TSUKUBA J. MATH.
Vol. 26 No. 2 (2002), 269–289

A CONSTRUCTION OF BRAIDED HOPF ALGEBRAS*

By

Shuan-hong WANG

Abstract. Under suitable assumption, we present a method to construct braided Hopf algebras (braided groups) $\overline{B}$ and $\underline{B}$ in Yetter-Drinfel'd category ${}_H\mathcal{YD}_1^H$ and ${}_H\mathcal{YD}_2^H$ respectively. As applications, we study some special cases in both module and comodule form for H quasitriangular and for H coquasitriangular respectively. Finally, some examples are given.

§ 1. Introduction and Preliminaries

There has been some interest in the theory of braided Hopf algebras (braided groups) or Hopf algebras in braided categories [AS1–2, D, Maij1–3]. Applications in physics include the spectrum generating quantum groups and the constructions of inhomogeneous quantum groups. Applications in pure mathematics include the proof of Schur's double centralizer theorems in [CFW], [FM], [XSW] and the complete classification of all pointed Hopf algebras of dimension p^2 or p^3 [AS1–2], and also linearly recursive sequences [NT].

Majid has introduced a procedure termed Bosonisation by which one can construct a braided Hopf algebras in the four categories denoted by ${}_H\mathcal{M}$, $\mathcal{M}_H$, ${}^H\mathcal{M}$ and $\mathcal{M}^H$ for left, right modules and left, right comodules respectively. Bosonisation generalise the Jordan-Wigner bosonisation transform for Z_2-graded systems in physics, and connectes to every Hopf algebra B in the braided category of representations of H an equivalent ordinary Hopf algebra $B \bowtie H$ (left handed cases) or $H \bowtie B$ (right handed cases).

We find that Bosonisation and Radford's biproducts [R] associated to Yetter-Drinfeld categories are not the same. This means that general braided Hopf algebras in the Yetter-Drinfeld category ${}_H\mathcal{YD}^H$ are much more general

*This research is Supported by a grant of NSF of China and also a grant awarded to him by NSF of Henan Provice, China.
Received December 11, 2000.
Revised July 30, 2001.

than the braided Hopf algebras in ${}_H\mathcal{M}$ for H quasitriangular or in ${}^H\mathcal{M}$ for H coquasitriangular. Thus, a natural and further question is: Does there exist a more general method to construct a braided Hopf algebra in the Yetter-Drinfeld categories?

We give a description about this in this paper, this is one motivation of this paper. Another motivation is due to [AS1–2] and [D] in which the authors investigate braided Hopf algebras of order p in the category ${}_H\mathcal{YD}^H$ of left-right Yetter-Drinfeld module over a more general Hopf algebra H and the trace formular for braided Hopf algebras respectively.

The present paper begins in Section 2 where we consider two braided monoidal category ${}_H\mathcal{YD}_1^H$ and ${}_H\mathcal{YD}_2^H$, and define one twisted algebra $\bar{B}$ for a bialgebra B which is both in ${}_H\mathcal{YD}_1^H$ and ${}_H\mathcal{YD}_2^H$, and then under suitable assumption, we show that $\bar{B}$ is a braided Hopf algebra in ${}_H\mathcal{YD}_2^H$. Similarly, it is proved that there exists another braided Hopf algebra $\underline{B}$ in ${}_H\mathcal{YD}_1^H$.

Section 3 is concerned with the conditions under which $\bar{B}$ and $\underline{B}$ above respectively become braided Hopf algebras.

The main results of the paper are some detailed calculation of the braided Hopf algebra $\bar{B}$ living in the category of modules of a quasitriangular Hopf algebra (H, R) associated to a dual pairing Hopf algebra (B, H, τ) and of comodules of a coquasitriangular Hopf algebra $(H, \langle | \rangle)$ associated to a dual R-Hopf algebra pair (B, H, R), these results are investigated in Section 4.

Throughout this paper, k denotes a fixed field and $(H, m_H, 1_H, \Delta_H, \varepsilon_H)$ a Hopf algebra over k with multiplication m_H, unit 1_H, comultiplcation $\Delta : H \to H \otimes H$ and counit $\varepsilon : H \to k$. We use Sweedler's Hopf algebra notation [Mont, Sw] and use the "sigma" notation for $\Delta : \Delta(h) = h_1 \otimes h_2$, for all $h \in H$, where we omit parentheses on subscripts and the sum notation. S_H denotes the antipode of H. S_H^{-1} denotes its composition inverse if S_H is bijective. All maps are k-linear, $\otimes$ means $\otimes_k$ unless otherwise specified, etc. Denote by ${}_H\mathcal{M}$ the category of left H-modules and by $\mathcal{M}^H$ the category of right H-comodules. For $(V, \rho) \in \mathcal{M}^H$, we use notation: $\rho(v) = v_{(0)} \otimes v_{(1)} \in V \otimes H$, for any $v \in V$. Also, for $(W, \delta) \in \mathcal{M}^H$, we use notation: $\delta(w) = w^{(0)} \otimes w^{(1)} \in W \otimes H$, for any $w \in W$.

By [RT] a left-right Yetter-Drinfeld category ${}_H\mathcal{YD}^H$, whose morphisms are simultanously module and comodule maps, is the category of objects $(M, \cdot, \delta)$ such that $(M, \cdot)$ is in ${}_H\mathcal{M}$ and (M, δ) is in $\mathcal{M}^H$ satisfying the compatibility condition:

$$h_1 \cdot m^{(0)} \otimes h_2 m^{(1)} = (h_2 \cdot m)^{(0)} \otimes (h_2 \cdot m)^{(1)} h_1 \tag{1.1}$$

for all $h \in H$ and $m \in M$.

If S_H is bijective, then the equation (1.1) is equivalent to:

$$\rho(h \cdot m) = h_2 \cdot m^{(0)} \otimes h_3 m^{(1)} S_H^{-1}(h_1).$$

If H is a finite-dimensional Hopf algebra, then the Yetter-Drinfeld ${}_H\mathcal{YD}^H$ can be identified with the category ${}_{D(H)}\mathcal{M}$ of left modules over the Drinfeld double $D(H)$, see [Mont, p. 214].

There exist two prebraiding monoidal structures on ${}_H\mathcal{YD}^H$ [RT] as follows. Let $V, W \in {}_H\mathcal{YD}^H$. For $v \otimes w \in V \otimes W$ and $h \in H$, one structure is defined by the following structure (1.2)–(1.4):

$$h \rightharpoonup (v \otimes w) = h_2 \cdot v \otimes h_1 \cdot w \tag{1.2}$$

$$\delta(v \otimes w) = v^{(0)} \otimes w^{(0)} \otimes v^{(1)} w^{(1)} \tag{1.3}$$

$$\tau'_{V,W}(v \otimes w) = v^{(1)} \cdot w \otimes v^{(0)} \tag{1.4}$$

and then the category ${}_H\mathcal{YD}_1^H$ denotes the category ${}_H\mathcal{YD}^H$ which is equipped with the above prebraiding monoidal structures. Then $(V \otimes W, \rightharpoonup, \delta)$ is in the ${}_H\mathcal{YD}_1^H$.

Another structure is given by the following structure (1.5)–(1.7):

$$h \rightharpoonup (v \otimes w) = h_1 \cdot v \otimes h_2 \cdot w \tag{1.5}$$

$$\rho(v \otimes w) = v_{(0)} \otimes w_{(0)} \otimes w_{(1)} v_{(1)} \tag{1.6}$$

$$\tau''_{V,W}(v \otimes w) = w_{(0)} \otimes w_{(1)} \cdot v \tag{1.7}$$

and by the category ${}_H\mathcal{YD}_2^H$ we denotes the category ${}_H\mathcal{YD}^H$ with the prebraiding monoidal structures (1.5), (1.6) and (1.7). Then $(V \otimes W, \rightharpoonup, \rho)$ is in the ${}_H\mathcal{YD}_2^H$.

Let A, H be two Hopf algebras. In [Milit], the author study a relationship between a Long dimodule category and some of D-equations. By a generalized Long dimodule category ${}_H\mathcal{L}^A$ we mean a triple $(M, \rightharpoonup, \delta)$ such that $(M, \rightharpoonup) \in {}_H\mathcal{M}$ and $(M, \delta) \in \mathcal{M}^A$ satisfying the compatibility condition:

$$\delta(h \rightharpoonup b) = h \rightharpoonup b^{(0)} \otimes b^{(1)} \tag{1.8}$$

whose morphisms are simultanously module and comodule maps. Especially, when $A = H$ we get a Long dimodule category ${}_H\mathcal{L}^H$.

A quasitriangular Hopf algebra is a pair (H, R), where H is a Hopf algebra over k and $R = R^{(1)} \otimes R^{(2)} \in H \otimes H$ is invertible such that the following ($r = R$):

$$(QT1) \quad \Delta R^{(1)} \otimes R^{(2)} = R^{(1)} \otimes r^{(1)} \otimes R^{(2)} r^{(2)}$$

$$(QT2) \quad R^{(1)} \otimes \Delta R^{(2)} = R^{(1)} r^{(1)} \otimes r^{(2)} \otimes R^{(2)}$$

$$(QT3) \quad \Delta^{cop}(h) = R \Delta(h) R^{-1}$$

is satisfied, where $\Delta^{cop}(h) = h_2 \otimes h_1$ for all $h \in H$. If $R^{-1} = R^{(2)} \otimes R^{(1)}$, then (H, R) is called a triangular Hopf algebra.

As a dual concept (see [DT, Sect 3]), a coquasitriangular Hopf algebra is a pair $(H, \langle|\rangle)$ where H is a Hopf algebra over k and $\langle|\rangle : H \otimes H \to k$ is a k-linear form which is convolution invertible in $Hom_k(H \otimes H, k)$ such that the following hold:

$$(BR1) \quad \langle h \,|\, gl\rangle = \langle h_1|l\rangle\langle h_2|g\rangle$$

$$(BR2) \quad \langle hg \,|\, l\rangle = \langle h|l_1\rangle\langle g|l_2\rangle$$

$$(BR3) \quad \langle h_1|g_1\rangle h_2 g_2 = g_1 h_1 \langle h_2|g_2\rangle.$$

If $\langle h_1|g_1\rangle\langle g_2|h_2\rangle = \varepsilon(g)\varepsilon(h)$ then $(H, \langle|\rangle)$ is called a cotriangular Hopf algebra.

A Hopf pairing (B, H, τ) means a triple (B, H, τ), where B, H are Hopf algebras and the $\tau : B \times H \to k$ is a (convolution) invertible bilinear form satisfying:

$$(DP1) \quad \tau(ab, h) = \tau(a, h_1)\tau(b, h_2)$$

$$(DP2) \quad \tau(a, hl) = \tau(a_1, h)\tau(a_2, l)$$

$$(DP3) \quad \tau(1, h) = \varepsilon(h)1$$

$$(DP4) \quad \tau(a, 1) = \varepsilon(a)1$$

It is easy to see that $(DP1)$ and $(DP2)$ yield

$$(DP1)' \quad \tau^{-1}(ab, h) = \tau^{-1}(a, h_2)\tau^{-1}(b, h_1)$$

$$(DP2)' \quad \tau^{-1}(a, hl) = \tau^{-1}(a_1, l)\tau^{-1}(a_2, h)$$

for $a, b \in B,\ h, l \in H$.

Let C be a coalgebra. The opposite coalgebra C^{cop} is C as a k-module with comultiplication given by $\Delta^{cop}(c) = c_2 \otimes c_1$ for $c \in C$, where we write $\Delta(c) = c_1 \otimes c_2$. Let H be a Hopf algebra. Suppose the antipode S of H is bijective (this holds true if H is quasitriangular or coquasitriangular). Then, H^{op} and H^{cop} are both Hopf algebras with antipode S^{-1}.

EXAMPLE 1.1. Let (B, H, τ) be skew-pairing Hopf algebra ([DT]). Then (B, H^{cop}, τ) and (B^{op}, H, τ) are Hopf pairings.

EXAMPLE 1.2. Let $(H, \langle|\rangle)$ be coquasitriangular Hopf algebra. Then $(H, H^{cop}, \langle|\rangle)$ and $(H^{op}, H, \langle|\rangle)$ are Hopf pairings.

EXAMPLE 1.3. Let H be a finite-dimensional Hopf algebra. Then $(H^*, H, \langle , \rangle)$ is Hopf pairing, where H^* is the dual Hopf algebra and $\langle , \rangle$ is the evalution map.

Dually, we define a dual R-Hopf algebra is a triple (B, H, R), where B, H are two Hopf algebras and the $R = R^{(1)} \otimes R^{(2)} \in B \otimes H$ is an invertible element such that the following $(r = R)$:

$$(QT1) \quad \Delta R^{(1)} \otimes R^{(2)} = R^{(1)} \otimes r^{(1)} \otimes R^{(2)} r^{(2)}$$

$$(QT2) \quad R^{(1)} \otimes \Delta R^{(2)} = R^{(1)} r^{(1)} \otimes R^{(2)} \otimes r^{(2)}$$

is satisfied, for all $h \in H$. It is not hard to show $R^{(1)} \otimes R^{(2)} = SR^{(1)} \otimes SR^{(2)} = S^2 R^{(1)} \otimes R^{(2)} = R^{(1)} \otimes S^2 R^{(2)}$ and $R^{-1} = SR^{(1)} \otimes R^{(2)} = R^{(1)} \otimes SR^{(2)}$.

EXAMPLE 1.4. Let (B, H, R) be a R-Hopf algebras ([WZJ]). Then (B^{op}, H, R) and (B, H^{cop}, R) are dual R-Hopf algebras.

EXAMPLE 1.5. Let (H, R) be quasitriangular Hopf algebra. Then (H^{op}, H, R) and (H, H^{cop}, R) are dual R-Hopf algebras.

EXAMPLE 1.6. Let H be a finite-dimensional Hopf algebra. Then (H, H^*, R) is dual R-Hopf algebras, where H^* is the dual Hopf algebra and $R = \sum_{i=1}^{n} h_i \otimes h_i^*$ here $\{h_i\}$ and $\{h_i^*\}$ are dual basis of H and H^*.

§ 2. Braided Bialgebras in ${}_H\mathscr{YD}^H$

Let $(A, \cdot, \delta_A)$ be an algebra in ${}_H\mathscr{YD}_1^H$ where $\cdot$ and δ_A is a left H-module structure and a right H-comodule structure on A respectively. We define $A^* = A$ as linear space, with a twisted multiplication given by:

$$a * b = (b^{(1)} \cdot a) b^{(0)}$$

PROPOSITION 2.1. *$(A^*, *)$ is an associative algebra.*

PROOF. It is easy to see that 1_A is a unit of A^*. As to associativity of $*$, one has:

$$(a * b) * c = (c^{(1)} \cdot [(b^{(1)} \cdot a) b^{(0)}]) c^{(0)}$$

$$\overset{(1.2)}{=} (c^{(1)}{}_2 b^{(1)} \cdot a)(c^{(1)}{}_1 \cdot b^{(0)}) c^{(0)} \quad \text{(since } A \text{ is } H\text{-module algebra)}$$

$$\overset{(1.1)}{=} [(c^{(1)}{}_2 \cdot b)^{(1)} c^{(1)}{}_1 \cdot a](c^{(1)}{}_2 \cdot b)^{(0)} c^{(0)}$$

$$= [(c^{(1)} \cdot b)^{(1)} c^{(0)(1)} \cdot a](c^{(1)} \cdot b)^{(0)} c^{(0)(0)}$$

$$\overset{(1.3)}{=} [((c^{(1)} \cdot b)c^{(0)})^{(1)} \cdot a]((c^{(1)} \cdot b)c^{(0)})^{(0)} = a * (c^{(1)} \cdot b)c^{(0)} = a * (b * c).$$

This concludes the proof.

REMARK. That $(A, \cdot, \delta_A)$ is an algebra in ${}_H\mathcal{YD}_1^H$ is not a necessary condition for $(A^*, *)$ to be an associative algebra. This is seen in the following (2.10) and the proof of Theorem 2.3.

Similarly, for any $(A, \cdot, \rho) \in {}_H\mathcal{YD}_2^H$ we define $A^\star = A$ as linear space with a twisted multiplication defined by:

$$a \star b = a_{(0)}(a_{(1)} \cdot b),$$

and we have the following proposition:

PROPOSITION 2.2. *$(A^\star, \star)$ is an associative algebra.*

Let $(B, \rightharpoonup, \delta)$ be an algebra in ${}_H\mathcal{YD}_1^H$ and $(B, \rightharpoonup, \rho)$ an algebra in ${}_H\mathcal{YD}_2^H$ such that $(B, \rightharpoonup, \rho)$ and $(B, \rightharpoonup, \delta)$ are in ${}_H\mathcal{L}^H$.

Now, we assume that the following condition (A) are satisfied:

CONDITION (A).

$$h \rightharpoonup (l \rightharpoonup b) = l \rightharpoonup (h \rightharpoonup b) \tag{2.1}$$

$$(h_1 \rightharpoonup b_1) \otimes (h_2 \rightharpoonup b_2) = \varepsilon(h) b_1 \otimes b_2 \tag{2.2}$$

$$\Delta(h \rightharpoonup b) = (h \rightharpoonup b_1) \otimes b_2 \tag{2.3}$$

$$\Delta(h \rightharpoonup b) = b_1 \otimes (h \rightharpoonup b_2) \tag{2.4}$$

$$b^{(1)} \otimes b^{(0)}{}_{(1)} \otimes b^{(0)}{}_{(0)} = b_{(0)}{}^{(1)} \otimes b_{(1)} \otimes b_{(0)}{}^{(0)} \tag{2.5}$$

$$b_{1(0)} \otimes b_2{}^{(0)} \otimes b_2{}^{(1)} b_{1(1)} = b_1 \otimes b_2 \otimes 1 \tag{2.6}$$

$$b_1{}^{(0)} \otimes b_2 \otimes b_1{}^{(1)} = b^{(0)}{}_1 \otimes b^{(0)}{}_2 \otimes b^{(1)} \tag{2.7}$$

$$b_1 \otimes b_{2(0)} \otimes b_{2(1)} = b_{(0)1} \otimes b_{(0)2} \otimes b_{(1)}, \tag{2.8}$$

for any $b \in B$ and $h, l \in H$.

Then, we define:

$$h \rightharpoonup b = h_1 \rightharpoonup (h_2 \rightharpoonup b) \tag{2.9}$$

$$a \star b = (b^{(1)} \rightharpoonup a)b^{(0)} \tag{2.10}$$

$$\chi_B(b) = b_{(0)}{}^{(0)} \otimes b_{(1)}b_{(0)}{}^{(1)}. \tag{2.11}$$

It is not hard to verify that $(B, \rightharpoonup)$ is a left H-module; that (B, χ_B) is a right H-comodule; and that $(B, \rightharpoonup, \chi_B)$ is an object in ${}_H\mathcal{YD}^H$. But the $(B, m_B, \rightharpoonup)$ is not an algebra in ${}_H\mathcal{YD}_2^H$. In fact, we have

$$\begin{aligned} h \rightharpoonup (ab) &= h_1 \rightharpoonup (h_2 \rightharpoonup (ab)) \overset{(1.5)}{=} h_1 \rightharpoonup [(h_2 \rightharpoonup a)(h_3 \rightharpoonup b)] \\ &\overset{(1.2)}{=} (h_2 \rightharpoonup (h_3 \rightharpoonup a))(h_1 \rightharpoonup (h_4 \rightharpoonup b)) = (h_2 \rightarrow a)(h_1 \rightharpoonup (h_3 \rightharpoonup b)) \\ &\neq (h_1 \rightarrow a)(h_2 \rightarrow b), \end{aligned}$$

and this proves that $(B, \rightharpoonup, \delta_B^r)$ is not an H-module algebra. Thus we cannot apply Proposition 2.1 to $(B, m_B, \rightharpoonup)$. However, one can calculate:

$$\begin{aligned} a \star (b \star c) &\overset{(2.10)}{=} a \star (c^{(1)} \rightarrow b)c^{(0)} \\ &\overset{(1.3)(2.10)}{=} [(c^{(1)} \rightarrow b)^{(1)}c^{(0)(1)} \rightarrow a](c^{(1)} \rightarrow b)^{(0)}c^{(0)(0)} \\ &\overset{(2.9)}{=} [(c^{(1)}{}_2 \rightharpoonup (c^{(1)}{}_3 \rightharpoonup b))^{(1)}c^{(1)}{}_1 \rightarrow a](c^{(1)}{}_2 \rightharpoonup (c^{(1)}{}_3 \rightharpoonup b))^{(0)}c^{(0)} \\ &\overset{(1.1)}{=} [c^{(1)}{}_2(c^{(1)}{}_3 \rightharpoonup b)^{(1)} \rightarrow a](c^{(1)}{}_1 \rightharpoonup (c^{(1)}{}_3 \rightharpoonup b)^{(0)})c^{(0)} \\ &\overset{(1.8)}{=} (c^{(1)}{}_2 b^{(1)} \rightarrow a)(c^{(1)}{}_1 \rightharpoonup (c^{(1)}{}_3 \rightharpoonup b^{(0)}))c^{(0)} \\ &\overset{(2.9)}{=} [c^{(1)}{}_2 \rightharpoonup (c^{(1)}{}_3 \rightharpoonup (b^{(1)} \rightarrow a))](c^{(1)}{}_1 \rightharpoonup (c^{(1)}{}_4 \rightharpoonup b^{(0)}))c^{(0)} \\ &\overset{(1.2)}{=} (c^{(1)}{}_1 \rightharpoonup [(c^{(1)}{}_2 \rightharpoonup (b^{(1)} \rightarrow a))(c^{(1)}{}_3 \rightharpoonup b^{(0)})])c^{(0)} \\ &= (c^{(1)}{}_1 \rightharpoonup (c^{(1)}{}_2 \rightharpoonup [(b^{(1)} \rightarrow a)b^{(0)})])c^{(0)} \\ &= (c^{(1)} \rightharpoonup [(b^{(1)} \rightarrow a)b^{(0)})])c^{(0)} = (a \star b) \star c, \end{aligned}$$

and this proves that $(B, \star)$ is an associative algebra with the unit 1_B.

THEOREM 2.3. *Let H be any Hopf algebra, and B a bialgebra. Let $(B, \rightharpoonup, \delta)$ be an algebra in ${}_H\mathcal{YD}_1^H$ and $(B, \rightharpoonup, \rho)$ an algebra in ${}_H\mathcal{YD}_2^H$ such that $(B, \rightharpoonup, \rho)$ and $(B, \rightharpoonup, \delta)$ are in ${}_H\mathcal{L}^H$. If the condition* (A) *hold, then there exists a bialgebra $\overline{B}$ in ${}_H\mathcal{YD}_2^H$, $\overline{B} = B$ as a linear space and as an object in ${}_H\mathcal{YD}_2^H$ with a H-module*

structure defined by (2.9), *and H-comodule structure defined by* (2.11) *respectively. The coalgebra structure and unit in* $\overline{B}$ *coincide with those of B. The multiplication is given by* (2.10).

PROOF. We show that $(B, \rightharpoonup, \star)$ is a left H-module algebra in ${}_H\mathcal{YD}_2^H$ as follows. Thus

$$\begin{aligned}
h \rightharpoonup (a \star b) &\overset{(2.10)(1.5)}{=} h_1 \rightharpoonup [(h_2 \rightharpoonup (b^{(1)} \rightharpoonup a))(h_3 \rightharpoonup b^{(0)})] \\
&\overset{(1.2)}{=} (h_2 \rightharpoonup (h_3 \rightharpoonup (b^{(1)} \rightharpoonup a)))(h_1 \rightharpoonup (h_4 \rightharpoonup b^{(0)})) \\
&\overset{(2.1)}{=} (h_3 \rightharpoonup (h_2 \rightharpoonup (b^{(1)} \rightharpoonup a)))(h_4 \rightharpoonup (h_1 \rightharpoonup b^{(0)})) \\
&\overset{(2.9)}{=} (h_3 \rightharpoonup (h_2 b^{(0)(1)} \rightharpoonup (b^{(1)} \rightharpoonup a)))(h_4 \rightharpoonup (h_1 \rightharpoonup b^{(0)(0)})) \\
&\overset{(1.1)}{=} (h_3 \rightharpoonup ((h_2 \rightharpoonup b^{(0)})^{(1)} h_1 \rightharpoonup (b^{(1)} \rightharpoonup a)))(h_4 \rightharpoonup (h_2 \rightharpoonup b^{(0)})^{(0)}) \\
&\overset{(2.1)}{=} ((h_2 \rightharpoonup b^{(0)})^{(1)} h_1 \rightharpoonup (h_3 b^{(1)} \rightharpoonup a))(h_4 \rightharpoonup (h_2 \rightharpoonup b^{(0)})^{(0)}) \\
&\overset{(1.1)}{=} ((h_3 \rightharpoonup b)^{(0)(1)} h_1 \rightharpoonup (h_3 \rightharpoonup b)^{(1)} h_2 \rightharpoonup a)(h_4 \rightharpoonup (h_3 \rightharpoonup b)^{(0)(0)}) \\
&\overset{(2.9)}{=} ((h_2 \rightharpoonup b)^{(1)} h_1 \rightharpoonup a)(h_3 \rightharpoonup (h_2 \rightharpoonup b)^{(0)}) \\
&\overset{(1.8)}{=} ((h_3 \rightharpoonup (h_2 \rightharpoonup b)^{(1)} h_1 \rightharpoonup a)(h_3 \rightharpoonup (h_2 \rightharpoonup b))^{(0)} \\
&\overset{(2.1)(2.9)}{=} ((h_2 \rightharpoonup b)^{(1)} h_1 \rightharpoonup a)(h_2 \rightharpoonup b)^{(0)} = (h_1 \rightharpoonup a) \star (h_2 \rightharpoonup b),
\end{aligned}$$

which is as required.

And then, we check that $(B, \star, \chi_B)$ is a right H-comodule algebra in ${}_H\mathcal{YD}_2^H$ according to the equation (1.6). In fact, one has

$$\begin{aligned}
\chi_B(a)\chi_B(b) &= a_{(0)}{}^{(0)} \star b_{(0)}{}^{(0)} \otimes b_{(1)} b_{(0)}{}^{(1)} a_{(1)} a_{(0)}{}^{(1)} \\
&= (b_{(0)}{}^{(0)(1)} \rightharpoonup a_{(0)}{}^{(0)}) b_{(0)}{}^{(0)(0)} \otimes b_{(1)} b_{(0)}{}^{(1)} a_{(1)} a_{(0)}{}^{(1)},
\end{aligned}$$

on the other hand,

$$\begin{aligned}
\chi_B(a \star b) &\overset{(2.10)}{=} \chi_B((b^{(1)} \rightharpoonup a) b^{(0)}) \\
&\overset{(2.11)}{=} ((b^{(1)} \rightharpoonup a) b^{(0)})_{(0)}{}^{(0)} \otimes ((b^{(1)} \rightharpoonup a) b^{(0)})_{(1)} ((b^{(1)} \rightharpoonup a) b^{(0)})_{(0)}{}^{(1)} \\
&\overset{(1.6)(1.3)}{=} (b^{(1)} \rightharpoonup a)_{(0)}{}^{(0)} b^{(0)}{}_{(0)}{}^{(0)} \otimes b^{(0)}{}_{(1)} (b^{(1)} \rightharpoonup a)_{(1)} (b^{(1)} \rightharpoonup a)_{(0)}^{(1)} b^{(0)}{}_{(0)}{}^{(1)}
\end{aligned}$$

$$\overset{(2.9)}{=} (b^{(1)}{}_1 \rightharpoonup (b^{(1)}{}_2 \rightharpoonup a))_{(0)}{}^{(0)} b^{(0)}{}_{(0)}{}^{(0)} \otimes b^{(0)}{}_{(1)} (b^{(1)}{}_1 \rightharpoonup (b^{(1)}{}_2 \rightharpoonup a))_{(1)}$$

$$(b^{(1)}{}_1 \rightharpoonup (b^{(1)}{}_2 \rightharpoonup a))_{(0)}{}^{(1)} b^{(0)}{}_{(0)}{}^{(1)}$$

$$\overset{(1.8)}{=} (b^{(1)}{}_1 \rightharpoonup (b^{(1)}{}_2 \rightharpoonup a)_{(0)})^{(0)} b^{(0)}{}_{(0)}{}^{(0)} \otimes b^{(0)}{}_{(1)} (b^{(1)}{}_2 \rightharpoonup a)_{(1)}$$

$$(b^{(1)}{}_1 \rightharpoonup (b^{(1)}{}_2 \rightharpoonup a)_{(0)})^{(1)} b^{(0)}{}_{(0)}{}^{(1)}$$

$$\overset{(2.5)}{=} (b_{(0)}{}^{(1)}{}_1 \rightharpoonup (b_{(0)}{}^{(1)}{}_2 \rightharpoonup a)_{(0)})^{(0)} b_{(0)}{}^{(0)(0)} \otimes b_{(1)} (b_{(0)}{}^{(1)}{}_2 \rightharpoonup a)_{(1)}$$

$$(b_{(0)}{}^{(1)}{}_1 \rightharpoonup (b_{(0)}{}^{(1)}{}_2 \rightharpoonup a)_{(0)})^{(1)} b_{(0)}{}^{(0)(1)}$$

$$\overset{(1.1)}{=} (b_{(0)}{}^{(1)}{}_1 \rightharpoonup (b_{(0)}{}^{(1)}{}_3 \rightharpoonup a)_{(0)}{}^{(0)}) b_{(0)}{}^{(0)} \otimes b_{(1)} (b_{(0)}{}^{(1)}{}_3 \rightharpoonup a)_{(1)}$$

$$b_{(0)}{}^{(1)}{}_2 (b_{(0)}{}^{(1)}{}_3 \rightharpoonup a)_{(0)}{}^{(1)}$$

$$\overset{(2.5)}{=} (b^{(1)}{}_1 \rightharpoonup (b^{(1)}{}_3 \rightharpoonup a)_{(0)}{}^{(0)}) b^{(0)}{}_{(0)} \otimes b^{(0)}{}_{(1)} (b^{(1)}{}_3 \rightharpoonup a)_{(1)}$$

$$\times b^{(1)}{}_2 (b^{(1)}{}_3 \rightharpoonup a)_{(0)}{}^{(1)}$$

$$\overset{(1.1)}{=} (b^{(1)}{}_1 \rightharpoonup (b^{(1)}{}_2 \rightharpoonup a_{(0)})^{(0)}) b^{(0)}{}_{(0)} \otimes b^{(0)}{}_{(1)} b^{(1)}{}_3 a_{(1)} (b^{(1)}{}_2 \rightharpoonup a_{(0)})^{(1)}$$

$$\overset{(1.8)}{=} (b^{(1)}{}_1 \rightharpoonup (b^{(1)}{}_2 \rightharpoonup a_{(0)}{}^{(0)})) b^{(0)}{}_{(0)} \otimes b^{(0)}{}_{(1)} b^{(1)}{}_3 a_{(1)} a_{(0)}{}^{(1)}$$

$$\overset{(2.9)}{=} (b^{(1)}{}_1 \rightharpoonup a_{(0)}{}^{(0)}) b^{(0)}{}_{(0)} \otimes b^{(0)}{}_{(1)} b^{(1)}{}_2 a_{(1)} a_{(0)}{}^{(1)}$$

$$\overset{(2.5)}{=} (b_{(0)}{}^{(0)(1)} \rightharpoonup a_{(0)}{}^{(0)}) b_{(0)}{}^{(0)(0)} \otimes b_{(1)} b_{(0)}{}^{(1)} a_{(1)} a_{(0)}{}^{(1)},$$

again as required.

It is easy to see that $(B, \Delta_B, \rightharpoonup)$ is a module coalgebra in ${}_H\mathcal{YD}_2^H$ by the conditions (2.2), (2.3) and (2.4). It follows from the formulae (2.6)–(2.8) that (B, Δ_B, χ_B) is a left H-comodule coalgebra in ${}_H\mathcal{YD}_2^H$.

In final, using the braiding τ'' in ${}_H\mathcal{YD}_2^H$ (see (1.7)), we can form a braided tensor product $\bar{B} \otimes \bar{B} : (a \otimes b)(c \otimes d) = ac_{(0)} \otimes (c_{(1)} \rightharpoonup b)d$ for any $a, b, c, d \in B$. We are prepared to show that Δ_B is an algebra map from $\bar{B}$ to $\bar{B} \otimes \bar{B}$. Thus, one does a calculation as follows:

$$\Delta_B(a \star b) = (b^{(1)} \rightarrow a)_1 b^{(0)}{}_1 \otimes (b^{(1)} \rightarrow a)_2 b^{(0)}{}_2$$

$$= (b^{(1)}{}_1 \rightarrow a_1) b^{(0)}{}_1 \otimes (b^{(1)}{}_2 \rightarrow a_2) b^{(0)}{}_2]$$

(since $(B, \Delta_B, \rightharpoonup) \in {}_H\mathcal{YD}_2^H$ is a module coalgebra)

$$\overset{(2.9)}{=} (b^{(1)}{}_1 \rightharpoonup (b^{(1)}{}_2 \rightharpoonup a_1)b^{(0)}{}_1 \otimes (b^{(1)}{}_3 \rightharpoonup b^{(1)}{}_4 \rightharpoonup a_2)b^{(0)}{}_2$$

$$\overset{(2.1)(2.2)}{=} (b^{(1)}{}_1 \rightharpoonup a_1)b^{(0)}{}_1 \otimes (b^{(1)}{}_2 \rightharpoonup a_2)b^{(0)}{}_2,$$

and on the other hand, we have

$$\begin{aligned}
&a_1 \star b_{1(0)}{}^{(0)} \otimes (b_{1(1)}b_{1(0)}{}^{(1)} \rightharpoonup a_2) \star b_2 \\
&\quad = (b_{1(0)}{}^{(0)(1)} \rightharpoonup a_1)b_{1(0)}{}^{(0)(0)} \otimes (b_2{}^{(1)}b_{1(1)}b_{1(0)}{}^{(1)} \rightharpoonup a_2)b_2{}^{(0)} \\
&\quad \overset{(2.5)}{=} (b_1{}^{(0)(1)} \rightharpoonup a_1)b_1{}^{(0)(0)}{}_{(0)} \otimes (b_2{}^{(1)}b_1{}^{(0)(0)}{}_{(1)}b_1{}^{(1)} \rightharpoonup a_2)b_2{}^{(0)} \\
&\quad = (b_1{}^{(1)}{}_1 \rightharpoonup a_1)b_1{}^{(0)}{}_{(0)} \otimes (b_2{}^{(1)}b_1{}^{(0)}{}_{(1)}b_1{}^{(1)}{}_2 \rightharpoonup a_2)b_2{}^{(0)} \\
&\quad \overset{(2.9)}{=} (b_1{}^{(1)}{}_1 \rightharpoonup b_1{}^{(1)}{}_2 \rightharpoonup a_1)b_1{}^{(0)}{}_{(0)} \otimes (b_2{}^{(1)}{}_1b_1{}^{(0)}{}_{(1)1}b_1{}^{(1)}{}_3 \\
&\qquad \rightharpoonup b_2{}^{(1)}{}_2b_1{}^{(0)}{}_{(1)2}b_1{}^{(1)}{}_4 \rightharpoonup a_2)b_2{}^{(0)} \\
&\quad \overset{(2.1)(2.2)}{=} (b_1{}^{(1)}{}_1 \rightharpoonup a_1)b_1{}^{(0)}{}_{(0)} \otimes (b_2{}^{(1)}{}_1b_1{}^{(0)}{}_{(1)1} \rightharpoonup b_2{}^{(1)}{}_2b_1{}^{(0)}{}_{(1)2}b_1{}^{(1)}{}_2 \rightharpoonup a_2)b_2{}^{(0)} \\
&\quad \overset{(2.7)}{=} (b^{(1)}{}_1 \rightharpoonup a_1)b^{(0)}{}_{1(0)} \otimes (b^{(0)}{}_2{}^{(1)}{}_1b^{(0)}{}_{1(1)1} \rightharpoonup b^{(0)}{}_2{}^{(1)}{}_2b^{(0)}{}_{1(1)2}b^{(1)}{}_2 \rightharpoonup a_2)b^{(0)}{}_2{}^{(0)} \\
&\quad = (b^{(1)}{}_1 \rightharpoonup a_1)b^{(0)}{}_{1(0)} \otimes ((b^{(0)}{}_2{}^{(1)}b^{(0)}{}_{1(1)})_1 \rightharpoonup (b^{(0)}{}_2{}^{(1)}b^{(0)}{}_{1(0)})_2b^{(1)}{}_2 \rightharpoonup a_2)b^{(0)}{}_2{}^{(0)} \\
&\quad \overset{(2.9)}{=} (b^{(1)}{}_1 \rightharpoonup a_1)b^{(0)}{}_{1(0)} \otimes (b^{(0)}{}_2{}^{(1)}b^{(0)}{}_{1(1)} \rightharpoonup (b^{(1)}{}_2 \rightharpoonup a_2))b^{(0)}{}_2{}^{(0)} \\
&\quad \overset{(2.6)}{=} (b^{(1)}{}_1 \rightharpoonup a_1)b^{(0)}{}_1 \otimes (b^{(1)}{}_2 \rightharpoonup a_2))b^{(0)}{}_2.
\end{aligned}$$

Thus, $(B, \Delta, \star)$ is a bialgebra in ${}_H\mathscr{YD}^H$ concluding the proof.

Similarly, we can make another definition as follows:

$$h \succ b = h_1 \rightharpoonup (h_2 \rightharpoonup b) \tag{2.12}$$

$$a \bar{\star} b = a_{(0)}(a_{(1)} \succ b) \tag{2.13}$$

$$\xi_B(b) = b_{(0)}{}^{(0)} \otimes b_{(0)}{}^{(1)}b_{(1)} \tag{2.14}$$

In what follows, we postulate the following condition (2.15) and (2.16) respectively instead of the equation (2.2) and (2.6). In the condition (A).

$$(h_2 \rightharpoonup b_1) \otimes (h_1 \rightharpoonup b_2) = \varepsilon(h)b_1 \otimes b_2 \tag{2.15}$$

$$b_{1(0)} \otimes b_2{}^{(0)} \otimes b_{1(1)}b_2{}^{(1)} = b_1 \otimes b_2 \otimes 1. \tag{2.16}$$

Then we have another main theorem of this section:

THEOREM 2.4. *Let H be any Hopf algebra, and B a bialgebra. Let $(B, \rightharpoonup, \delta)$ be an algebra in ${}_H\mathcal{YD}_1^H$ and $(B, \rightharpoonup, \rho)$ an algebra in ${}_H\mathcal{YD}_2^H$ such that $(B, \rightharpoonup, \rho)$ and $(B, \rightharpoonup, \delta)$ are in ${}_H\mathcal{L}^H$. Assume that the conditions* (2.1), (2.3)–(2.5), (2.7)–(2.8), *and the conditions* (2.15)–(2.16) *hold. Then there exists a bialgebra $\underline{B}$ in ${}_H\mathcal{YD}_1^H$, $\underline{B} = B$ as a linear space and as an object in ${}_H\mathcal{YD}_1^H$ with a H-module structure defined by* (2.12), *and H-comodule structure defined by* (2.14) *respectively. The coalgebra structure and unit in $\underline{B}$ coincide with those of B. The multiplication is given by* (2.13).

PROOF. Similar to that of the theorem 2.3.

REMARK 2.5. The left Yetter-Drinfeld modules form the braided category ${}^H_H\mathcal{YD}$, see [Mont, p. 214]. Similarly, the right such modules form $\mathcal{YD}^H_H$. We have natural identification of braided categories,

$$ {}_H\mathcal{YD}_1^H = {}^{H^{cop}}_{H^{cop}}\mathcal{YD}, \quad {}_H\mathcal{YD}_2^H = \mathcal{YD}^{H^{op}}_{H^{op}}. $$

Replace H by H^{op}, and identify $H^{op,cop}$ with H via $S : H \xrightarrow{\cong} H^{op,cop}$. Thus, if M is an object in ${}_{H^{op}}\mathcal{YD}_1^{H^{op}}$ with structures $(h^{op}, m) \mapsto h^{op}m$, $H^{op} \otimes M \to M$ and $m \mapsto m^{(0)} \otimes m^{(1)}$, $M \to M \otimes H$, then it is identified with an object in ${}^H_H\mathcal{YD}$ with the structures given by

$$ hm = S(h)^{op}m, \quad \lambda(m) = S^{-1}(m^{(1)}) \otimes m^{(0)} \in H \otimes M. $$

Theorem 2.3 is translated as follows.

Let B be a bialgebra. Suppose the algebra B is further an algebra object in ${}^H_H\mathcal{YD}$ and also in $\mathcal{YD}^H_H$. Suppose that each pair of structures indicated by

$$ {}_HB_H, \quad {}^HB^H, \quad {}_HB^H, \quad {}^HB_H $$

commutes with each other (the commutativity in ${}^HB^H$, for example, means that B is an H-bicomodule). Denote the left and the right H-comodule structures on B by

$$ \lambda(b) = b^{(-1)} \otimes b^{(0)}, \quad \rho(b) = b^{(0)} \otimes b^{(1)} \quad (b \in B), $$

respectively, and suppose further that

$$ b_1h \otimes b_2 = b_1 \otimes hb_2, \quad \rho(b_1) \otimes b_2 = b_1 \otimes \lambda(b_2); $$

$$ \Delta(hb) = hb_1 \otimes b_2, \quad \Delta(bh) = b_1 \otimes b_2h; $$

$$ \lambda(b_1) \otimes b_2 = b^{(-1)} \otimes \Delta(b^{(0)}), \quad b_1 \otimes \rho(b_2) = \Delta(b^{(0)}) \otimes b^{(1)}, $$

where $b \in B$, $h \in H$. Then the coalgebra B forms a bialgebra in ${}^{H}_{H}\mathcal{YD}$, given the following new structures:

$$b \leftarrow h := S^{-1}(h_1)bh_2;$$

$$a \star b := (a \leftarrow S(b^{(-1)}))b^{(0)} = (b^{(-1)}aS(b^{(-2)}))b^{(0)};$$

$$b \mapsto b^{(0)} \otimes S(b^{(-1)})b^{(1)}, \quad B \to B \otimes H,$$

where $a, b \in B$, $h \in H$.

Similarly, Theorem 2.4 can be reformulated in a symmetric form, which will give a construction of bialgebras in ${}^{H}_{H}\mathcal{YD}$. These reformulated statements look simpler than the original, although here one has to assume that the antipode S of H is bijective.

§ 3. Braided Hopf Algebras

Let H be a Hopf algebra with a bijective antipode S, and B a Hopf algebra with an antipode S_B. In this section we give a sufficient condition for the braided bialgebras as defined in § 2 to be a braided Hopf algebra. At first, we assume that the following condition (B) is satisfied:

CONDITIONS (B).

$$S_B(h \rightharpoonup b) = h \rightharpoonup S_B(b) \tag{3.1}$$

$$S_B(h \rightharpoondown b) = S^{-2}h \rightharpoondown S_B(b) \tag{3.2}$$

$$(S_B(b))^{(0)} \otimes (S_B(b))^{(1)} = S_B b_{(0)} \otimes S^{-2}(b_{(1)}) \tag{3.3}$$

$$(S_B(b))_{(0)} \otimes (S_B(b))_{(1)} = S_B b^{(0)} \otimes b^{(1)}, \tag{3.4}$$

where the S^{-2} denotes the $(S^{-1})^2$.

PROPOSITION 3.1. *In the situation of the Theorem* 2.3. *Assume that the condition* (B) *holds. If B has an antipode then $\bar{B}$ has an antipode in the category ${}_{H}\mathcal{YD}^{H}_{2}$. It is defined by*

$$\bar{S}(b) = b_{(1)} \rightharpoonup S_B(b_{(0)}).$$

PROOF. We need to prove that $\bar{S}$ is a morphism in ${}^{H}_{H}\mathcal{YD}_2$. For this, we have

$$\begin{aligned}
\bar{S}(h \rightharpoonup b) &= (h \rightharpoonup b)_{(1)} \rightharpoonup S_B((h \rightharpoonup b)_{(0)}) \\
&\overset{(2.9)(1.8)}{=} (h_2 \rightharpoondown b)_{(1)} \rightharpoonup S_B(h_1 \rightharpoonup (h_2 \rightharpoondown b)_{(0)}) \\
&\overset{(1.1)}{=} h_4 b_{(1)} S^{-1}(h_2) \rightharpoonup S_B(h_1 \rightharpoonup h_3 \rightharpoondown b_{(0)})
\end{aligned}$$

$$\overset{(2.9)}{=} h_5 b_{(1)1} S^{-1}(h_3) \rightharpoonup h_6 b_{(1)2} S^{-1}(h_2) \rightharpoonup S_B(h_1 \rightharpoonup h_4 \rightharpoonup b_{(0)})$$

$$\overset{(3.1)-(3.4)}{=} h_5 b_{(1)1} S^{-1}(h_3) \rightharpoonup h_6 b_{(1)2} S^{-1}(h_2) h_1 \rightharpoonup S^{-2}(h_4) \rightharpoonup S_B b_{(0)}$$

$$\overset{(2.1)}{=} h_1 b_{(1)1} \rightharpoonup h_2 b_{(1)2} \rightharpoonup S_B b_{(0)}$$

$$\overset{(2.9)}{=} h \rightarrow \bar{S}(b),$$

and this prove $\bar{S}$ is H-module map.

Also, one has

$$\begin{aligned}
\xi_B \circ \bar{S}(b) &= (b_{(1)} \rightarrow S_B b_{(0)})_{(0)}{}^{(0)} \otimes (b_{(1)} \rightarrow S_B b_{(0)})_{(1)} (b_{(1)} \rightarrow S_B b_{(0)})_{(0)}{}^{(1)} \\
&= (b_{(1)1} \rightharpoonup b_{(1)3} \rightharpoonup (S_B b_{(0)})_{(0)})^{(0)} \otimes b_{(1)4} (S_B b_{(0)})_{(1)} \\
&\quad S^{-1} b_{(1)2} (b_{(1)1} \rightharpoonup b_{(1)3} \rightharpoonup (S_B b_{(0)})_{(0)})^{(1)} \\
&= (b_{(1)5} \rightharpoonup b_{(1)2} \rightharpoonup S_B(b_{(0)}{}^{(0)}{}_{(0)}) \otimes b_{(1)4} b_{(0)}{}^{(1)} \\
&\quad S^{-1}(b_{(1)4}) b_{(1)3} S^{-2}(b_{(0)}{}^{(0)}{}_{(1)}) S^{-1}(b_{(1)1}) \\
&= b^{(0)}{}_{(0)(1)} \rightarrow S_B(b^{(0)}{}_{(0)(0)}) \otimes b^{(0)}{}_{(1)} b^{(1)} \\
&= (\bar{S} \otimes id)(b^{(0)}{}_{(0)} \otimes b^{(0)}{}_{(1)} b^{(1)}) = (\bar{S} \otimes id) \xi_B(b),
\end{aligned}$$

where the second equation follows (1.8) and (1.1); the third follows using the (2.1), (1.8) and (1.1); the forth follows equations (2.5) and (2.9), completing the $\bar{S}$ is a morphism in ${}_H\mathcal{YD}_2^H$.

Using (2.6), one has that $\bar{S} b_1 \star b_2 = \varepsilon(b)$ holds. We also have

$$\begin{aligned}
b_1 \star \bar{S}(b_2) &= [(b_{2(1)} \rightarrow S_B b_{2(0)})^{(1)} \rightarrow b_1](b_{2(1)} \rightarrow S_B b_{2(0)})^{(0)} \\
&= [(b_{2(1)3} (S_B b_{2(0)})^{(1)} S^{-1} b_{2(1)1} \rightarrow b_1][b_{2(1)4} \rightharpoonup b_{2(1)2} \rightharpoonup (S_B b_{2(0)})^{(0)}] \\
&= [(b_{2(1)2} \rightharpoonup b_{2(1)3} \rightharpoonup b_1][b_{2(1)3} \rightharpoonup b_{2(1)1} \rightharpoonup S_B b_{2(0)}] \\
&= b_{2(1)} \rightarrow b_1 S_B b_{2(0)} \overset{(2.8)}{=} b_{(1)} \rightarrow b_{(0)1} S_B b_{(0)2} = \varepsilon(b),
\end{aligned}$$

here the second identity follows the formulae (2.9), (1.1), and (1.8); the third follows (3.3) and (2.9); the forth is given via the formulae (2.1), (1.2), (1.5) and (2.9). As required.

This completes the proof of the proposition 3.1.

Similarly, we postulate the following

CONDITIONS (C).

$$h \rightharpoonup S_B b = S_B(h \rightharpoonup b) \tag{3.5}$$

$$S^{-2}h \rightharpoonup S_B b = S_B(h \rightharpoonup b) \tag{3.6}$$

$$(S_B(b))^{(0)} \otimes (S_B(b))^{(1)} = S_B b_{(0)} \otimes b_{(1)}) \tag{3.7}$$

$$(S_B(b))_{(0)} \otimes (S_B(b))_{(1)} = S_B b^{(0)} \otimes S^{-2} b^{(1)} \tag{3.8}$$

hold. Thus we have the following result similar to the proposition 3.1.

PROPOSITION 3.2. *In the situation of the theorem* 2.4. *Assume that the condition* (C) *hold. If* B *has an antipode then* $\underline{B}$ *has an antipode in the category* ${}_H\mathscr{YD}_1^H$. *It is defined by*

$$\underline{S}(b) = b^{(1)} \succ S_B(b^{(0)}).$$

PROOF. The proof is similar to that of Proposition 3.1.

§ 4. Applications and Examples

In this section we give some of the braided Hopf algebras in the category ${}_H\mathscr{M}$ for a quasitriangular Hopf algebra H and the category $\mathscr{M}^H$ for a coquasitriangular Hopf algebra H.

When (H, R) is quasitriangular, By [CFW], the category ${}_H\mathscr{M}$ of the left modules over the quasitriangular Hopf algebra (H, R) is endowed with the following structure:

$$\tau''(m \otimes n) = R^{(2)} \cdot n \otimes R^{(1)} \cdot m, \quad h \cdot (m \otimes n) = (h_1 \cdot m) \otimes (h_2 \cdot n),$$

for any $m \in M \in {}_H\mathscr{M}$, $n \in N \in {}_H\mathscr{M}$, and $h \in H$.

In this case, the category ${}_H\mathscr{YD}_2^H$ contains this braided monoidal category ${}_H\mathscr{M}$ as its subcategory. In our setting, ${}_H\mathscr{M}_2 = {}_H\mathscr{M}$ with the well known coaction: $\rho(m) = R^{(2)} \cdot m \otimes R^{(1)}$.

Similarly, we have a braided monoidal subcategory ${}_H\mathscr{M}_1$ with the well known coaction: $\rho(m) = R^{(2)} \cdot m \otimes R^{(1)}$, and the braiding $\tau' : m \otimes n \mapsto R^{(1)} \cdot n \otimes R^{(2)} \cdot m$, and the monoidal structure $h \cdot (m \otimes n) = h_2 \cdot m \otimes h_1 \cdot n$ for any $m \in M \in {}_H\mathscr{M}_1$, $n \in N \in {}_H\mathscr{M}_1$ and $h \in H$.

When $(H, \langle | \rangle)$ is coquasitriangular, the category ${}_H\mathscr{YD}_2^H$ contains a braided monoidal subcategory $\mathscr{M}_2^H$ which is endowed with the following structure, $h \cdot m = \langle h | m_{(1)} \rangle m_{(0)}$, $\tau'' : m \otimes n \mapsto \langle n_{(1)} | m_{(1)} \rangle n_{(0)} \otimes m_{(0)}$, $\rho(m \otimes n) = m_{(0)} \otimes n_{(0)} \otimes n_{(1)} m_{(1)}$ for any $m \in M \in \mathscr{M}_2^H$ and $n \in N \in \mathscr{M}_2^H$.

Similarly, we have a braided monoidal subcategory ${}^H_1\mathcal{M}$ armed with the following structure, $h \cdot m = \langle h | m_{(1)} \rangle m_{(0)}$, $\tau' : m \otimes n \mapsto \langle m^{(1)} | n^{(1)} \rangle n^{(0)} \otimes m^{(0)}$, $\delta(m \otimes n) = m^{(0)} \otimes n^{(0)} \otimes m^{(1)} n^{(1)}$ for any $m \in M \in {}^H_1\mathcal{M}$ and $n \in N \in {}^H_1\mathcal{M}$.

In what follows, we construct two class of braided Hopf algebra in the categories ${}_H\mathcal{M}_1$, ${}_H\mathcal{M}_2$, and ${}^H_1\mathcal{M}$, and ${}^H_2\mathcal{M}$.

Let (H, R) be a quasitriangular Hopf algebra. Let (B, H, τ) be a Hopf pairing. We define $h \rightharpoonup b = \tau(Sb_1, h)b_2$ and $h \leftharpoonup b = \tau(b_2, h)b_1$ for all $b \in B$, $h \in H$. In ${}_H\mathcal{M}_2$, it is natural that we have $\delta(b) = R^{(2)} \leftharpoonup b \otimes R^{(1)} \overset{def.}{=} b^{(0)} \otimes b^{(1)}$ and $\rho(b) = R^{(2)} \triangleright b \otimes R^{(1)} \overset{def.}{=} b_{(0)} \otimes b_{(1)}$.

It is easy to verify that $(B, \rightharpoonup, \delta)$ is an algebra in ${}_H\mathcal{YD}^H_1$, and that $(B, \leftharpoonup, \rho)$ is an algebra in ${}_H\mathcal{YD}^H_2$. Obviously, $(B, \rightharpoonup, \rho)$ and $(B, \leftharpoonup, \delta)$ are objects in Long dimodule category ${}_H\mathcal{L}^H$.

Thus, by (2.9)–(2.10) and the proposition 3.1, we obtain

$$h \rightarrow b = h_1 \rightharpoonup (h_2 \leftharpoonup b) = \tau(S(b_1)b_3, h)b_2 \tag{4.1}$$

$$a \star b = (b^{(1)} \rightarrow a)b^{(0)} = \tau(S(a_1)a_3, R^{(1)})a_2 b_2 \tau(S(b_1), R^{(2)}) \tag{4.2}$$

$$\bar{S}(b) = b_{(1)} \rightarrow S_B(b_{(0)}) = \tau(b_2, R^{(2)})(R^{(1)} \rightarrow S_B(b_1)), \tag{4.3}$$

for all $a, b \in B$ and $h \in H$.

We now have the following:

THEOREM 4.1. *Let H be quasitriangular. With the notation above. Then there is a braided Hopf algebra $\bar{B}$ in ${}_H\mathcal{M}_2$, $\bar{B} = B$ as a linear space and as an object in ${}_H\mathcal{M}_2$ with a H-module structure defined by* (4.1). *The coalgebra structure and unit in $\bar{B}$ coincide with those of B. The multiplication is given by* (4.2). *Its antipode is given by* (4.3).

PROOF. Firstly, in order to apply the Theorem 2.3, we need to check the Condition (B) hold. A routine check can show that the conditions (2.1)–(2.4) are satisfied. Then, by definition, and using (2.1), we have

$$\begin{aligned} b^{(1)} \otimes b^{(0)}{}_{(1)} \otimes b^{(0)}{}_{(0)} &= R^{(1)} \otimes (R^{(2)} \leftharpoonup b)_{(1)} \otimes (R^{(2)} \leftharpoonup b)_{(0)} \\ &= R^{(1)} \otimes r^{(1)} \otimes r^{(2)} \leftharpoonup (R^{(2)} \leftharpoonup b)) \\ &= R^{(1)} \otimes r^{(1)} \otimes R^{(2)} \leftharpoonup (r^{(2)} \leftharpoonup b)) \\ &= (r^{(2)} \leftharpoonup b)^{(1)} \otimes r^{(1)} \otimes (r^{(2)} \leftharpoonup b)^{(0)} = b_{(0)}{}^{(1)} \otimes b_{(1)} \otimes b_{(0)}{}^{(0)}, \end{aligned}$$

and the formula (2.5) is proven.

The following computation:

$$b_{1(0)} \otimes b_2{}^{(0)} \otimes b_2{}^{(1)} b_{1(1)} = R^{(2)} \rightharpoonup b_1 \otimes r^{(2)} \rightharpoonup b_2 \otimes r^{(1)} R^{(1)}$$

$$\overset{(QT2)}{=} R^{(1)}{}_1 \rightharpoonup b_1 \otimes R^{(1)}{}_2 \rightharpoonup b_2 \otimes R^{(1)} \overset{(2.2)}{=} b_1 \otimes b_2 \otimes 1,$$

shows the equation (2.6).

Then, using definitions, and (2.4), we can obtain:

$$b_1{}^{(0)} \otimes b_2 \otimes b_1{}^{(1)} = R^{(2)} \rightharpoonup b_1 \otimes b_2 \otimes R^{(1)}$$

$$= (R^{(2)} \rightharpoonup b)_1 \otimes (R^{(2)} \rightharpoonup b)_2 \otimes R^{(1)} = b^{(0)}{}_1 \otimes b^{(0)}{}_2 \otimes b^{(1)},$$

and this checkes the (2.7), and similarly, one has the (2.8). It is easy to that $\bar{S}(b) = b^{(2)} \rightarrow S_B(b^{\langle 1 \rangle}) = \tau(b_4, R^{(2)})\tau(S^2(b_3)S(b_1), R^{(1)})b_2$.

In final, it is not hard to check the condition (B) hold, concluding the proof.

For a quasitriangular Hopf algebra H. Let (B, H, τ) be a Hopf pairing. Similarly, we can define: $h \rightharpoonup b = \tau^{-1}(b_1, h)b_2$ and $h \triangleright b = \tau^{-1}(Sb_2, h)b_1$ for all $b \in B$, $h \in H$. In ${}_H\mathcal{M}_1$, it is natural that we have $\delta(b) = R^{(2)} \rightharpoonup b \otimes R^{(1)} \overset{def.}{=} b^{(0)} \otimes b^{(1)}$ and $\rho(b) = R^{(2)} \rightharpoonup b \otimes R^{(1)} \overset{def.}{=} b_{(0)} \otimes b_{(1)}$.

Thus, by (2.12)–(2.13) and the proposition 3.2 we have:

$$h \succ b = h_1 \rightharpoonup (h_2 \rightharpoonup b) = \tau^{-1}(b_1 S b_3, h) b_2 \tag{4.4}$$

$$a \bar{\star} b = a_{(0)}(a_{(1)} \succ b) = \tau^{-1}(b_1 S b_3, R^{(1)}) a_1 b_2 \tau^{-1}(S(a_2), R^{(2)}) \tag{4.5}$$

$$\underline{S}(b) = b^{(1)} \succ S_B(b^{(0)}) = \tau^{-1}(b_1, R^{(2)})(R^{(1)} \succ S_B(b_2)). \tag{4.6}$$

We now have the following

THEOREM 4.2. *Let H be quasitriangular. With the notation above. Then there is a braided Hopf algebra $\underline{B}$ in ${}_H\mathcal{M}_1$, $\underline{B} = B$ as a linear space and as an object in ${}_H\mathcal{M}_1$ with a H-module structure defined by* (4.4). *The coalgebra structure and unit in $\underline{B}$ coincide with those of B. The multiplication is given by* (4.5). *Its antipode is defined by* (4.6)

PROOF. Similar to Theorem 4.1.

Let $(H, \langle | \rangle)$ be coquasitriangular, and let B be a Hopf algebra. Assume that $f : B \to H$ is a Hopf algebra map. Define $\delta(b) = b_2 \otimes S^{-1} f(b_1) \overset{def.}{=}$

$b^{(0)} \otimes b^{(1)}$, and $\rho(b) = b_1 \otimes f(b_2) \overset{def.}{=} b_{(0)} \otimes b_{(1)}$ for $b \in B$. Then we have $h \rightharpoonup b = \langle h \mid S^{-1}f(b_1)\rangle b_2$ and $h \rightharpoondown b = \langle h \mid f(b_2)\rangle b_1$ for $h \in H$, $b \in B$. It is easy to check that $(B^{op}, \rightharpoonup, \delta)$ is an algebra in ${}_H\mathcal{YD}_1^H$, and $(B^{op}, \rightharpoondown, \rho)$ an algebra in ${}_H\mathcal{YD}_2^H$ such that $(B^{op}, \rightharpoonup, \rho)$ is in ${}_H\mathcal{L}^H$, and $(B^{op}, \rightharpoondown, \delta)$ is in ${}_H\mathcal{L}^H$.

Then, by (2.10)–(2.11) and the proposition 3.1, one has

$$\chi_B(b) = b_{(0)}{}^{(0)} \otimes b_{(1)}b_{(0)}{}^{(1)} = b_2 \otimes S^{-1}(f(b_1))f(b_3) \tag{4.7}$$

$$a \star b = (b^{(1)} \rightharpoonup a)b^{(0)} = \langle S^{-1}(f(b_1)) \mid f(a_3)S^{-1}(f(a_1))\rangle b_2 a_2 \tag{4.8}$$

$$\bar{S}(b) = b_{(1)} \rightharpoondown S_B(b_{(0)}) = \langle f(b_4) \mid f(b_3)f(Sb_1)\rangle S_B(b_2). \tag{4.9}$$

It is easy to show the condition (A) and the condition (B) are satisfied, and so by the theorem 2.3 and the proposition 3.1, we have:

THEOREM 4.3. *Let $(H, \langle|\rangle)$ be coquasitriangular, and B a Hopf algebra. Let $f : B \to H$ be a Hopf algebra map. Then there is a braided Hopf algebra $\overline{B}$ in $\mathcal{M}_2^H$, $\overline{B} = B$ as a linear space and as an object in $\mathcal{M}_2^H$ with a H-comodule structure defined by* (4.7). *The coalgebra structure and counit in $\overline{B}$ coincide with those of B. The multiplication is given by* (4.8). *Its antipode is defined by* (4.9).

Let B be any bialgebra, and $f : H \to B$ a bialgebra map. If f is a convolution invertible map with an inverse f^{-1}, then $f^{-1} : H \to B$ is an anti-bialgebra map, i.e., $f^{-1}(hl) = f^{-1}(l)f^{-1}(h)$ and $\Delta_B f^{-1}(h) = f^{-1}(h_2) \otimes f^{-1}(h_1)$.

EXAMPLE. If H is a Hopf algebra, then $f^{-1} = fS_H$ is a convolution invertible map.

Similarly, let $(H, \langle|\rangle)$ be coquasitriangular, and B a Hopf algebra. Let $B \to H$ be a Hopf algebra map. Define $\delta(b) = b_2 \otimes Sf(b_1) \overset{def.}{=} b^{(0)} \otimes b^{(1)}$, and $\rho(b) = b_1 \otimes f(b_2) \overset{def.}{=} b_{(0)} \otimes b_{(1)}$ for $b \in B$. Naturaly, we get: $h \rightharpoonup b = \langle h \mid Sf(b_1)\rangle b_2$ and $h \rightharpoondown b = \langle h \mid f(b_2)\rangle b_1$ for $h \in H$, $b \in B$. It is easy to check that $(B^{op}, \rightharpoonup, \delta) \in {}_H\mathcal{YD}_1^H$ is an algebra, and $(B^{op}, \rightharpoondown, \rho) \in {}_H\mathcal{YD}_2^H$ an algebra such that $(B^{op}, \rightharpoonup, \rho)$ is in ${}_H\mathcal{L}^H$, and $(B^{op}, \rightharpoondown, \delta)$ is in ${}_H\mathcal{L}^H$.

Thus, by (2.13)–(2.14) and the proposition 3.2, one has

$$\xi_B(b) = b_{(0)}{}^{(0)} \otimes b_{(0)}{}^{(1)}b_{(1)} = b_2 \otimes S(f(b_1))f(b_3) \tag{4.10}$$

$$a \bar{\star} b = a_{(0)}(a_{(1)} \succ b) = \langle S(f(b_1)) \mid f(a_3)S(f(a_1))\rangle b_2 a_2 \tag{4.11}$$

$$\underline{S}(b) = b^{(1)} \succ S_B(b^{(0)}) = \langle f(b_1) \mid f(b_2)f(Sb_4)\rangle S_B b_3. \tag{4.12}$$

In final, it is not hard to the conditions (2.1)–(2.2), (2.4)–(2.5), (2.7)–(2.8),

(2.15), (2.16) and the condition (C) are satisfied. By the theorem 2.4 and the proposition 3.2, we have:

THEOREM 4.4. *Let $(H, \langle|\rangle)$ be coquasitriangular and B a Hopf algebra. Let $f : B \to H$ be a Hopf algebra map. Then there is a braided Hopf algebra $\underline{B}$ in $\mathscr{M}_1^H$, $\underline{B} = B$ as a linear space and as an object in $\mathscr{M}_1^H$ with a H-comodule structure defined by* (4.10). *The coalgebra structure and counit in $\underline{B}$ coincide with those of B. The multiplication is given by* (4.11). *Its antipode is defined by* (4.12).

By the theorem 4.1, we give an example explicitly as follows.

REMARK 4.5. If (H, R) is a quasitriangular Hopf algebra, then H^{cop} is a Hopf algebra, which has the quasitriangular structure $R^t = R^{(2)} \otimes R^{(1)}$. The braided category ${}_H\mathscr{M}_1$ is identified with ${}_{H^{cop}}\mathscr{M}_2$, and hence with $\mathscr{M}_{H^{bop}2}$, the 2nd kind (or the natural) braided category of right modules over $H^{bop} := (H^{op})^{cop}$. In addition, if $\tau : B \otimes H \to k$ is a Hopf pairing, then $\tau^{-1} : B \otimes H^{bop} \to k$ is a Hopf pairing as shown by $(DP1)'$ and $(DP2)'$. Therefore, it is not hard to check that Theorem 4.2 follows from a variation of Theorem 4.1 which gives a construction of Hopf algebras in $\mathscr{M}_{H2}$.

A similar investigation and verification would be given for Theorem 4.3 and 4.4.

EXAMPLE 4.6. Let H_4 be the Sweedler's 4-dimensional Hopf algebra, i.e., H_4 is a free k-module with basis $1, x, y, z$ and its Hopf algebra structure is defined by

$$x^2 = 1, \quad y^2 = 0, \quad xy = z, \quad xz = -zx = y$$

$$\Delta(x) = x \otimes x, \quad \Delta(y) = y \otimes x + 1 \otimes y, \quad \Delta(z) = z \otimes 1 + x \otimes z$$

$$\varepsilon(x) = 1, \quad \varepsilon(y) = \varepsilon(z) = 0, \quad S(x) = x, \quad S(y) = z, \quad S(z) = -y$$

For any $\alpha \in k$. Then (H_4, σ_α) has uniquely coquasitriangular Hopf algebra, where σ_α is given via:

σ_α	1	x	y	z
1	1	1	0	0
x	1	-1	0	0
y	0	0	α	$-\alpha$
z	0	0	α	α

Note that H_4 is also a quasitriangular Hopf algebra with

$$R_\alpha = 1/2(1\otimes 1 + 1\otimes x + x\otimes 1 - x\otimes x) + \alpha/2(y\otimes y + y\otimes z + z\otimes z - z\otimes y).$$

Then two actions H_4 on H_4^{cop} is respectively defined by:

$\rightharpoonup$	1	x	y	z
1	1	x	y	z
x	1	$-x$	$-y$	z
y	0	0	$\alpha 1$	$-\alpha x$
z	0	0	$\alpha 1$	αx

and

$\rightharpoondown$	1	x	y	z
1	1	x	y	z
x	1	$-x$	y	$-z$
y	0	0	αx	$\alpha 1$
z	0	0	$-\alpha x$	$\alpha 1$

Thus, by the formula (4.1), the H_4^{cop} is a left H_4-module where the H_4-module structure is given by:

$\rightarrow$	1	x	y	z
1	1	x	y	z
x	1	x	$-y$	$-z$
y	0	0	$\alpha(x+1)$	$\alpha(x+1)$
z	0	0	$\alpha(x+1)$	$\alpha(x+1)$

By the equation (4.2), the multiplication on H_4^{cop} is obtained by the following table:

$\star$	1	x	y	z
1	1	x	y	z
x	x	1	z	y
y	y	z	$\alpha^3(x+1)$	$\alpha^3(x+1)$
z	z	y	$\alpha^3(x+1)$	$\alpha^3(x+1)$

Therefore, by the theorem 4.1, the $(H_4, \Delta_{H_4}^{cop}, \star)$ is a braided Hopf algebra in ${}_{H_4}\mathscr{M}_2$. Its antipode is defined by

$$\bar{S}(1) = 1, \quad \bar{S}(x) = x, \quad \bar{S}(y) = z, \quad \bar{S}(z) = y.$$

Similarly, applying the theorem 4.2, we have

EXAMPLE 4.6. Let H_4 be the Sweedler's 4-dimensional Hopf algebra. Then the $(H_4, \Delta_{H_4}^{cop}, \bar{\star})$ is a braided Hopf algebra in ${}_{H_4}\mathscr{M}_2$. Its antipode is defined by

$$\underline{S}(1) = 1, \quad \underline{S}(x) = x, \quad \underline{S}(y) = -z, \quad \underline{S}(z) = -y.$$

The H_4-module structure and the multiplication on H_4^{cop} is given respectively by the following tables:

$\succ$	1	x	y	z
1	1	x	y	z
x	1	x	$-y$	$-z$
y	0	0	$\alpha(x+1)$	$-\alpha(x+1)$
z	0	0	$\alpha(x+1)$	$-\alpha(x+1)$

and

$\bar{\star}$	1	x	y	z
1	1	x	y	z
x	x	1	$-z$	$-y$
y	y	$-z$	$\alpha^3(x+1)$	$-\alpha^3(x+1)$
z	z	$-y$	$-\alpha^3(x+1)$	$\alpha^3(x+1)$

REMARK. In the example 4.6, our two H_4-module structures associated to the H_4-module structure $\succ$ is given by respectively:

$\rightharpoonup$	1	x	y	z
1	1	x	y	z
x	1	$-x$	$-y$	z
y	0	0	$\alpha 1$	$-\alpha x$
z	0	0	$\alpha 1$	αx

and

$\rightharpoonup$	1	x	y	z
1	1	x	y	z
x	1	$-x$	y	$-z$
y	0	0	$-\alpha x$	$-\alpha 1$
z	0	0	αx	$-\alpha 1$.

Acknowledgment

The author would like to thank Professor M. Takeuchi for the valuable suggestions. I am indebted to the referees for valuable comments, in particular Remark 2.5 and 4.5.

References

[AS1] N. Andruskiewitsch and H. J. Schneider, Hopf algebras of order p^2 and braided Hopf algebras of order p, J. Algebra **199**(2) (1998), 430–454.

[AS2] N. Andruskiewitsch and H. J. Schneider, Lifting of quantum linear spaces and pointed Hopf algebras of order p^3, J. Algebra **209**(2) (1998), 658–691.

[CFW] M. Cohen, D. Fischman, and S. Westreich, Schur's double centralizer theorem for triangular Hopf algebras, Proc. Amer. Math. Soc. **122** (1994), 19–29.

[D] Y. Doi, The trace formula for braided Hopf algebras, Comm. in Algebra **28**(4) (2000), 1881–1895.

[DT] Y. Doi and M. Takeuchi, Multiplication alteration by two-cocycles—the quantum version, Comm. in Algebra **22** (1994), 5715–5732.

[FM] D. Fischman and S. Montgomery, A Schur's double centralizer theorem for cotriangular Hopf algebras and generalized Lie algebras, J. Algebra **168** (1994), 594–614.

[JS] A. Joyal and R. Street, Braided tensor categories, Adv. in Math. **102** (1993), 20–78.

[Maji1] S. Majid, Examples of braided groups and braided matrices, J. Math. Phys. **32** (1991), 3246–3253.

[Maji2] S. Majid, Braided groups and algebraic quantum field theories, Lett. Math. Phys. **22** (1991), 167–176.

[Maji3] S. Majid, Transmutation theory and rank for quantum braided groups, Math. Proc. Camb. Phil. Soc. **113** (1993), 45–70.

[Milit] G. Militaru, The Long dimodule category and nonlinear equations, Algebras and Representation Theory **2** (1999), 177–200.

[Mont] S. Montgomery, "Hopf algebras and Their Actions on Rings," CBMS Series in Math., Vol. 82, Am. Math. Soc. Providence, 1993.

[NT] Siu-Hung Ng and E. J. Taft, Quantum convolution of linearly recursive sequences, J. Algebra **198**(1) (1997), 101–119.

[R] D. E. Radford, The structure of Hopf algebra with a projetion, J. Algebra **92** (1985), 322–347.

[RT] D. E. Radford and J. Towber, Yetter-Drinfield categories associated to an arbitrary bialgebra, J. of Pure and Applied Alg. **87** (1993), 259–279.

[Sw] M. E. Sweedler, "Hopf Algebras," Benjamin, New York, 1969.

[WZJ] S. H. Wang, W. Z. Zhao, and Z. M. Jiao, On the structure of Hopf algebra H^R with normalized 2-cocycle, Preprint.

[XSW] Xu Yonghua, Kar-Ping Shum and Shuan-hong Wang, Schur's double centralizer theorem for Hopf algebras, Preprint.

Department of Mathematics, Henan Normal University
Henan Xinxiang 453002, P.R. of China
E-mail: shuanhwang@263.net