha_{1,2}\}, \\ \{\alpha_{2,1}, \alpha_{2,2}\} &= \{4 - \alpha_{2,1}, 4 - \alpha_{2,2}\}, \\ \{\alpha_{3,1}, \alpha_{3,2}\} &= \{6 - \alpha_{3,1}, 6 - \alpha_{3,2}\}. \end{aligned} \tag{5.13}$$

By the definition of $b_{f_1}(s)$, it follows that $b_1((s,0)) = b_{f_1}(s)$. Combining this with Proposition 5.9, we have

$$\{\alpha_{1,1},\alpha_{1,2},\alpha_{3,1},\alpha_{3,2}\} = \left\{1,2,\frac{5}{2},\frac{7}{2}\right\}. \tag{5.14}$$

By (5.13) and (5.14), we get $\{\alpha_{1,1},\alpha_{1,2}\} = \{1,2\}$ and $\{\alpha_{3,1},\alpha_{3,2}\} = \{5/2,7/2\}$. Since we also have that $b_2((0,s)) = b_{f_2}(s)$, $\{\alpha_{2,1},\alpha_{2,2},5/2,7/2\}$ are the roots of the polynomial $b_{f_2}(s)$. However, in view of Lemma 1.6 (1), either $\alpha_{2,1}$ or $\alpha_{2,2}$ must be equal to 1, and thus we get $\{\alpha_{2,1},\alpha_{2,2}\} = \{1,3\}$ by (5.13). We therefore obtain the following results:

$$b_1(\underline{s}) = 2^4(s_1+1)(s_1+2)\left(s_1+s_2+\frac{5}{2}\right)\left(s_1+s_2+\frac{7}{2}\right),$$

$$b_2(\underline{s}) = 2^4(s_2+1)(s_2+3)\left(s_1+s_2+\frac{5}{2}\right)\left(s_1+s_2+\frac{7}{2}\right).$$

Finally, Lemma 2.8 leads us to the following theorem.

THEOREM 5.11. *Let* $\underline{f} = (f_1, f_2)$ *be the fundamental relative invariants of the prehomogeneous vector space* (28) $(GL(1)^2 \times Sp(3), \Lambda_3 \oplus \Lambda_1, V(14) \oplus V(6))$, *and* $\underline{f}^\vee = (f_1^\vee, f_2^\vee)$ *the fundamental relative invariants of the dual prehomogeneous vector space. We normalize* f_1 (*resp.* $f_2, f_1^\vee, f_2^\vee$) *as* (5.1) (*resp.* (5.2), (5.7), (5.10)). *Then we have*

$$\begin{aligned} b_{\underline{f}^{\underline{m}}}(s) = 2^{4(m_1+m_2)} &\left\{\prod_{v=0}^{m_1-1}(m_1 s+1+v)(m_1 s+2+v)\right\} \\ &\times\left\{\prod_{v=0}^{m_2-1}(m_2 s+1+v)(m_2 s+3+v)\right\} \\ &\times\left\{\prod_{v=0}^{m_1+m_2-1}\left((m_1+m_2)s+\frac{5}{2}+v\right)\left((m_1+m_2)s+\frac{7}{2}+v\right)\right\}. \end{aligned}$$

Hence the conjecture of S. Kasai [8, pp. 65] *is true.*

6. Expansion of Pfaffians

In this section, we shall give a formula for expansion of Pfaffians. First let us recall the definition of Pfaffians (cf. [21]).

Let $\mathrm{Alt}(m)$ be the space of alternating matrices of degree m. We take an $X \in \mathrm{Alt}(m)$. If m is odd, obviously $\det X = 0$. When m is even, say $m = 2n$, there exists a polynomial $\mathrm{Pf}(X)$ of degree n such that

(1) $\det X = \mathrm{Pf}(X)^2$,
(2) $\mathrm{Pf}(J_n) = (-1)^{n(n-1)/2}$ for $J_n = \begin{pmatrix} 0_n & I_n \\ -I_n & 0_n \end{pmatrix}$.

The polynomial $\mathrm{Pf}(X)$ is called the *Pfaffian* of $X \in \mathrm{Alt}(2n)$.

Now we introduce a notation to simplify the expression of an alternating matrix. Let E_{ij} be the (i, j)-th matrix unit of $M(m)$, that is, the $m \times m$ matrix such that the (i, j)-th component is 1 and the other components are 0. When there is a possible ambiguity on the size of the matrix, we write $E_{ij}^{(m)}$ in place of E_{ij}. We define $D_{ij} \in \mathrm{Alt}(m)$ by $D_{ij} = E_{ij} - E_{ji}$. As before, we write $D_{ij}^{(m)}$ in place of D_{ij}, if we want to make the size clear.

Take an arbitrary alternating matrix of size $2n$;

$$X = \begin{pmatrix} 0 & x_{12} & x_{13} & \cdots & x_{1,2n} \\ -x_{12} & 0 & x_{23} & \cdots & x_{2,2n} \\ -x_{13} & -x_{23} & 0 & \cdots & x_{3,2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -x_{1,2n} & -x_{2,2n} & -x_{3,2n} & \cdots & 0 \end{pmatrix} \in \mathrm{Alt}(2n).$$

Then the above X is expressed as

$$X = \sum_{1 \le i \le j \le 2n} x_{ij} D_{ij}^{(2n)}.$$

Now we take natural numbers $k_1, k_2, \ldots, k_l$ such that $1 \le k_\mu \le 2n$ and $k_\mu \ne k_\nu$ if $\mu \ne \nu$. Then we define the $l \times l$ alternating matrix

$$X(k_1, k_2, k_3, \ldots, k_l) \tag{6.1}$$

by the following formula:

$$\begin{aligned} X(k_1, k_2, k_3, \ldots, k_l) &= \sum_{1 \le \mu < \nu < l} x_{k_\mu, k_\nu} D_{\mu\nu}^{(l)} \\ &= \begin{pmatrix} 0 & x_{k_1,k_2} & x_{k_1,k_3} & \cdots & x_{k_1,k_l} \\ -x_{k_1,k_2} & 0 & x_{k_2,k_3} & \cdots & x_{k_2,k_l} \\ -x_{k_1,k_3} & -x_{k_2,k_3} & 0 & \cdots & x_{k_3,k_l} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -x_{k_1,k_l} & -x_{k_2,k_l} & -x_{k_3,k_l} & \cdots & 0 \end{pmatrix} \in \mathrm{Alt}(l). \end{aligned}$$

Here we interpret the symbol x_{k_μ, k_ν} with $k_\mu > k_\nu$ as $-x_{k_\nu, k_\mu}$. So, in this notation, the original $X \in \mathrm{Alt}(2n)$ is expressed as

$$X(1, 2, 3, \dots, 2n). \tag{6.2}$$

Now let us give some examples. In the following, $\check{\ }$ denotes that this element is removed.

EXAMPLE 6.1. (1) We have

$$X(1, \dots, \check{i}, \dots, 2n) = \begin{pmatrix} 0 & x_{12} & \cdots & x_{1,i-1} & x_{1,i+1} & \cdots & x_{1,2n} \\ -x_{12} & 0 & \cdots & x_{2,i-1} & x_{2,i+1} & \cdots & x_{2,2n} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ -x_{1,i-1} & -x_{2,i-1} & \cdots & 0 & x_{i-1,i+1} & \cdots & x_{i-1,2n} \\ -x_{1,i+1} & -x_{2,i+1} & \cdots & -x_{i-1,i+1} & 0 & \cdots & x_{i+1,2n} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ -x_{1,2n} & -x_{2,2n} & \cdots & -x_{i-1,2n} & -x_{i+1,2n} & \cdots & 0 \end{pmatrix}.$$

(2) If we interchange ith and jth rows of X, and in addition, ith and jth columns of X, we obtain the alternating matrix

$$X(1, \dots, \overset{i}{j}, \dots, \overset{j}{i}, \dots, 2n) = \begin{matrix} \\ \\ i \\ \\ j \\ \\ \\ \end{matrix} \overset{\begin{matrix} & & i & & j & & \end{matrix}}{\begin{pmatrix} 0 & \cdots & x_{1j} & \cdots & x_{1i} & \cdots & x_{1,2n} \\ \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ -x_{1j} & \cdots & 0 & \cdots & -x_{ij} & \cdots & x_{j,2n} \\ \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ -x_{1i} & \cdots & x_{ij} & \cdots & 0 & \cdots & x_{i,2n} \\ \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ -x_{1,2n} & \cdots & -x_{j,2n} & \cdots & -x_{i,2n} & \cdots & 0 \end{pmatrix}}.$$

REMARK 6.2. (1) As for the notation such as (6.1) and (6.2), the author was much inspired by the book of R. Hirota [4], especially, Chapter 2. However, we slightly modify his notation to make it fitting for our calculation in the present paper.

(2) The expansion formula in Proposition 6.5 was discovered in some various contexts. For example, see [1, 4, 6]. The author, however, shall give a proof of the formula for the sake of completeness.

Since $\mathrm{Pf}(AX{}^tA) = \det A \cdot \mathrm{Pf}(X)$ for $A \in GL(2n)$ and $X \in \mathrm{Alt}(2n)$, we have the following lemma.

Lemma 6.3.

$$\mathrm{Pf}\left[X(1,\dots,\overset{i}{j},\dots,\overset{j}{i},\dots,2n)\right] = (-1)\cdot \mathrm{Pf}(X).$$

The following formula is proved in [21, pp. 95].

Proposition 6.4. *For $X \in \mathrm{Alt}(2n)$, we have*

$$\mathrm{Pf}(X) = \sum \mathrm{sgn}\begin{pmatrix} 1 & 2 & \cdots & 2n-1 & 2n \\ i_1 & i_2 & \cdots & i_{2n-1} & i_{2n}\end{pmatrix} x_{i_1 i_2}\cdots x_{i_{2n-1} i_{2n}},$$

where the sum runs over $(i_1, i_2, \dots, i_{2n-1}, i_{2n}) \in \boldsymbol{Z}_{>0}^{2n}$ such that

(i) $\{i_1, i_2, \dots, i_{2n-1}, i_{2n}\}$ *is a permutation of* $\{1, 2, \dots, 2n-1, 2n\}$, *and*
(ii) $i_1 < i_2, i_3 < i_4, \dots, i_{2n-1} < i_{2n}$, *and*
(iii) $i_1 < i_3 < i_5 < \cdots < i_{2n-1}$.

By Proposition 6.4, we obtain the following formula, which we call *the expansion of* $\mathrm{Pf}(X)$ *along the* i_0*th hock.*

Proposition 6.5. *We fix a number i_0 with $1 \le i_0 \le 2n$. Then* $\mathrm{Pf}(X)$ *is expressed as*

$$\mathrm{Pf}(X) = \sum_{j=1}^{i_0-1} (-1)^{i_0+j-1} x_{j i_0}\,\mathrm{Pf}[X(1,\dots,\check{j},\dots,\check{i}_0,\dots,2n)]$$
$$+ \sum_{j=i_0+1}^{2n} (-1)^{i_0+j-1} x_{i_0 j}\,\mathrm{Pf}[X(1,\dots,\check{i}_0,\dots,\check{j},\dots,2n)].$$

Proof. Let $\mathscr{S}$ be the set of indices $(i_1, i_2, \dots, i_{2n-1}, i_{2n}) \in \boldsymbol{Z}_{>0}^{2n}$ with the conditions (i), (ii), (iii) in Proposition 6.4. For $2 \le j \le 2n$, we denote by $\mathscr{S}_j$

the subset of $\mathscr{S}$ consisting of indices $(i_1, i_2, \dots, i_{2n-1}, i_{2n})$ with $i_1 = 1$ and $i_2 = j$. It is easy to see that $\mathscr{S} = \mathscr{S}_2 \sqcup \cdots \sqcup \mathscr{S}_{2n}$. By Proposition 6.4, we have

$$\begin{aligned}&\mathrm{Pf}[X(2, 3, \dots, \check{j}, \dots, 2n)] \\ &\quad = \sum \mathrm{sgn}\begin{pmatrix} 2 & 3 & \cdots & j-1 & j+1 & \cdots & 2n \\ k_1 & k_2 & \cdots & k_{j-2} & k_{j-1} & \cdots & k_{2n-2} \end{pmatrix} x_{k_1 k_2} \cdots x_{k_{2n-3}, k_{2n-2}}.\end{aligned}$$

Thus we observe that

$$\begin{aligned}&\sum_{\mathscr{S}_j} \mathrm{sgn}\begin{pmatrix} 1 & 2 & \cdots & 2n-1 & 2n \\ i_1 & i_2 & \cdots & i_{2n-1} & i_{2n} \end{pmatrix} x_{i_1 i_2} \cdots x_{i_{2n-1} i_{2n}} \\ &\quad = \mathrm{sgn}\begin{pmatrix} 1 & 2 & 3 & 4 & \cdots & j-1 & j & j+1 & \cdots & 2n-1 & 2n \\ 1 & j & 2 & 3 & \cdots & j-2 & j-1 & j+1 & \cdots & 2n-1 & 2n \end{pmatrix} \\ &\qquad \times x_{1j} \times \mathrm{Pf}[X(2, 3, \dots, \check{j}, \dots, 2n)] \\ &\quad = (-1)^{j-2} \cdot x_{1j} \cdot \mathrm{Pf}[X(2, 3, \dots, \check{j}, \dots, 2n)].\end{aligned}$$

Hence it follows from Proposition 6.4 that

$$\begin{aligned}\mathrm{Pf}(X) &= \sum_{j=2}^{2n} \sum_{\mathscr{S}_j} \mathrm{sgn}\begin{pmatrix} 1 & 2 & \cdots & 2n-1 & 2n \\ i_1 & i_2 & \cdots & i_{2n-1} & i_{2n} \end{pmatrix} x_{i_1 i_2} \cdots x_{i_{2n-1} i_{2n}} \\ &= \sum_{j=2}^{2n} (-1)^j \cdot x_{1j} \cdot \mathrm{Pf}[X(2, 3, \dots, \check{j}, \dots, 2n)].\end{aligned}$$

This proves the proposition for $i_0 = 1$. Now we have

$$\mathrm{Pf}[X(1, 2, \dots, i_0, \dots, 2n)] = (-1)^{i_0 - 1} \cdot \mathrm{Pf}[X(i_0, 1, 2, \dots, i_0 - 1, i_0 + 1, \dots, 2n)]$$

by Lemma 6.3. Noticing that

$$\begin{aligned}&X(i_0, 1, 2, \dots, i_0 - 1, i_0 + 1, \dots, 2n) \\ &\quad = \begin{pmatrix} 0 & -x_{1 i_0} & -x_{2 i_0} & \cdots & -x_{i_0 - 1, i_0} & x_{i_0, i_0 + 1} & \cdots & x_{i_0, 2n} \\ x_{1 i_0} & 0 & x_{12} & \cdots & x_{1, i_0 - 1} & x_{1, i_0 + 1} & \cdots & x_{1, 2n} \\ x_{2 i_0} & -x_{12} & 0 & \cdots & x_{2, i_0 - 1} & x_{2, i_0 + 1} & \cdots & x_{2, 2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \end{pmatrix},\end{aligned}$$

we have

$$\begin{aligned}
&\mathrm{Pf}[X(i_0, 1, 2, \dots, i_0 - 1, i_0 + 1, \dots, 2n)] \\
&\quad = \sum_{j=1}^{i_0-1} (-1)^{j+1} \cdot (-x_{ji_0}) \cdot \mathrm{Pf}[X(1, \dots, \check{j}, \dots, \check{i}_0, \dots, 2n)] \\
&\qquad + \sum_{j=i_0+1}^{2n} (-1)^{j} \cdot x_{i_0 j} \cdot \mathrm{Pf}[X(1, \dots, \check{i}_0, \dots, \check{j}, \dots, 2n)].
\end{aligned}$$

We therefore obtain our assertion. □

In §7, we shall calculate the value of $\operatorname{grad}\log f$ at a generic point for some relative invariants f related to Pfaffians, by using the expansion formula above. Let us illustrate our method with the following example.

Example 6.6. We consider the irreducible regular prehomogeneous vector space $(H, \sigma, W) = (GL(2n), \Lambda_2, V(n(2n-1)))$. When we identify $W = V(n(2n-1))$ with $\mathrm{Alt}(2n)$, the representation $\sigma = \Lambda_2$ is explicitly given as follows: $\sigma(A)X = AX{}^tA$ for $A \in H$ and $X \in W$. Then $f(X) := \mathrm{Pf}(X)$ is the irreducible relative invariant of this space and $X_0 := J_n$ is a generic point. Let us define a bilinear form on $\mathrm{Alt}(2n)$ by $\langle X, Y \rangle = (1/2)\,\mathrm{tr}\,{}^tXY$. We identify the dual space $W^\vee$ of W with $\mathrm{Alt}(2n)$ by this bilinear form. The action $\sigma^\vee$ of H on $W^\vee$ is given by $\sigma^\vee(A)Y = {}^tA^{-1}YA^{-1}$ for $H \in H$ and $Y \in W^\vee = \mathrm{Alt}(2n)$. We take $D_{ij} = E_{ij} - E_{ji}$ $(1 \le i < j \le 2n)$ as a basis of $W = \mathrm{Alt}(2n)$. Its dual basis of $W^\vee$ is given by

$$(D_{ij})^\vee = (E_{ij} - E_{ji})^\vee = E_{ij} - E_{ji} \quad (1 \le i < j \le 2n).$$

Then we have

$$(\operatorname{grad}\log f)(X) = \sum_{1 \le i < j \le 2n} \frac{1}{f(X)} \cdot \frac{\partial f}{\partial x_{ij}}(X) \cdot D_{ij} \in W^\vee = \mathrm{Alt}(2n)$$

for $X \in W = \mathrm{Alt}(2n)$. By Proposition 6.5, it follows that

$$\frac{\partial f}{\partial x_{ij}}(X) = (-1)^{i+j-1} \cdot \mathrm{Pf}[X(1, \dots, \check{i}, \dots, \check{j}, \dots, 2n)]$$

for $1 \le i < j \le 2n$, and so

$$\frac{\partial f}{\partial x_{ij}}(X_0) = (-1)^{i+j-1} \cdot \mathrm{Pf}[J_n(1, \dots, \check{i}, \dots, \check{j}, \dots, 2n)].$$

By an easy consideration, we see that $\mathrm{Pf}[J_n(1,\dots,\check{i},\dots,\check{j},\dots,2n)]$ is not equal to zero if and only if $1 \leq i \leq n$ and $j = n+i$. Moreover, we obtain

$$\left(\frac{\partial f}{\partial x_{i,n+i}}\right)(X_0) = (-1)^{i+(n+i)-1} \cdot \mathrm{Pf}(J_{n-1}) = (-1)^{n(n-1)/2}.$$

Together with $f(X_0) = \mathrm{Pf}(J_n) = (-1)^{n(n-1)/2}$, we have

$$\begin{aligned}(\operatorname{grad}\log f)(X_0) &= \sum_{1 \leq i < j \leq 2n} \frac{1}{f(X_0)} \cdot \frac{\partial f}{\partial x_{ij}}(X_0) \cdot D_{ij} \\ &= \sum_{i=1}^{n} \frac{1}{\mathrm{Pf}(J_n)} \cdot \left(\frac{\partial f}{\partial x_{i,n+i}}\right)(X_0) \cdot D_{i,n+i} \\ &= \sum_{i=1}^{n} D_{i,n+i} = J_n.\end{aligned}$$

7. The Space (37)

In this section, we shall determine the b-function of the regular simple prehomogeneous vector space

(37) $(GL(1)^4 \times SL(2n+1), \Lambda_2 \oplus \Lambda_1 \oplus \Lambda_1 \oplus \Lambda_1, V(n(2n+1)) \oplus V(2n+1) \oplus V(2n+1) \oplus V(2n+1))$.

First we give the prehomogeneous vector space above explicitly. We identify $V(n(2n+1))$ with the space $\mathrm{Alt}(2n+1)$ of alternating matrices of degree $(2n+1)$, and $V(2n+1)$ with $\boldsymbol{C}^{2n+1}$. Let $G = GL(1)^4 \times SL(2n+1)$ and $V = \mathrm{Alt}(2n+1) \oplus \boldsymbol{C}^{2n+1} \oplus \boldsymbol{C}^{2n+1} \oplus \boldsymbol{C}^{2n+1}$. We take $\tilde{g} = (\alpha,\beta,\gamma,\delta;A) \in G$ and $\tilde{x} = (X;y,z,w) \in V$. Then we define the action ρ of G on V by

$$\rho(\tilde{g})\tilde{x} = (\alpha A X {}^t A; \beta A y, \gamma A z, \delta A w).$$

By [14, Proposition 2.5], the triplet $(G,\rho,V) = (GL(1)^4 \times SL(2n+1), \Lambda_2 \oplus \Lambda_1 \oplus \Lambda_1 \oplus \Lambda_1, \mathrm{Alt}(2n+1) \oplus \boldsymbol{C}^{2n+1} \oplus \boldsymbol{C}^{2n+1} \oplus \boldsymbol{C}^{2n+1})$ is a regular simple prehomogeneous vector space. When we want to distinguish one copy of $\boldsymbol{C}^{2n+1}$ in V from the others, we write $V = V_1 \oplus V_2 \oplus V_3 \oplus V_4$ with $V_1 \cong \mathrm{Alt}(2n+1)$ and $V_\nu \cong \boldsymbol{C}^{2n+1}$ for $\nu = 2,3,4$. The relative invariants of (G,ρ,V) are given explicitly in [14]; let

$$f_1(\tilde{x}) = \mathrm{Pf}\begin{pmatrix} X & y \\ -{}^t y & 0 \end{pmatrix},$$

$$f_2(\tilde{x}) = \mathrm{Pf}\begin{pmatrix} X & z \\ -{}^t z & 0 \end{pmatrix},$$

$$f_3(\tilde{x}) = \mathrm{Pf}\begin{pmatrix} X & w \\ -{}^t w & 0 \end{pmatrix},$$

$$f_4(\tilde{x}) = \mathrm{Pf}\begin{pmatrix} X & y & z & w \\ -{}^t y & 0 & 0 & 0 \\ -{}^t z & 0 & 0 & 0 \\ -{}^t w & 0 & 0 & 0 \end{pmatrix}. \tag{7.1}$$

Then $f_1(\tilde{x}), \dots, f_4(\tilde{x})$ are the fundamental relative invariants of (G, ρ, V). For $1 \le i \le 4$, denote by ϕ_i the character of G corresponding to f_i. Then we have

$$\phi_1 = \alpha^n \beta, \quad \phi_2 = \alpha^n \gamma, \quad \phi_3 = \alpha^n \delta, \quad \phi_4 = \alpha^{n-1}\beta\gamma\delta.$$

For $1 \le i, j \le 2n+1$, let $E_{ij}^{(2n+1)}$ be the (i,j)-th matrix unit of $M(2n+1)$, and let $D_{ij}^{(2n+1)} = E_{ij}^{(2n+1)} - E_{ji}^{(2n+1)} \in \mathrm{Alt}(2n+1)$. A basis of $V_1 \cong \mathrm{Alt}(2n+1)$ is given by $D_{ij}^{(2n+1)}$ $(1 \le i < j \le 2n+1)$. For $v = 2, 3, 4$, we denote by $\{e_1^{(v)}, e_2^{(v)}, \dots, e_{2n+1}^{(v)}\}$ the standard basis of V_v. Let

$$X_0 = \sum_{i=2}^{n+1} D_{i,n+i}^{(2n+1)} = \left(\begin{array}{c|c|c} 0 & 0_{1,n} & 0_{1,n} \\ \hline 0_{n,1} & 0_n & I_n \\ \hline 0_{n,1} & -I_n & 0_n \end{array}\right) \in \mathrm{Alt}(2n+1). \tag{7.2}$$

and

$$\begin{aligned} y_0 &= e_1^{(2)} = {}^t(1|0,\dots,0|0,\dots,0) \in V_2 = \boldsymbol{C}^{2n+1}, \\ z_0 &= e_1^{(3)} + e_2^{(3)} = {}^t(1|1,0,\dots,0|0,\dots,0) \in V_3 = \boldsymbol{C}^{2n+1}, \\ w_0 &= e_1^{(4)} + e_{n+2}^{(4)} = {}^t(1|0,\dots,0|1,0,\dots,0) \in V_4 = \boldsymbol{C}^{2n+1}. \end{aligned} \tag{7.3}$$

We put $\tilde{x}_0 = (X_0; y_0, z_0, w_0) \in V$. By calculating the isotropy subalgebra at $\tilde{x}_0$, we see that $\tilde{x}_0$ is a generic point of (G, ρ, V). Let us define a bilinear form on V by

$$\langle \tilde{x}, \tilde{x}^\vee \rangle = \frac{1}{2} \operatorname{tr} {}^t X X^\vee + {}^t y y^\vee + {}^t z z^\vee + {}^t w w^\vee.$$

We identify the dual space $V^\vee$ of V with V by this bilinear form. Then the action $\rho^\vee$ of G on $V^\vee$ is given by

$$\rho^\vee(\tilde{g})\tilde{x}^\vee = (\alpha^{-1}\,{}^t A^{-1} X^\vee A^{-1}; \beta^{-1}\,{}^t A^{-1} y^\vee, \gamma^{-1}\,{}^t A^{-1} z^\vee, \delta^{-1}\,{}^t A^{-1} w^\vee)$$

for $\tilde{g} = (\alpha, \beta, \gamma, \delta; A) \in G$ and $\tilde{x}^\vee = (X^\vee; y^\vee, z^\vee, w^\vee) \in V^\vee$. Replacing $\tilde{x}$ in the polynomials $f_1(\tilde{x}), \dots, f_4(\tilde{x})$ in (7.1) by $\tilde{x}^\vee \in V^\vee$, we obtain the fundamental relative invariants $f_1^\vee(\tilde{x}^\vee), \dots, f_4^\vee(\tilde{x}^\vee)$ of $(G, \rho^\vee, V^\vee)$. Moreover, we have

$$(\operatorname{grad} \log f_\mu)(\tilde{x}) = (X^\vee(\tilde{x}); y^\vee(\tilde{x}), z^\vee(\tilde{x}), w^\vee(\tilde{x}))$$
$$\in \operatorname{Alt}(2n+1) \oplus \boldsymbol{C}^{2n+1} \oplus \boldsymbol{C}^{2n+1} \oplus \boldsymbol{C}^{2n+1},$$

for $\tilde{x} \in V$ and $\mu = 1, 2, 3, 4$, where

$$X^\vee(\tilde{x}) = \sum_{1 \le i < j \le 2n+1} \frac{1}{f_\mu(\tilde{x})} \cdot \frac{\partial f_\mu}{\partial x_{ij}}(\tilde{x}) \cdot D_{ij}^{(2n+1)},$$

$$y^\vee(\tilde{x}) = \sum_{k=1}^{2n+1} \frac{1}{f_\mu(\tilde{x})} \cdot \frac{\partial f_\mu}{\partial y_k}(\tilde{x}) \cdot e_k^{(2)},$$

$$z^\vee(\tilde{x}) = \sum_{l=1}^{2n+1} \frac{1}{f_\mu(\tilde{x})} \cdot \frac{\partial f_\mu}{\partial z_l}(\tilde{x}) \cdot e_l^{(3)},$$

$$w^\vee(\tilde{x}) = \sum_{m=1}^{2n+1} \frac{1}{f_\mu(\tilde{x})} \cdot \frac{\partial f_\mu}{\partial w_m}(\tilde{x}) \cdot e_m^{(4)}.$$

The following proposition is critical in our calculation.

PROPOSITION 7.1. *For* $1 \le i < j \le 2n+1$, *let* $D_{ij}^{(2n+1)} = E_{ij}^{(2n+1)} - E_{ji}^{(2n+1)} \in \operatorname{Alt}(2n+1)$. *We define the* $n \times n$ *matrix* I_{n-1}^0 *by*

$$I_{n-1}^0 = \left(\begin{array}{c|c} 0 & 0_{1,n-1} \\ \hline 0_{n-1,1} & I_{n-1} \end{array}\right) \in M(n).$$

and the vector v_0 *by* $v_0 = {}^t(1, 0, \dots, 0) \in \boldsymbol{C}^n$. *Then we have*

(1) $(\operatorname{grad} \log f_1)(\tilde{x}_0) = (X_1^\vee; e_1^{(2)}, 0_{n,1}, 0_{n,1})$.
(2) $(\operatorname{grad} \log f_2)(\tilde{x}_0) = (X_2^\vee; 0_{n,1}, e_1^{(3)}, 0_{n,1})$.
(3) $(\operatorname{grad} \log f_3)(\tilde{x}_0) = (X_3^\vee; 0_{n,1}, 0_{n,1}, e_1^{(4)})$.
(4) $(\operatorname{grad} \log f_4)(\tilde{x}_0) = (X_4^\vee; e_1^{(2)} - e_2^{(2)} - e_{n+2}^{(2)}, e_2^{(3)}, e_{n+2}^{(4)})$.

Here $X_1^\vee = X_0$ *in* (7.2) *and*

$$X_2^\vee = -D_{1,n+2}^{(2n+1)} + \sum_{i=2}^{n+1} D_{i,n+i}^{(2n+1)} = \left(\begin{array}{c|c|c} 0 & 0_{1,n} & -{}^t v_0 \\ \hline 0_{n,1} & 0_n & I_n \\ \hline v_0 & -I_n & 0_n \end{array}\right) \in \operatorname{Alt}(2n+1),$$

$$X_3^\vee = D_{1,2}^{(2n+1)} + \sum_{i=2}^{n+1} D_{i,n+i}^{(2n+1)} = \left(\begin{array}{c|c|c} 0 & {}^t v_0 & 0_{1,n} \\ \hline -v_0 & 0_n & I_n \\ \hline 0_{n,1} & -I_n & 0_n \end{array}\right) \in \mathrm{Alt}(2n+1),$$

$$X_4^\vee = \sum_{i=3}^{n+1} D_{i,n+i}^{(2n+1)} = \left(\begin{array}{c|c|c} 0 & 0_{1,n} & 0_{1,n} \\ \hline 0_{n,1} & 0_n & I_{n-1}^0 \\ \hline 0_{n,1} & -I_{n-1}^0 & 0_n \end{array}\right) \in \mathrm{Alt}(2n+1).$$

Proof. First we shall calculate $f_1(\tilde{x}_0), \ldots, f_4(\tilde{x}_0)$. We make use of Proposition 6.5 to expand the Pfaffians along the 1st hock, and obtain

$$f_1(\tilde{x}_0) = f_2(\tilde{x}_0) = f_3(\tilde{x}_0) = (-1)^{n(n-1)/2}. \tag{7.4}$$

As for $f_4(\tilde{x}_0)$, we use Proposition 6.5 two times:

$$\begin{aligned} &f_4(\tilde{x}_0) \\ &\quad = \mathrm{Pf}\left(\sum_{i=2}^{n+1} D_{i,n+i}^{(2n+4)} + D_{1,2n+2}^{(2n+4)} + D_{1,2n+3}^{(2n+4)} + D_{2,2n+3}^{(2n+4)} + D_{1,2n+4}^{(2n+4)} + D_{n+2,2n+4}^{(2n+4)}\right) \\ &\quad = (-1)^{1+(2n+2)-1} \cdot \mathrm{Pf}\left(\sum_{i=1}^{n} D_{i,n+i}^{(2n+2)} + D_{1,2n+1}^{(2n+2)} + D_{n+1,2n+2}^{(2n+2)}\right) \\ &\quad = (-1)^{1+(2n+1)-1} \cdot \mathrm{Pf}\left(\sum_{i=1}^{n} D_{i,n+i}^{(2n)}\right) = (-1) \cdot \mathrm{Pf}(J_n). \end{aligned}$$

Thus we obtain

$$f_4(\tilde{x}_0) = (-1)^{(n^2-n+2)/2}. \tag{7.5}$$

For $1 \le k \le 2n+1$, we define the polynomial function $g_k(X)$ on $\mathrm{Alt}(2n+1)$ by

$$g_k(X) = \mathrm{Pf}[X(1, \ldots, \check{k}, \ldots, 2n+1)].$$

By Proposition 6.5, we have that

$$f_1(\tilde{x}) = \sum_{k=1}^{2n+1} (-1)^{k+(2n+2)-1} \cdot y_k \cdot g_k(X),$$

and thus

$$\frac{\partial f_1}{\partial x_{ij}}(\tilde{x}) = \sum_{k=1}^{2n+1} (-1)^{k-1} \cdot y_k \cdot \left(\frac{\partial g_k}{\partial x_{ij}}\right)(X)$$

for $1 \le i < j \le 2n+1$. On account of (7.3), we see that only $(\partial g_1/\partial x_{ij})(X_0)$ contributes to the value $(\partial f_1/\partial x_{ij})(\tilde{x}_0)$. Again by Proposition 6.5, it follows that

$$\frac{\partial g_1}{\partial x_{ij}}(X) = (-1)^{(i-1)+(j-1)-1} \cdot \mathrm{Pf}[X(2,\dots,\check{i},\dots,\check{j},\dots,2n+1)].$$

By an easy consideration, we observe that $\mathrm{Pf}[X_0(2,\dots,\check{i},\dots,\check{j},\dots,2n+1)]$ does not vanish if and only if $2 \le i \le n+1$ and $j = n+i$, and that

$$\left(\frac{\partial g_1}{\partial x_{i,n+i}}\right)(X_0) = (-1)^{(i-1)+(n+i-1)-1} \cdot \mathrm{Pf}(J_{n-1}) = (-1)^{n(n-1)/2}.$$

Together with (7.4), we obtain

$$\sum_{1 \le i<j \le 2n+1} \frac{1}{f_1(\tilde{x}_0)} \cdot \frac{\partial f_1}{\partial x_{ij}}(\tilde{x}_0) \cdot D_{ij}^{(2n+1)} = \sum_{i=2}^{n+1} D_{i,n+i}^{(2n+1)} = X_1^{\vee}.$$

Since all the components of the 1st row (column) of X_0 are 0, it follows that $(\partial f_1/\partial y_k)(\tilde{x}_0) = 0$ for $2 \le k \le 2n+1$. Moreover, by Proposition 6.5,

$$\left(\frac{\partial f_1}{\partial y_1}\right)(\tilde{x}_0) = (-1)^{1+(2n+2)-1} \cdot \mathrm{Pf}(J_n) = (-1)^{n(n-1)/2},$$

and thus we get the assertion (1).

Now we turn to $(\operatorname{grad} \log f_2)(\tilde{x}_0)$. Taking (7.3) into account, we observe that

$$\left(\frac{\partial f_2}{\partial x_{ij}}\right)(\tilde{x}_0) = \left(\frac{\partial g_1}{\partial x_{ij}}\right)(X_0) + (-1) \cdot \left(\frac{\partial g_2}{\partial x_{ij}}\right)(X_0)$$

for $1 \le i < j \le 2n+1$. We have already considered $(\partial g_1/\partial x_{ij})(X_0)$. Notice that, in the $2n \times 2n$ alternating matrix $X_0(1,3,4,\dots,2n+1)$, the 1st row (column) and the $(n+1)$-th row (column) are zero vectors. Hence we observe that $(\partial g_2/\partial x_{ij})(X_0) \ne 0$ if and only if $i = 1$ and $j = n+2$, and that

$$\left(\frac{\partial g_2}{\partial x_{1,n+2}}\right)(X_0) = (-1)^{1+(n+1)-1} \cdot \mathrm{Pf}[X_0(\check{1},3,4,\dots,\check{n+2},\dots,2n+1)]$$

$$= (-1)^{1+(n+1)-1} \cdot \mathrm{Pf}(J_{n-1}) = (-1)^{n(n-1)/2}.$$

Together with (7.4), we obtain

$$\sum_{1 \le i<j \le 2n+1} \frac{1}{f_2(\tilde{x}_0)} \cdot \frac{\partial f_2}{\partial x_{ij}}(\tilde{x}_0) \cdot D_{ij}^{(2n+1)} = -D_{1,n+2}^{(2n+1)} + \sum_{i=2}^{n+1} D_{i,n+i}^{(2n+1)} = X_2^{\vee}.$$

In addition, we see that

$$\sum_{l=1}^{2n+1} \frac{1}{f_2(\tilde{x}_0)} \cdot \frac{\partial f_2}{\partial z_l}(\tilde{x}_0) \cdot e_l^{(3)} = e_1^{(3)},$$

$$\sum_{k=1}^{2n+1} \frac{1}{f_2(\tilde{x}_0)} \cdot \frac{\partial f_2}{\partial y_k}(\tilde{x}_0) \cdot e_k^{(2)} = \sum_{m=1}^{2n+1} \frac{1}{f_2(\tilde{x}_0)} \cdot \frac{\partial f_2}{\partial w_m}(\tilde{x}_0) \cdot e_m^{(4)} = 0_{n,1}.$$

and thus we get the assertion (2). We omit the proof of (3), since it is quite the same as (2).

Let $\mathscr{T}$ be the index set defined by

$$\mathscr{T} = \{(k,l,m) \in \mathbf{Z}^3_{>0};\, 1 \le k,l,m \le 2n+1, k \ne l, l \ne m, m \ne k\}.$$

For $(k,l,m) \in \mathscr{T}$, we define the polynomial function $h_{k,l,m}(X)$ on $\mathrm{Alt}(2n+1)$ by

$$h_{k,l,m}(X) = \mathrm{Pf}[X(1,\dots,\check{k},\dots,\check{l},\dots,\check{m},\dots,2n+1)].$$

By using Proposition 6.5 three times, we obtain

$$f_4(\tilde{x}) = \sum_{\mathscr{T}} \varepsilon(k,l,m) \cdot y_k z_l w_m \cdot h_{k,l,m}(X),$$

where $\varepsilon(k,l,m)$ is a function on $\mathscr{T}$ such that $\varepsilon(k,l,m)$ takes ± 1 as its values. On account of (7.3), we see that, among $\{(\partial h_{k,l,m}/\partial x_{ij})(X_0)\}_{(k,l,m)\in\mathscr{T}}$, only

$$\left(\frac{\partial h_{1,2,n+2}}{\partial x_{ij}}\right)(X_0)$$

contributes to the value $(\partial f_4/\partial x_{ij})(\tilde{x}_0)$. It follows from Proposition 6.5 that

$$\varepsilon(1,2,n+2) = (-1)^{1+(2n+2)-1} \cdot (-1)^{1+(2n+1)-1} \cdot (-1)^{n+2n-1} = (-1)^n,$$

and that

$$\left(\frac{\partial h_{1,2,n+2}}{\partial x_{ij}}\right)(X_0) = \delta(i,j)\,\mathrm{Pf}[X_0(3,4,\dots,\check{i},\dots,n\check{+}2,\dots,\check{j},\dots,2n+1)]$$

for $1 \le i < j \le 2n+1$. Here $\delta(i,j)$ is a function of i and j, taking ± 1 as its values. By an easy consideration,

$$\mathrm{Pf}[X_0(3,4,\dots,\check{i},\dots,n\check{+}2,\dots,\check{j},\dots,2n+1)] \tag{7.6}$$

is not equal to zero if and only if $3 \le i \le n+1$ and $j = n+i$. Moreover, if this follows, $\delta(i, n+i) = (-1)^{(i-2)+(n+i-3)-1}$ and (7.6) is equal to $\mathrm{Pf}(J_{n-2})$. Hence $(\partial f_4/\partial x_{ij})(\tilde{x}_0) \neq 0$ if and only if $3 \leq i \leq n+1$ and $j = n+i$. Furthermore,

$$\begin{aligned}\left(\frac{\partial f_4}{\partial x_{i,n+i}}\right)(\tilde{x}_0) &= \varepsilon(1,2,n+2)\cdot\left(\frac{\partial h_{1,2,n+2}}{\partial x_{i,n+i}}\right)(X_0)\\ &= (-1)^n\cdot(-1)^{(i-2)+(n+i-3)-1}\cdot\mathrm{Pf}(J_{n-2})\\ &= (-1)^{(n-2)(n-3)/2}\end{aligned}$$

for $3 \leq i \leq n+1$. Together with (7.5), we have

$$\frac{1}{f_4(\tilde{x}_0)}\cdot\frac{\partial f_4}{\partial x_{i,n+i}}(\tilde{x}_0) = (-1)^{n^2-3n+4} = 1,$$

and consequently

$$\sum_{1\leq i<j\leq 2n+1}\frac{1}{f_4(\tilde{x}_0)}\cdot\frac{\partial f_4}{\partial x_{ij}}(\tilde{x}_0)\cdot D_{ij}^{(2n+1)} = \sum_{i=3}^{n+1} D_{i,n+i}^{(2n+1)} = X_4^\vee.$$

We can evaluate the values $(\partial f_4/\partial y_k)(\tilde{x}_0)$, $(\partial f_4/\partial z_l)(\tilde{x}_0)$, $(\partial f_4/\partial w_m)(\tilde{x}_0)$, by using Proposition 6.5. The results are the following:

$$\sum_{k=1}^{2n+1}\frac{1}{f_4(\tilde{x}_0)}\cdot\frac{\partial f_4}{\partial y_k}(\tilde{x}_0)\cdot e_k^{(2)} = e_1^{(2)} - e_2^{(2)} - e_{n+2}^{(2)},$$

$$\sum_{l=1}^{2n+1}\frac{1}{f_4(\tilde{x}_0)}\cdot\frac{\partial f_4}{\partial z_l}(\tilde{x}_0)\cdot e_l^{(3)} = e_2^{(3)},$$

$$\sum_{m=1}^{2n+1}\frac{1}{f_4(\tilde{x}_0)}\cdot\frac{\partial f_4}{\partial w_m}(\tilde{x}_0)\cdot e_m^{(4)} = e_{n+2}^{(4)}.$$

Hence we have proved the assertion (4). □

Now we shall calculate the a-function $a_{\underline{m}}(\underline{s})$. Let $X_s = s_1X_1^\vee + s_2X_2^\vee + s_3X_3^\vee + s_4X_4^\vee$ and $y_s = (s_1+s_4)e_1 + (-s_4)e_2 + (-s_4)e_{n+2}$. Then we have

$$\begin{aligned}f_1^\vee(\operatorname{grad}\log \underline{f}^{\underline{s}}(\tilde{x}_0)) &= \mathrm{Pf}\begin{pmatrix} X_s & y_s \\ -{}^t y_s & 0\end{pmatrix}\\ &= (-1)^{n(n-1)/2}\cdot s_1\cdot(s_1+s_2+s_3+s_4)^n.\end{aligned}$$

by Proposition 6.5. Since $f_1(\tilde{x}_0) = (-1)^{n(n-1)/2}$ by (7.4), we have

$$a_1(\underline{s}) = f_1^\vee(\operatorname{grad}\log \underline{f}^{\underline{s}}(\tilde{x}_0))f_1(\tilde{x}_0) = s_1(s_1+s_2+s_3+s_4)^n.$$

Similarly, we can calculate the a-functions $a_2(\underline{s}), a_3(\underline{s}), a_4(\underline{s})$, and obtain the following proposition.

Proposition 7.2.

$$a_{\underline{m}}(\underline{s}) = s_1^{m_1} s_2^{m_2} s_3^{m_3} s_4^{2m_4} (s_1 + s_2 + s_3 + s_4)^{n(m_1+m_2+m_3+m_4)}.$$

Now we use the same notation as in Lemma 2.2. Let

$$\gamma_1(\underline{s}) = s_1, \quad \gamma_2(\underline{s}) = s_2, \quad \gamma_3(\underline{s}) = s_3, \quad \gamma_4(\underline{s}) = s_4, \quad \gamma_5(\underline{s}) = s_1 + s_2 + s_3 + s_4.$$

Then we have $\mu_1 = \mu_2 = \mu_3 = 1$, $\mu_4 = 2$, and $\mu_5 = n$. Note that the above a-function satisfies Assumption 2.4 and consequently we can apply Lemmas 2.6–2.8 to the b-functions. By Lemma 2.6, the b-functions $b_i(\underline{s})$ for $i = 1, 2, 3, 4$ can be expressed as

$$b_1(\underline{s}) = (s_1 + \alpha_{1,1}) \cdot \prod_{k=1}^{n} (s_1 + s_2 + s_3 + s_4 + \alpha_{5,k}),$$

$$b_2(\underline{s}) = (s_2 + \alpha_{2,1}) \cdot \prod_{k=1}^{n} (s_1 + s_2 + s_3 + s_4 + \alpha_{5,k}),$$

$$b_3(\underline{s}) = (s_3 + \alpha_{3,1}) \cdot \prod_{k=1}^{n} (s_1 + s_2 + s_3 + s_4 + \alpha_{5,k}),$$

$$b_4(\underline{s}) = (s_4 + \alpha_{4,1})(s_4 + \alpha_{4,2}) \cdot \prod_{k=1}^{n} (s_1 + s_2 + s_3 + s_4 + \alpha_{5,k}),$$

with some $\alpha_{j,r} \in \boldsymbol{Q}_{>0}$. Let us describe the functional equations explicitly. We see that $\underline{\kappa}$ in Lemma 2.10 is given by $\underline{\kappa} = (1, 1, 1, 2n)$. Put

$$\beta_{\gamma_1}(u) = u + \alpha_{1,1}, \quad \beta_{\gamma_2}(u) = u + \alpha_{2,1}, \quad \beta_{\gamma_3}(u) = u + \alpha_{3,1},$$

$$\beta_{\gamma_4}(u) = (u + \alpha_{4,1})(u + \alpha_{4,2}), \quad \beta_{\gamma_5}(u) = \prod_{k=1}^{n} (u + \alpha_{5,k}).$$

Then, by Theorem 2.12, we have

$$\beta_{\gamma_1}(u) = (-1)\beta_{\gamma_1}(-u-2), \quad \beta_{\gamma_2}(u) = (-1)\beta_{\gamma_2}(-u-2),$$

$$\beta_{\gamma_3}(u) = (-1)\beta_{\gamma_3}(-u-2), \quad \beta_{\gamma_4}(u) = \beta_{\gamma_4}(-u-2n-1),$$

$$\beta_{\gamma_5}(u) = (-1)^n \cdot \beta_{\gamma_5}(-u-2n-4).$$

Hence we observe that

$$\alpha_{1,1} = \alpha_{2,1} = \alpha_{3,1} = 1, \tag{7.7}$$

and that

$$\begin{aligned} \{\alpha_{4,1}, \alpha_{4,2}\} &= \{2n+1-\alpha_{4,1}, 2n+1-\alpha_{4,2}\}, \\ \{\alpha_{5,1}, \dots, \alpha_{5,n}\} &= \{2n+4-\alpha_{5,1}, \dots, 2n+4-\alpha_{5,n}\}. \end{aligned} \tag{7.8}$$

Now we quote the following result from Kimura [13].

PROPOSITION 7.3. *Let $f(X) = \mathrm{Pf}(X)$ be the irreducible relative invariant of the irreducible regular prehomogeneous vector space $(GL(2n), \Lambda_2, V(n(2n-1)))$, which we gave explicitly in Example* 6.6. *Then the b-function $b_f(s)$ of f is given by*

$$b_f(s) = \prod_{k=1}^{n}(s+2k-1) = (s+1)(s+3)\cdots(s+2n-1).$$

We see that each of $f_1(\tilde{x})$, $f_2(\tilde{x})$, $f_3(\tilde{x})$ in (7.1) can be regarded as the irreducible relative invariant of the prehomogeneous vector space $(GL(2n+2), \Lambda_2, V((n+1)(2n+1)))$. So we obtain

$$b_{f_i}(s) = \prod_{k=1}^{n+1}(s+2k-1) = (s+1)(s+3)\cdots(s+2n+1). \tag{7.9}$$

for $i = 1,2,3$. By the definition of $b_{f_1}(s)$, it follows that $b_1(s\varepsilon_1) = b_{f_1}(s)$. Combining this with (7.7) and (7.9), we get

$$\{\alpha_{5,1}, \alpha_{5,2}, \dots, \alpha_{5,n}\} = \{3, 5, \dots, 2n+1\}. \tag{7.10}$$

It remains to determine $\alpha_{4,1}$ and $\alpha_{4,2}$. Since we also have that $b_4(s\varepsilon_4) = b_{f_4}(s)$ by definition, we observe that the roots of the polynomial of $b_{f_4}(s)$ are given by

$$\{\alpha_{4,1}, \alpha_{4,2}, \alpha_{5,1}, \alpha_{5,2}, \dots, \alpha_{5,n}\} = \{\alpha_{4,1}, \alpha_{4,2}, 3, 5, \dots, 2n+1\}.$$

However, in view of Lemma 1.6 (1), either $\alpha_{4,1}$ or $\alpha_{4,2}$ must be equal to 1, and hence we have $\{\alpha_{4,1}, \alpha_{4,2}\} = \{1, 2n\}$ by (7.8). We therefore obtain the following results:

$$b_1(\underline{s}) = (s_1+1)\cdot\prod_{k=2}^{n+1}(s_1+s_2+s_3+s_4+2k-1),$$

$$b_2(\underline{s}) = (s_2+1)\cdot\prod_{k=2}^{n+1}(s_1+s_2+s_3+s_4+2k-1),$$

$$b_3(\underline{s}) = (s_3+1)\cdot\prod_{k=2}^{n+1}(s_1+s_2+s_3+s_4+2k-1),$$

$$b_4(\underline{s}) = (s_4+1)(s_4+2n)\cdot\prod_{k=2}^{n+1}(s_1+s_2+s_3+s_4+2k-1).$$

Finally, Lemma 2.8 leads us to the following theorem.

Theorem 7.4. *Let* $\underline{f} = (f_1, f_2, f_3, f_4)$ *be the fundamental relative invariants of the regular simple prehomogeneous vector space* (37) $(GL(1)^4 \times SL(2n+1), \Lambda_2 \oplus \Lambda_1 \oplus \Lambda_1 \oplus \Lambda_1, V(n(2n+1)) \oplus V(2n+1) \oplus V(2n+1) \oplus V(2n+1))$. *Then the b-function of* $\underline{f}^{\underline{m}} = \prod_{i=1}^4 f_i^{m_i}$ *is given by*

$$b_{\underline{f}^{\underline{m}}}(s) = \left\{\prod_{v=0}^{m_1-1}(m_1 s+1+v)\right\}\left\{\prod_{v=0}^{m_2-1}(m_2 s+1+v)\right\}\left\{\prod_{v=0}^{m_3-1}(m_3 s+1+v)\right\}$$
$$\times\left\{\prod_{v=0}^{m_4-1}(m_4 s+1+v)(m_4 s+2n+v)\right\}$$
$$\times\left\{\prod_{v=0}^{m_1+m_2+m_3+m_4-1}\prod_{k=2}^{n+1}((m_1+m_2+m_3+m_4)s+2k-1+v)\right\}.$$

Remark 7.5. It is known that the space (37) decomposes into infinitely many orbits (see [17]). On the other hand, it is possible to apply the method of Kimura [15] to the space (37). However, the explicit form of the Igusa local zeta function of the space (37) is still open (cf. [5]). So the explicit form of the b-function has not been determined by the other methods.

8. Final Remarks

As indicated in the Introduction, our method is mainly due to K. Ukai [31, 32]. Ukai determined the b-functions of the prehomogeneous vector spaces arising from nilpotent elements of exceptional Lie algebra, via the Dynkin-Kostant theory. A. Gyoja and Y. Kaneko also determined the b-functions of prehomogeneous vector spaces of Dynkin-Kostant type for classical groups, by using the castling transform ([7]).

Recently, S. Wakatsuki and the present author started explicit calculation of the b-functions associated with regular 2-simple prehomogeneous vector spaces of type I, which are listed in [16, pp. 395–398]. At present, we have determined those b-functions *except* for the following five cases:

- $(GL_1^2 \times SL_5 \times GL_2, \Lambda_2 \otimes \Lambda_1 + (\Lambda_1^* + \Lambda_1^*) \otimes 1)$.
- $(GL_1 \times SL_5 \times GL_8, \Lambda_2 \otimes \Lambda_1 + 1 \otimes \Lambda_1^*)$.
- $(GL_1 \times SL_5 \times GL_9, \Lambda_2 \otimes \Lambda_1 + 1 \otimes \Lambda_1^*)$.
- $(GL_1 \times Spin_{10} \times GL_{14}, \Lambda' \otimes \Lambda_1 + 1 \otimes \Lambda_1^*)$.
- $(GL_1 \times Spin_{10} \times GL_{15}, \Lambda' \otimes \Lambda_1 + 1 \otimes \Lambda_1^*)$.

The details will be discussed in the forthcoming papers [28, 29, 33].

Appendix: Tables of the b-Functions

(I) Irreducible case $(v = 1)$

In this case, we have $l = 1$. Thus we shall give the list of $d = \deg f$ and α_j. The following results are due to Kimura [13] and Ozeki [20].

	d	α_j
(1)	n	$\frac{k+1}{2}$ $(k = 1, \ldots, n)$
(2)	n	$2k - 1$ $(k = 1, \ldots, n)$
(3)	4	$1^{\times 2}, \frac{5}{6}, \frac{7}{6}$
(4)	4	$1, \frac{5}{2}, \frac{7}{2}, 5$
(5)	7	$1, 2, \frac{5}{2}, \frac{7}{2}, 3, 4, 5$
(6)	16	$1, \frac{3}{2}^{\times 2}, \frac{11}{6}, 2^{\times 3}, \frac{13}{6}, \frac{7}{3}, \frac{5}{2}^{\times 3}, \frac{8}{3}, 3^{\times 2}, \frac{7}{2}$
(7)	2	$1, \frac{n}{2}$
(8)	4	$1, 2, \frac{5}{2}, \frac{7}{2}$
(9)	2	$1, 4$
(10)	2	$1, 8$
(11)	4	$1, \frac{7}{2}, \frac{11}{2}, 8$
(12)	4	$1, \frac{7}{2}, \frac{11}{2}, 8$
(13)	8	$1, \frac{5}{2}, \frac{7}{2}, 4, 5, \frac{11}{2}, \frac{13}{2}, 8$
(14)	2	$1, \frac{7}{2}$
(15)	3	$1, 5, 9$
(16)	4	$1, \frac{11}{2}, \frac{19}{2}, 14$

(II) Non-irreducible case ($\nu \geq 2$).

(II-1) $l = 1$.

	d	α_j
(17)	2	$1, n$
(18)	n	$k \ (k = 1, \dots, n)$
(19)	$n+1$	$2k-1 \ (k = 1, \dots, n+1)$
(20)	2	$1, 2n$
(21)	4	$1, 4, 5, 8$

(II-2) $l = 2$.

In what follows, notations in the tables below are the same as in §2: Let $f_1, \dots, f_l$ be the relative invariants. Their degrees are given by $d_i = \deg f_i$. The b-function of the relative invariant $\underline{f} = f_1^{m_1} \cdots f_l^{m_l}$ $(m_1, \dots, m_l \in \boldsymbol{Z}_{\geq 0})$ is given by

$$b_{\underline{m}}(\underline{s}) = \prod_{j=1}^{N} \prod_{\nu=0}^{\gamma_j(\underline{m})-1} \prod_{r=1}^{\mu_j} (\gamma_j(\underline{s}) + \alpha_{j,r} + \nu).$$

The ordering of relative invariants $f_1, \dots, f_l$ is the same as in Kimura [14]. In addition, we shall ignore the constant multiple $\underline{A}^{\underline{m}}$ of the b-function $b_{\underline{m}}(\underline{s})$.

	d_i	γ_j	$\alpha_{j,r}$	
(22)	n $n+1$	s_1 s_2 $s_1 + s_2$	1 $1, 2n$ $2k-1 \ (k = 2, \dots, n)$	[8, 10]
(23)	n 3	s_1 s_2 $s_1 + s_2$	$2k-1 \ (k = 1, \dots, n-1)$ $1, 2n$ $2n-1$	[8, 10]
(24)	n $n+1$	s_1 s_2 $s_1 + s_2$	1 $1, \frac{n}{2}$ $\frac{k+1}{2} \ (k = 2, \dots, n)$	[8, 30]
(25)	n 3	s_1 s_2 $s_1 + s_2$	$\frac{k+1}{2} \ (k = 1, \dots, n-1)$ $1, \frac{n}{2}$ $\frac{n+1}{2}$	[8, 30]

	d_i	γ_j	$\alpha_{j,r}$	
(26)	7 6	s_1 s_2 s_1+s_2	$1,2,3$ $1,\frac{7}{2}$ $\frac{5}{2},\frac{7}{2},4,5$	[8]
(27)	7 5	s_1 s_2 s_1+s_2	$1,2,\frac{5}{2},\frac{7}{2}$ $1,\frac{7}{2}$ $3,4,5$	[8, 9]
(28)	4 4	s_1 s_2 s_1+s_2	$1,2$ $1,3$ $\frac{5}{2},\frac{7}{2}$	§5
(29)	2 3	s_1 s_2 s_1+s_2	1 $1,8$ 5	[8, 11]

The four spaces listed blow are direct sums of some regular prehomogeneous vector spaces whose b-functions are known.

	d_i	γ_j	$\alpha_{j,r}$
(30)	n 2	s_1 s_2	$2k-1\ \ (k=1,\dots,n)$ $1,2$
(31)	2 2	s_1 s_2	$1,\frac{7}{2}$ $1,4$
(32)	2 2	s_1 s_2	$1,4$ $1,4$
(33)	2 4	s_1 s_2	$1,6$ $1,\frac{7}{2},\frac{11}{2},8$

(II-3) $l=3$.

	d_i	γ_j	$\alpha_{j,r}$	
(34)	2 2 2	s_1 s_2 s_3 $s_1+s_2+s_3$	1 1 1 2	[8, 27]

(II-4) $l \geq 4$.

For the space (35), we have $l = n+1$ and $d_1 = \cdots = d_{n+1} = n$ (see [14]).

	d_i	γ_j	$\alpha_{j,r}$	
(35)	n $\vdots$ n	s_1 $\vdots$ s_{n+1} $s_1 + \cdots + s_{n+1}$	1 $\vdots$ 1 $k \ (k = 2, \dots, n)$	[5, 8, 15]

For the space (36), we have $l = n+1$ and $d_1 = \cdots = d_n = 2$ and $d_{n+1} = n$ (see [14]).

	d_i	γ_j	$\alpha_{j,r}$	
(36)	2 $\vdots$ 2 n	s_1 $\vdots$ s_n s_{n+1} $s_1 + \cdots + s_{n+1}$	1 $\vdots$ 1 $k \ (k = 1, \dots, n-1)$ n	[5, 8, 15]
(37)	$n+1$ $n+1$ $n+1$ $n+2$	s_1 s_2 s_3 s_4 $s_1 + s_2 + s_3 + s_4$	1 1 1 $1, 2n$ $2k-1 \ (k = 2, \dots, n+1)$	§7
(38)	$n+1$ 2 2 3	s_1 s_2 s_3 s_4 $s_1 + s_2 + s_3 + s_4$	$2k-1 \ (k = 1, \dots, n)$ 1 1 $1, 2n$ $2n+1$	[5, 15]

Added in proof. The expansion formula for Pfaffians (Proposition 6.5) is also stated in the following paper: J. Igusa, On the arithmetic of Pfaffians, Nagoya Math. J. **47** (1972), 169–198.

Acknowledgment

The author would like to express his hearty thanks to Professor Tatsuo Kimura for his helpful advice and encouragement. The author wishes to ac-knowledge Professor Hiroyuki Ochiai for careful reading and comments, which helped the author to improve the paper.

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