

## ON AN ALGEBRA OF SIEGEL MODULAR FORMS ASSOCIATED WITH THE THETA GROUP $\Gamma_2(1, 2)$

By

Takeyoshi KOGISO and Koji TSUSHIMA

**Abstract.** In this note, we shall calculate some homogeneous polynomials explicitly based on B. Runge's conjecture in [4] p. 203 and give the explicit structure of the graded ring  $A(\Gamma_2(1, 2))$  of Siegel modular forms of genus two belonging to the discrete subgroup  $\Gamma_2(1, 2)$  of  $Sp(2, \mathbf{R})$ .

### §1. Notations and preliminaries

Throughout this note, we will use the following notations.

We denote by  $H_g$  the Siegel upper half space of genus  $g$  defined by

$$H_g := \{Z \in M(g, \mathbf{C}) \mid Z : \text{symmetric, } \text{Im}(Z) > 0\}.$$

We denote by  $Sp(g, \mathbf{R})$  the usual real symplectic group of size  $2g$  defined by

$$Sp(g, \mathbf{R}) := \left\{ X \in M(2g, \mathbf{R}) \mid XJX = J, J = \begin{bmatrix} 0 & I_g \\ -I_g & 0 \end{bmatrix} \right\}.$$

Furthermore we consider the following discrete subgroup of  $Sp(g, \mathbf{R})$ ,

$$\Gamma_g := Sp(g, \mathbf{Z}).$$

We call this  $\Gamma_g$  the Siegel modular group, and we put

$$\Gamma_g(n) := \left\{ \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in \Gamma_g \mid \begin{bmatrix} A & B \\ C & D \end{bmatrix} \equiv \begin{bmatrix} I_g & 0 \\ 0 & I_g \end{bmatrix} \pmod{n} \right\}$$

which is called the principal congruence subgroup of level  $n$ .

We put

$$\Gamma_g(n, 2n) := \left\{ \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in \Gamma_g(n) \mid (A^t B)_0 \equiv (D^t C)_0 \equiv 0 \pmod{2n} \right\},$$

where for  $M = (m_{ij}) \in M(g, \mathbf{C})$ ,  $M_0$  denotes the column vector  $M_0 = \begin{bmatrix} m_{11} \\ \vdots \\ m_{gg} \end{bmatrix}$  of diagonal elements of  $M$ .

$\Gamma_g(1, 2)$  is called the theta group.

The symplectic group  $Sp(g, \mathbf{R})$  acts transitively on the Siegel upper half space  $H_g$  by

$$X(Z) := (AZ + B)(CZ + D)^{-1} \quad \text{for } X = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in Sp(g, \mathbf{R}), \quad Z \in H_g.$$

We assume that  $g \geq 2$ .

For any non-negative integer  $k$ , we call  $f(Z)$  a Siegel modular form of weight  $k$  if

- (i)  $f$  is a holomorphic function on  $H_g$ ,
- (ii)  $f(X(Z)) = \det(CZ + D)^k f(Z)$  for any  $X = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in \Gamma_g$ .

The space of all modular forms of the same weight  $k$  has a structure of a vector space and we denote it by  $[\Gamma_g, k]$ . The direct sum of them becomes the finite generated graded algebra and we denote it by  $A(\Gamma_g) := \bigoplus_k [\Gamma_g, k]$ .

We use the classical notation for the thetas, i.e. for  $(\tau, z) \in H_g \times \mathbf{C}^g$ ,

$$\theta \begin{bmatrix} \alpha \\ \beta \end{bmatrix} (\tau, z) = \sum_{x \in \mathbf{Z}^g} \exp 2\pi i \left( \frac{1}{2} \tau \left[ x + \frac{1}{2} \alpha \right] + \left\langle x + \frac{1}{2} \alpha, z + \frac{1}{2} \beta \right\rangle \right)$$

where  $\langle x, y \rangle$  denotes the standard scalar product,  $\tau[x] = {}^t x \tau x$  and  $i$  denotes  $\sqrt{-1}$ .

The elements  $\alpha$  and  $\beta$  of  $\mathbf{Z}^g$  are also regarded as elements in  $F_2^g$  with entries 0 and 1. Here we remark that  $\theta \begin{bmatrix} \alpha + 2p \\ \beta + 2q \end{bmatrix} = \theta \begin{bmatrix} \alpha \\ \beta \end{bmatrix} (-1)^{\langle \alpha, q \rangle}$ .

It is well known that  $\theta \begin{bmatrix} \alpha \\ \beta \end{bmatrix} (\tau, z)$  is an analytic function on  $H_g \times \mathbf{C}^g$ .

The  $\theta \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \theta \begin{bmatrix} \alpha \\ \beta \end{bmatrix} (\tau, 0)$  are called the theta constants.

In the case of  $g = 2$ , there exists the following 10 even pairs of  $\alpha, \beta$ , i.e.,  $(-1)^{\langle \alpha, \beta \rangle} = 1$ :

$$\begin{aligned} &\theta \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \theta \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \theta \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \theta \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}, \theta \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \theta \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \theta \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \\ &\theta \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \theta \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \theta \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}. \end{aligned}$$

Now we define  $\Theta := \prod_{\text{even}} \theta \begin{bmatrix} \alpha \\ \beta \end{bmatrix}$ . Here we note that the degree of  $\Theta$  is 10.

We use this  $\Theta$  to describe our problem (posed by B. Runge).

Finally we have to prepare the definition of a code. A code  $C$  means a linear subspace  $C$  of  $F_2^n$ , when its dimension  $k$  and the minimal weight  $d$ , it is denoted by  $[n, k, d]$ . Where the weight  $d$  means the number of entries 1 in an element  $\alpha$  of a code  $C$ . A code is called doubly-even if the weight of every element of the code is divisible by 4. A code  $C$  is called the self-dual when it coincides with its dual  $C^\perp = \{x \in F_2^n \mid \langle x, y \rangle = 0 \text{ for any } y \in C\}$ , where  $\langle x, y \rangle$  means the standard inner product.

**§ 2. The relation between the Siegel modular group  $\Gamma_g$  and the corresponding finite group**

We can choose the following generator of  $\Gamma_g$ :

$$\Gamma_g = \left\langle \begin{bmatrix} I_g & S \\ 0 & I_g \end{bmatrix}, \begin{bmatrix} 0 & I_g \\ -I_g & 0 \end{bmatrix}, {}^tS = S \in M(g, \mathbf{Z}) \right\rangle.$$

On the other hand, the thetas of second order are given by

$$f_a(\tau) := \theta \begin{bmatrix} a \\ 0 \end{bmatrix} (2\tau) = \sum_{x \in \mathbf{Z}^g} \exp 2\pi i \left( \tau \left[ x + \frac{1}{2} a \right] \right)$$

for  $a \in \mathbf{Z}^g$ , where  $i$  denotes  $\sqrt{-1}$ .

The function  $f_a$  depends only on  $a \pmod{2}$ , here  $a$  is considered also as the element of  $F_2^g$ .

The action of  $\Gamma_g$  on thetas of second order  $f_a = \theta \begin{bmatrix} a \\ 0 \end{bmatrix} (2\tau, 0)$  is the following.

PROPOSITION 2.1 (see [3], p. 60). (1) For  $\sigma = \begin{bmatrix} I_g & S \\ 0 & I_g \end{bmatrix} \in \Gamma_g$ ,  $\sigma(f_a) = i^{S[a]} f_a$

with  $S[a] = {}^t aSa$ .

(2) We define the  $2^g \times 2^g$ -matrix  $T_g$  by  $T_g := \left( \frac{1+i}{2} \right)^g ((-1)^{\langle a, b \rangle})_{a, b \in F_2^g}$ .

Then  $J = \begin{bmatrix} 0 & I_g \\ -I_g & 0 \end{bmatrix}$  acts on  $f_a$  as follows:

$$J(f_a) = \sqrt{\det(-\tau)} \sum_{b \in F_2^g} (T_g)_{a, b} f_b$$

for  $f_a = \theta \begin{bmatrix} a \\ 0 \end{bmatrix} (2\tau, 0)$ .

Then we can consider the following  $2^g \times 2^g$ -matrix  $D_S$ :

$D_S = \text{diag}(i^{S[a]})$  for  $a \in \mathbf{F}_2^g$ , and we define  $H_g := \langle T_g, D_S \rangle$ .

Here  $H_g$  acts faithfully on the vector space generated by  $f_a$ ,  $a \in \mathbf{F}_2^g$  and  $H_g/\{\pm 1\}$  acts on the algebra  $\mathbf{C}[f_a f_b]_{a,b \in \mathbf{F}_2^g}$ .

Then the following theorem is known.

PROPOSITION 2.2 (see [3], p. 60). *There exists the surjective group homomorphism  $\phi : \Gamma_g \rightarrow H_g/\{\pm 1\}$  satisfying  $\phi(J) = T_g$  and  $\phi(\sigma) = \sigma \left( \begin{bmatrix} I_g & S \\ 0 & I_g \end{bmatrix} \right) = D_S$  with the notations in proposition 2.1.*

By the surjective group homomorphism  $\phi$  in above proposition 2.2, the subgroup  $\Gamma_2(1,2)$  of  $\Gamma_2$  corresponds to the following finite group  $G$ :

$$G = \left\langle S_4, \text{diag}(\pm 1, \pm 1, \pm 1, \pm 1), M_1 = \left( \frac{1+i}{2} \right) \begin{bmatrix} I_2 & I_2 \\ I_2 & -I_2 \end{bmatrix} \right\rangle$$

where  $S_4$  denotes the symmetric group of degree 4.

§3. Runge’s conjecture

From above proposition 2.1 and proposition 2.2 §2, B. Runge proved the following theorem 3.1 about the correspondence between the ring of modular forms and the invariant ring of some finite group by using the theta functions (see [3]).

THEOREM 3.1 (B. Runge). *For any element  $a = (a_1, \dots, a_g)$  in  $\mathbf{F}_2^g$ , we define the function*

$$Y_a : \overbrace{\mathbf{F}_2^n \times \dots \times \mathbf{F}_2^n}^g \rightarrow \mathbf{Z}_+$$

by

$$Y_a(\gamma_1, \dots, \gamma_g) := \# \{i \mid a = (\gamma_{i,1}, \dots, \gamma_{i,g})\}$$

where

$$(\gamma_1, \dots, \gamma_g) = \begin{bmatrix} \gamma_{11} & \dots & \gamma_{1g} \\ \vdots & & \vdots \\ \gamma_{n1} & \dots & \gamma_{ng} \end{bmatrix}.$$

Then for a self-dual doubly-even code  $C$ , the polynomial of  $2^g$ -variables

$$P_g(C) := \sum_{(\gamma_1, \dots, \gamma_g) \in C^g} \prod_{a \in \mathbb{F}_2^g} f_a^{Y_a(\gamma_1, \dots, \gamma_g)},$$

with the thetas of second order

$$f_a = \theta \begin{bmatrix} a \\ 0 \end{bmatrix} (2\tau, 0)$$

becomes a modular form (for the full modular group) of weight  $k = n/2$ .

From the fact in [1], p. 177, the fact in [2], theorem 7.1 in p. 200, the statement of p. 201 and Lemma 2.1 of p. 180 in [4], the space generated by the polynomials  $P_2(C)$  for  $C = C^\perp$  becomes the invariant ring  $\mathbb{C}[f_a; a \in \mathbb{F}_2^{2g}]^{G'}$  for  $G' = \left\langle S_4, \text{diag}(\pm 1, \pm 1, \pm 1, \pm 1), W = 1/\sqrt{2} \begin{bmatrix} I_2 & I_2 \\ I_2 & -I_2 \end{bmatrix} \right\rangle$ , where the matrix  $M_1$  of  $G = \phi(\Gamma_2(1, 2))$  and  $W$  of  $G'$  differ by an eight root of unity. Furthermore, from the argument in p. 202, p. 203 in [4], for our case of  $G = \phi(\Gamma_2(1, 2))$ ,  $A(\Gamma_2(1, 2)) = \bigoplus_k [\Gamma_2(1, 2), k]$  corresponds to a certain subring of  $\mathbb{C}[f_a, a \in \mathbb{F}_2^2, \Theta]$  where  $\Theta = \Pi_{\text{even}} \theta \begin{bmatrix} \alpha \\ \beta \end{bmatrix}$ .

Here, let us recall the related facts from the invariant theory.

**PROPOSITION 3.2** (in [3], p. 76). *Let  $G$  be a finite group acting on a polynomial ring  $A = \mathbb{C}[X_1, \dots, X_r]$  and if  $R = A^G$  is the invariant ring, it is known that  $R$  is of the type:*

$$R = \mathbb{C}[g_1, \dots, g_r] \oplus \mathbb{C}[g_1, \dots, g_r]g_{r+1} \oplus \dots \oplus \mathbb{C}[g_1, \dots, g_r]g_n \quad (r \leq n)$$

where each  $g_i$  is a homogeneous polynomial of degree  $d_i$ .

We denote by  $R_{(l)}$  the homogeneous part of degree  $l$  in  $R$ . Then we have

$$\Phi_G(\lambda) = \sum_{l \geq 0} \dim_{\mathbb{C}} R_{(l)} \lambda^l = \frac{1 + \sum_{i=r+1, \dots, n} \lambda^{d_i}}{\prod_{i=1, \dots, r} (1 - \lambda^{d_i})}.$$

In [4], p. 203, B. Runge calculated  $\Phi_{\phi(\Gamma_2(1,2))}(\sqrt{\lambda})$  as follows:

$$\Phi_{\phi(\Gamma_2(1,2))}(\sqrt{\lambda}) = \frac{1 + \lambda^6 + \lambda^8 + \lambda^{10} + \lambda^{19} + \lambda^{21} + \lambda^{23} + \lambda^{29}}{(1 - \lambda^4)^2(1 - \lambda^6)(1 - \lambda^{12})}$$

and he remarked that this Poincaré series accords with [2], p. 405. Then he described the following conjecture:

CONJECTURE (Runge) (in [4], p. 203). *The ring  $\oplus_k[\Gamma_2(1,2),k]$  has the following graded ring structure.*

$$\begin{aligned} \oplus_k[\Gamma_2(1,2),k] &= \mathbb{C}[P_2^4, P_2^3P_6, P_2^2P_6^2, P_2P_6^3, P_6^4, P_8, P_{12}, P_{28}\Theta, P_{32}\Theta, P_{36}\Theta] \\ &= \mathbb{C}[P_2^4, P_6^4, P_8, P_{12}] \\ &\quad \oplus \mathbb{C}[P_2^4, P_6^4, P_8, P_{12}]P_2^3P_6 \\ &\quad \oplus \mathbb{C}[P_2^4, P_6^4, P_8, P_{12}]P_2^2P_6^2 \\ &\quad \oplus \mathbb{C}[P_2^4, P_6^4, P_8, P_{12}]P_2P_6^3 \\ &\quad \oplus \mathbb{C}[P_2^4, P_6^4, P_8, P_{12}]P_{28}\Theta \\ &\quad \oplus \mathbb{C}[P_2^4, P_6^4, P_8, P_{12}]P_{32}\Theta \\ &\quad \oplus \mathbb{C}[P_2^4, P_6^4, P_8, P_{12}]P_{36}\Theta \\ &\quad \oplus \mathbb{C}[P_2^4, P_6^4, P_8, P_{12}]Q_{58} \end{aligned}$$

where  $\Theta = \Pi_{\text{even}}\theta \begin{bmatrix} \alpha \\ \beta \end{bmatrix}$ ,  $P_i$  is a certain homogeneous polynomial of degree  $i$  and  $Q_{58}$  is a certain polynomial of degree 58, product of such  $P_i$ 's and  $\Theta$ .

In this note, we give an affirmative answer about the conjecture.

Actually B. Runge showed in the case of even weight related to this conjecture that:

THEOREM 3.3 (Runge) (see [4], p. 202). *The ring  $\oplus_{2|k}[\Gamma_2(1,2),k]$  has the following graded ring structure.*

$$\begin{aligned} \oplus_{2|k}[\Gamma_2(1,2),k] &= \mathbb{C}[P_2^4, P_2^3P_6, P_2^2P_6^2, P_2P_6^3, P_6^4, P_8, P_{12}] \\ &= \mathbb{C}[P_2^4, P_6^4, P_8, P_{12}] \\ &\quad \oplus \mathbb{C}[P_2^4, P_6^4, P_8, P_{12}]P_2^3P_6 \\ &\quad \oplus \mathbb{C}[P_2^4, P_6^4, P_8, P_{12}]P_2^2P_6^2 \\ &\quad \oplus \mathbb{C}[P_2^4, P_6^4, P_8, P_{12}]P_2P_6^3 \end{aligned}$$

where each  $P_i$  are the following polynomials:

$$P_2 = (2),$$

$$P_6 = (6) - 5(4, 2) + 30(2, 2, 2),$$

$$P_8 = (8) + 14(4, 4) + 168(2, 2, 2, 2),$$

and

$$P_{12} = (12) - 33(8, 4) + 330(4, 4, 4) + 792(6, 2, 2, 2)$$

where

$$(a_1, a_2, a_3, a_4) := \sum_{\sigma \in S_4 / \text{Stab}(a_1, a_2, a_3, a_4)} \prod_{i=1, \dots, 4} (f_{i-1}^{a_i})^\sigma.$$

Here we put  $f_0 = f_{[0]}$ ,  $f_1 = f_{[1]}$ ,  $f_2 = f_{[2]}$ ,  $f_3 = f_{[3]}$ .

In this case, the corresponding finite group  $G = \phi(\Gamma_2(1, 2))$  is given as follows.

$$G = \left\langle S_4, \begin{bmatrix} \pm 1 & & & \\ & \pm 1 & & \\ & & \pm 1 & \\ & & & \pm 1 \end{bmatrix}, M_1 = \frac{1+i}{2} \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{bmatrix} \right\rangle \subset GL(4, \mathbb{C}).$$

By theorem 3.3, what we have to consider are homogeneous polynomials of odd weight.

#### §4. The calculations and the result

By the invariance of the action of  $S_4$  and  $\text{diag}(\pm 1, \pm 1, \pm 1, \pm 1)$ , we may put

$$P_{28} = \sum_{\substack{a_1+a_2+a_3+a_4=28 \\ a_1, a_2, a_3, a_4 \text{ are different odd integers}}} C_{a_1, a_2, a_3, a_4}(a_1, a_2, a_3, a_4)^A,$$

$$P_{32} = \sum_{\substack{a_1+a_2+a_3+a_4=32 \\ a_1, a_2, a_3, a_4 \text{ are different odd integers}}} C_{a_1, a_2, a_3, a_4}(a_1, a_2, a_3, a_4)^A$$

and

$$P_{36} = \sum_{\substack{a_1+a_2+a_3+a_4=36 \\ a_1, a_2, a_3, a_4 \text{ are different odd integers}}} C_{a_1, a_2, a_3, a_4}(a_1, a_2, a_3, a_4)^A$$

where  $C_{a_1, a_2, a_3, a_4}$ 's are the constants of rational integers and

$$(a_1, a_2, a_3, a_4)^A := \sum_{\sigma \in S_4} \text{sign}(\sigma) \prod_{i=1, \dots, 4} (f_{i-1}^{a_i})^\sigma.$$

By the invariance of the action of  $M_1$ , we calculate using the computer, nonzero alternating polynomials  $P_{28}$ ,  $P_{32}$  and  $P_{36}$  as follows:

$$\begin{aligned} P_{28} = & (19, 5, 3, 1)^A - 3(17, 7, 3, 1)^A + (15, 9, 3, 1)^A \\ & - 6(15, 7, 5, 1)^A + 5(13, 11, 3, 1)^A + 16(13, 9, 5, 1)^A \\ & - 54(13, 7, 5, 3)^A + 13(11, 9, 7, 1)^A - 39(11, 9, 5, 3)^A, \end{aligned}$$

$$\begin{aligned} P_{32} = & (21, 7, 3, 1)^A - 7(19, 9, 3, 1)^A + 21(17, 11, 3, 1)^A - 63(17, 7, 5, 3)^A \\ & - 35(15, 13, 3, 1)^A - 18(15, 9, 7, 1)^A + 203(15, 9, 5, 3)^A + 63(13, 11, 7, 1)^A \\ & - 294(13, 11, 5, 3)^A - 322(13, 9, 7, 3)^A + 2457(11, 9, 7, 5)^A \end{aligned}$$

and

$$\begin{aligned} P_{36} = & 15(25, 7, 3, 1)^A - 75(23, 9, 3, 1)^A + 46(23, 7, 5, 1)^A + 120(21, 11, 3, 1)^A \\ & - 276(21, 9, 5, 1)^A - 1074(21, 7, 5, 3)^A + 644(19, 13, 3, 1)^A - 135(19, 9, 7, 1)^A \\ & + 1610(19, 9, 5, 3)^A - 210(17, 15, 3, 1)^A - 644(17, 13, 5, 1)^A \\ & + 266(17, 11, 5, 3)^A - 1665(17, 9, 7, 3)^A + 1254(15, 13, 7, 1)^A \\ & - 1330(15, 13, 5, 3)^A + 1956(15, 11, 9, 1)^A - 8736(15, 11, 7, 3)^A \\ & + 34690(15, 9, 7, 5)^A - 15750(13, 11, 9, 3)^A + 20930(13, 11, 7, 5)^A. \end{aligned}$$

So we have  $\oplus_k [\Gamma_2(1, 2), k]$  contains  $\mathbf{C}[P_2^4, P_2^3 P_6, P_2^2 P_6^2, P_2 P_6^3, P_6^4, P_8, P_{12}, P_{28} \Theta, P_{32} \Theta, P_{36} \Theta]$  as a subring.

Furthermore we remark that

$$\begin{aligned} \Phi_{\phi(\Gamma_2(1,2))}(\sqrt{\lambda}) &= \frac{1 + \lambda^6 + \lambda^8 + \lambda^{10} + \lambda^{19} + \lambda^{21} + \lambda^{23} + \lambda^{29}}{(1 - \lambda^4)^2 (1 - \lambda^6) (1 - \lambda^{12})} \\ &= 1 + 2\lambda^4 + 2\lambda^6 + 4\lambda^8 + 5\lambda^{10} + 9\lambda^{12} + 9\lambda^{14} + 15\lambda^{16} + 17\lambda^{18} + \lambda^{19} + 23\lambda^{20} \\ &\quad + \lambda^{21} + 27\lambda^{22} + 3\lambda^{23} + 36\lambda^{24} + 3\lambda^{25} + 39\lambda^{26} + 6\lambda^{27} + 7\lambda^{29} + \dots, \end{aligned}$$

and we checked that

$$\begin{aligned}
Q_{48} := P_2^3 P_6 P_{36} = & 15(37, 7, 3, 1)^A - 105(35, 9, 3, 1)^A + 16(35, 7, 5, 1)^A \\
& + 15(33, 11, 3, 1)^A - 128(33, 9, 5, 1)^A - 821(33, 7, 5, 3)^A \\
& + 615(31, 13, 3, 1)^A + 144(31, 11, 5, 1)^A + 3590(31, 9, 7, 1)^A \\
& + 5681(31, 9, 5, 3)^A - 405(29, 15, 3, 1)^A + 512(29, 13, 5, 1)^A \\
& - 2633(29, 11, 7, 1)^A + 7258(29, 11, 5, 3)^A - 21320(29, 9, 7, 3)^A \\
& - 1485(27, 17, 3, 1)^A - 944(27, 15, 5, 1)^A + 103(27, 13, 7, 1)^A \\
& + 182(27, 13, 5, 3)^A - 31106(27, 11, 9, 1)^A - 111872(27, 11, 7, 3)^A \\
& - 68689(27, 9, 7, 5)^A + 1275(25, 19, 3, 1)^A - 640(25, 17, 5, 1)^A \\
& + 18826(25, 15, 7, 1)^A + 377(25, 15, 5, 3)^A - 12800(25, 13, 9, 1)^A \\
& + 88000(25, 13, 7, 3)^A - 136786(25, 11, 9, 3)^A - 104221(25, 11, 9, 5)^A \\
& + 1875(23, 21, 3, 1)^A + 2000(23, 19, 5, 1)^A - 24412(23, 17, 7, 1)^A \\
& - 29837(23, 17, 5, 3)^A + 11122(23, 15, 9, 1)^A + 346624(23, 15, 7, 3)^A \\
& + 57819(23, 13, 11, 1)^A + 207430(23, 13, 9, 3)^A - 296200(23, 15, 7, 3)^A \\
& + 688701(23, 11, 9, 5)^A - 36913(21, 19, 7, 1)^A - 30788(21, 19, 5, 3)^A \\
& - 85376(21, 17, 9, 1)^A + 82478(21, 17, 7, 3)^A + 178738(21, 15, 11, 1)^A \\
& + 209085(21, 15, 9, 3)^A - 1453960(21, 15, 7, 5)^A - 417746(21, 13, 11, 3)^A \\
& + 243712(21, 13, 9, 5)^A + 4563377(21, 11, 9, 7)^A + 92575(19, 17, 11, 1)^A \\
& - 108025(19, 17, 9, 3)^A - 1194129(19, 17, 7, 5)^A + 230501(19, 15, 13, 1)^A \\
& - 874752(19, 15, 11, 3)^A - 1942435(19, 15, 9, 5)^A - 2084558(19, 13, 11, 5)^A \\
& + 8801090(19, 13, 9, 7)^A - 319143(17, 15, 13, 3)^A - 2492914(17, 15, 11, 5)^A \\
& + 4411534(17, 15, 9, 7)^A + 4772515(17, 13, 11, 7)^A + 1161508(15, 13, 11, 9)^A
\end{aligned}$$

is different from any linear combinations of the following polynomials:

$$\begin{aligned}
P_2^4 P_{12} P_{28} &= (39, 5, 3, 1)^A + (37, 7, 3, 1)^A - 38(35, 9, 3, 1)^A + 813(35, 7, 5, 1)^A \\
&\quad - 38(33, 11, 3, 1)^A + 760(33, 9, 5, 1)^A + 733(33, 7, 5, 3)^A \\
&\quad + 142(31, 13, 3, 1)^A + \dots,
\end{aligned}$$

$$\begin{aligned}
P_8 P_{12} P_{28} &= (39, 5, 3, 1)^A - 3(37, 7, 3, 1)^A - 18(35, 9, 3, 1)^A + 805(35, 7, 5, 1)^A \\
&\quad + 62(33, 11, 3, 1)^A - 2360(33, 9, 5, 1)^A - 2319(33, 7, 5, 3)^A \\
&\quad - 518(31, 13, 3, 1)^A + \dots,
\end{aligned}$$

$$\begin{aligned}
P_8 P_2^4 P_{32} &= (37, 7, 3, 1)^A - 3(35, 9, 3, 1)^A + 4(35, 7, 5, 1)^A + 13(33, 11, 3, 1)^A \\
&\quad - 16(33, 9, 5, 1)^A - 71(33, 7, 5, 3)^A - 31(31, 13, 3, 1)^A + \dots,
\end{aligned}$$

$$\begin{aligned}
P_8^2 P_{32} &= (37, 7, 3, 1)^A - 7(35, 9, 3, 1)^A + 49(33, 11, 3, 1)^A - 63(33, 7, 5, 3)^A \\
&\quad - 231(31, 13, 3, 1)^A + 28(31, 13, 3, 1)^A + \dots,
\end{aligned}$$

$$\begin{aligned}
P_2^8 P_{32} &= (37, 7, 3, 1)^A + (35, 9, 3, 1)^A - 7(33, 11, 3, 1)^A + 56(33, 9, 5, 1)^A \\
&\quad - 63(33, 7, 5, 3)^A - 596(31, 13, 3, 1)^A + \dots,
\end{aligned}$$

and

$$\begin{aligned}
P_{12} P_{36} &= 15(37, 7, 3, 1)^A - 75(35, 9, 3, 1)^A + 46(35, 7, 5, 1)^A - 375(33, 11, 3, 1)^A \\
&\quad - 276(33, 9, 5, 1)^A - 1074(33, 7, 5, 3)^A + 2475(31, 13, 3, 1)^A + \dots
\end{aligned}$$

Furthermore we get these 7 polynomials  $\times \Theta$  as a basis of the 7-dimensional vector space  $R_{29}$ , i.e.

$$\begin{aligned}
R_{29} &= \langle Q_{48} \Theta = P_2^3 P_6 P_{36} \Theta, P_2^4 P_{12} P_{28} \Theta, P_8 P_{12} P_{28} \Theta, P_8 P_2^4 P_{32} \Theta, \\
&\quad P_8^2 P_{32} \Theta, P_2^8 P_{32} \Theta, P_{12} P_{36} \Theta \rangle.
\end{aligned}$$

(We checked that  $P_2^2 P_6^2 P_{32} \Theta$  and  $P_2 P_6^3 P_{28} \Theta$  are linear combinations of above 7 polynomials  $\times \Theta$ .)

So we have the following decomposition:

$$\begin{aligned}
 & C[P_2^4, P_2^3 P_6, P_2^2 P_6^2, P_2 P_6^3, P_6^4, P_8, P_{12}, P_{28}\Theta, P_{32}\Theta, P_{36}\Theta] \\
 &= C[P_2^4, P_6^4, P_8, P_{12}] \\
 &\quad \oplus C[P_2^4, P_6^4, P_8, P_{12}] P_2^3 P_6 \\
 &\quad \oplus C[P_2^4, P_6^4, P_8, P_{12}] P_2^2 P_6^2 \\
 &\quad \oplus C[P_2^4, P_6^4, P_8, P_{12}] P_2 P_6^3 \\
 &\quad \oplus C[P_2^4, P_6^4, P_8, P_{12}] P_{28}\Theta \\
 &\quad \oplus C[P_2^4, P_6^4, P_8, P_{12}] P_{32}\Theta \\
 &\quad \oplus C[P_2^4, P_6^4, P_8, P_{12}] P_{36}\Theta \\
 &\quad \oplus C[P_2^4, P_6^4, P_8, P_{12}] Q_{48}\Theta \\
 &\quad \oplus \dots\dots
 \end{aligned}$$

and using this result, we see that the Poincaré series of  $\oplus_k[\Gamma_2(1, 2), k]$  and that of this subring coincide. This implies that

$$\begin{aligned}
 \oplus_k[\Gamma_2(1, 2), k] &= C[P_2^4, P_2^3 P_6, P_2^2 P_6^2, P_2 P_6^3, P_6^4, P_8, P_{12}, P_{28}\Theta, P_{32}\Theta, P_{36}\Theta] \\
 &= C[P_2^4, P_6^4, P_8, P_{12}] \\
 &\quad \oplus C[P_2^4, P_6^4, P_8, P_{12}] P_2^3 P_6 \\
 &\quad \oplus C[P_2^4, P_6^4, P_8, P_{12}] P_2^2 P_6^2 \\
 &\quad \oplus C[P_2^4, P_6^4, P_8, P_{12}] P_2 P_6^3 \\
 &\quad \oplus C[P_2^4, P_6^4, P_8, P_{12}] P_{28}\Theta \\
 &\quad \oplus C[P_2^4, P_6^4, P_8, P_{12}] P_{32}\Theta \\
 &\quad \oplus C[P_2^4, P_6^4, P_8, P_{12}] P_{36}\Theta \\
 &\quad \oplus C[P_2^4, P_6^4, P_8, P_{12}] P_2^3 P_6 P_{36}\Theta.
 \end{aligned}$$

**Acknowledgement**

The authors are grateful to Prof. B. Runge for suggesting this work and to Prof. T. Kimura for discussing about this work.

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Institute of Mathematics  
University of Tsukuba  
Ibaraki, 305, Japan