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SPECTRAL PROPERTIES OF QUASIPERIODIC KRONIG-PENNEY MODEL

By

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Abstract. We study some spectral properties of quasiperiodic Kronig-Penney model. We show the absence of point spectrum in some situations, and derive a relationship between the spectrum of this model and that of the corresponding tight-binding model.

1. Introduction

In 1984, Schechtman et al. [22] discovered the quasicrystalline phase in a metallic solid (an Al-Mn alloy), which is neither crystalline nor amorphous. It has symmetrical properties inconsistent with crystal structure, and thus it is called "quasicrystal". The behavior of electrons in the quasicrystal is described by the Schrödinger operator with quasiperiodic potential, of which the spectral theory is far from complete despite intensive researches by many mathematical physicists since eighties. In fact, most of basic problems are unsolved for dimension larger than one.

However, in one-dimensional case, the quasiperiodic Schrödinger operators such as the following tight-binding Hamiltonian on $l^2(\boldsymbol{Z})$ are known to have interesting spectral properties.

$$(h_\theta(\lambda)u)(n) := u(n+1) + u(n-1) + \lambda v_\theta(n)u(n), \tag{1.1}$$

$$v_\theta(n) := \chi_A(\Phi(\alpha n) + \theta), \tag{1.2}$$

where λ is a real constant, χ_A is the characteristic function of an interval A on the torus $\boldsymbol{R}/\boldsymbol{Z}$, Φ is the canonical projection from $\boldsymbol{R}$ onto $\boldsymbol{R}/\boldsymbol{Z}$, $\alpha \in (0,1)$, and $\theta \in \boldsymbol{R}/\boldsymbol{Z}$. The operator (1.1) is proposed by Kohmoto et al. [14] and Ostlund et al. [18] in the case of $\alpha = (\sqrt{5}-1)/2$, $A = \Phi([1-\alpha, 1))$, and $\theta = 0$. When α is irrational, $v_\theta(n)$ is quasiperiodic and by Luck and Petritis [17], the oper-

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ator $h_\theta(\lambda)$ is interpreted as a model describing the phonon spectra in one-dimensional quasicrystals. In the above-mentioned case (*i.e.*, $\alpha = (\sqrt{5}-1)/2$, $A = \Phi([1-\alpha, 1))$, and $\theta = 0$), Sütő ([23], [24]) proved that the spectrum of $h_0(\lambda)$ is a Cantor set (*i.e.*, nowhere dense closed set without isolated points) of zero Lebesgue measure and purely singular continuous. Furthermore, Bellissard et al. [4] extended Sütő's result to the case where α is an any irrational number. Delyon-Petritis [7] and Kaminaga [11] proved the absence of point spectrum for a.e. θ with respect to the Lebesgue measure under certain conditions.

The operator we study in this paper is a continuous analogue of (1.1) formally given by

$$H_\theta := -\frac{d^2}{dx^2} + \sum_{j \in \boldsymbol{Z}} V_\theta(j)\delta(x-j), \tag{1.3}$$

where δ denotes the one-dimensional delta-function, and $\{V_\theta(j)\}_{j \in \boldsymbol{Z}}$ is a sequence which takes only two positive values λ_1, λ_2 $(\lambda_1, \lambda_2 > 0, \lambda_1 \neq \lambda_2)$:

$$V_\theta(j) := \lambda_1 \chi_A(\Phi(\alpha j) + \theta) + \lambda_2 \chi_{A^c}(\Phi(\alpha j) + \theta), \quad j \in \boldsymbol{Z}. \tag{1.4}$$

An explicit definition of (1.3) will be given in the next section. Some interesting spectral properties of H_θ (or variant of that), such as the hierarchical structure, are discussed in physics literature (*e.g.*, [3], [8], [10], [15], [25], [26]) partly with the aid of numerical computations. When $\lambda_1 = \lambda_2 = \lambda$, H_θ is called the Kronig-Penney model and its spectrum is known to coincide with the following set

$$\Sigma(\lambda) := \left\{ E \in (0, \infty) \,|\, 2\cos\sqrt{E} + \lambda \frac{\sin\sqrt{E}}{\sqrt{E}} \in [-2, 2] \right\}, \tag{1.5}$$

which consists of infinitely many closed intervals ([1], Theorem III.2.3.1). Kirsch-Martinelli [13] studied the spectrum of the random Kronig-Penney model

$$H^\omega := -\frac{d^2}{dx^2} + \sum_{j \in \boldsymbol{Z}} V^\omega(j)\delta(x-j),$$

where $\{V^\omega(j)\}_{j \in \boldsymbol{Z}}$ are independent, identically distributed random variables on a probability space $(\Omega, \mathscr{F}, P)$, and $V^\omega(0)$ satisfies $0 < c_1 \leq V_\omega(0) \leq c_2 < \infty$ for some constants $c_1, c_2 > 0$ almost surely. They proved that the spectrum of H^ω is given by

$$\Sigma = \Sigma(V_{\min}), \tag{1.6}$$

almost surely, where $V_{\min} = \text{ess-inf}\{q \,|\, q \in \text{supp } P\}$.

In this paper, we prove the absence of point spectrum of H_θ under certain conditions and show that, in the case of $A = \Phi([1-\alpha, 1))$, $\sigma(H_\theta)$ (the spectrum of H_θ) can be represented in terms of $\sigma(h_0(\lambda))$. Our main result (Theorem 4.1) implies that $\sigma(H_\theta)$ is related to the *family* of $\sigma(h_0(\lambda)) : \{\sigma(h_0(\lambda)) \mid \lambda \in C\}$, where C is a certain subset of $\boldsymbol{R}$, and not to $\sigma(h_0(\lambda_0))$ for a particular value of λ_0 alone. Moreover, $\sigma(H_\theta)$ is even related to $\sigma(h(0))$: the spectrum of free discrete Laplacian (Cor. 4.1 (1)). We remark that a similar relation is already found in [5], [26]. Their argument to derive it is to transform the transfer matrix of H_θ into that of a certain tight-binding Hamiltonian. On the other hand, our argument described below is to compare corresponding discrete dynamical systems.

The rest of this paper is organized as follows. In section 2, we clarify the definition of the quasiperiodic Kronig-Penney model. In section 3, we prove the absence of point spectrum for almost all θ under certain conditions (Theorem 3.2) by applying the arguments in [7], [11]. In section 4, we prove that $\sigma(H_\theta)$ is independent of θ, and give an explicit representation of $\sigma(H_\theta)$ in terms of $\sigma(h_0(\lambda))$. The argument in section 4 is based on the works of Sütő [23], and Bellissard et al. [4] where they proved $\sigma(h_0(\lambda))$ is equal to the set (which is called the dynamical spectrum) such that the sequence of traces of transfer matrices is bounded. We show that a similar fact also holds for the sequence of transfer matrices of H_θ which satisfies the same recursive equation as that of $h_0(\lambda)$, and thus derive a relationship between $\sigma(h_0(\lambda))$ and $\sigma(H_0)$ by comparing initial values of them.

2. Definition of the Quasiperiodic Kronig-Penney Model

In this section, we define the quasiperiodic Kronig-Penney model, and derive the transfer matrix of the solution of the equation $H_\theta\psi = k^2\psi$. We define the formal Schrödinger operator

$$H_\theta = -\frac{d^2}{dx^2} + \sum_{j \in \boldsymbol{Z}} V_\theta(j)\delta(x-j)$$

as the one-dimensional Laplacian

$$-\triangle_\theta = -\frac{d^2}{dx^2},$$

with domain

$$\mathscr{D}(-\triangle_\theta) := \{\psi \in H^1(\boldsymbol{R}) \cap H^2(\boldsymbol{R}\backslash\boldsymbol{Z}) \mid \psi'(j+0) - \psi'(j-0) = V_\theta(j)\psi(j), j \in \boldsymbol{Z}\}.$$

$H^p(D)$ denotes the Sobolev space of order p over an open set D $(\subset \boldsymbol{R})$. The operator $-\triangle_\theta$ is self-adjoint on $L^2(\boldsymbol{R})$, and coincides with the unique self-adjoint operator on $L^2(\boldsymbol{R})$ associated with the closed quadratic form on $H^1(\boldsymbol{R})$ given by

$$Q(\psi, \phi) := (\psi', \phi') + \sum_{j \in \boldsymbol{Z}} V_\theta(j)\psi(j)\overline{\phi(j)}.$$

$(\cdot\,,\cdot)$ is the inner product on $L^2(\boldsymbol{R})$. For proofs of these known facts, we refer to [1], [9], [13]. By the definition of H_θ, the solution of the equation $H_\theta\psi = k^2\psi$ is given by

$$\psi(x) = \begin{cases} \alpha_j \sin k(x-j) + \beta_j \cos k(x-j), & (\text{if } k > 0), \\ \alpha_j + \beta_j(x-j), & (\text{if } k = 0), \end{cases} \tag{2.1}$$

$$x \in (j, j+1), \quad \alpha_j, \beta_j \in \boldsymbol{C},$$

which satisfies

$$\psi(j+0) = \psi(j-0),$$

$$\psi'(j+0) - \psi'(j-0) = V_\theta(j)\psi(j), \quad j \in \boldsymbol{Z}.$$

Let

$$\mathbf{w}_j :=^t (\alpha_j, \beta_j) \in \boldsymbol{C}^2, \quad j \in \boldsymbol{Z},$$

$$R_V(k^2) := \begin{cases} \begin{pmatrix} \cos k + (V/k)\sin k & -\sin k + (V/k)\cos k \\ \sin k & \cos k \end{pmatrix}, & (\text{if } k > 0), \\ \begin{pmatrix} 1 & 1 \\ V & 1+V \end{pmatrix}, & (\text{if } k = 0). \end{cases} \tag{2.2}$$

$$R(j, k^2) := R_{V_\theta(j)}(k^2), \quad j \in \boldsymbol{Z}, k \geq 0.$$

Then we have the following equation

$$\mathbf{w}_j = R(j, k^2)\mathbf{w}_{j-1}, \quad j \in \boldsymbol{Z}, k \geq 0. \tag{2.3}$$

We call $R(j, k^2)$ the transfer matrix of H_θ.

3. Absence of Point Spectrum

In this section, we prove that H_θ has no point spectrum for Lebesgue-a.e. θ under certain conditions. Let us consider the continued fraction expansion of α,

$$\alpha = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \ddots}}$$

with $a_n \in \boldsymbol{N}$. The associated rational approximations p_n/q_n obey

$$p_{n+1} = a_{n+1}p_n + p_{n-1}, \tag{3.1}$$

$$q_{n+1} = a_{n+1}q_n + q_{n-1}, \quad n \geq 0, \tag{3.2}$$

with $p_0 = 0$, $q_0 = 1$, $p_{-1} = 1$, and $q_{-1} = 0$. The following facts are well-known (*e.g.*, [16]).

$$\left|\alpha - \frac{p_n}{q_n}\right| \leq \frac{1}{q_n q_{n+1}}, \quad n \geq 1, \tag{3.3}$$

$$\|q_n\alpha\| = (-1)^n(q_n\alpha - p_n) < \|k\alpha\|, \quad k = 1, \ldots, q_{n+1} - 1, k \neq q_n, n \geq 1, \tag{3.4}$$

where $\|x\| := \inf_{n \in \boldsymbol{Z}}|x - n|$ is the distance from x to $\boldsymbol{Z}$. As for the spectrum of $h_\theta(\lambda)$, the following result is known.

THEOREM 3.1 (Kaminaga [11]). *Suppose that* $\limsup_{n\to\infty} a_n \geq 4$ *or* $A = \Phi([1-\alpha, 1))$. *Then* $h_\theta(\lambda)$ *has no point spectrum for almost every* θ *with respect to the Lebesgue measure.*

REMARK 3.1. *Damanik-Lenz* [6] *has shown that* $h_\theta(\lambda)$ *has no point spectrum for all* $\theta \in \boldsymbol{R}/\boldsymbol{Z}$ *under the condition*: $A = \Phi([1-\alpha, 1))$ *and* $\limsup_{n\to\infty} a_n \neq 2$.

By using the arguments in [7], [11], we can show the following result which means the absence of point spectrum under the same assumptions as in Theorem 3.1.

THEOREM 3.2. *Suppose that* $\limsup_{n\to\infty} a_n \geq 4$ *or* $A = \Phi([1-\alpha, 1))$. *Then* H_θ *has no point spectrum for almost every* θ *with respect to the Lebesgue measure.*

PROOF OF THEOREM 3.2. We first consider the eigenvalue equation $H_\theta\psi = k^2\psi$ $(k > 0)$. By (2.1), the solution of this equation is given by

$$\psi(x) = \alpha_j \sin k(x-j) + \beta_j \cos k(x-j), \quad x \in (j, j+1),$$

for some $(\alpha_j, \beta_j) \in \boldsymbol{C}^2$. Then $k^2 \in \sigma_p(H_\theta)$ $(k > 0)$ if and only if the following condition holds.

$$\|\psi\|_2^2 = \sum_{j\in \boldsymbol{Z}} \int_0^1 |\alpha_j \sin kx + \beta_j \cos kx|^2 \, dx < \infty.$$

$\|\psi\|_2$ is the $L^2(\boldsymbol{R})$-norm of ψ. It is easy to verify

$$\int_0^1 |\alpha_j \sin kx + \beta_j \cos kx|^2 \, dx = {}^t\,\overline{\mathbf{w}}_j \begin{pmatrix} 1/2 - \sin 2k/(4k) & (1-\cos 2k)/(4k) \\ (1-\cos 2k)/(4k) & 1/2 + \sin 2k/(4k) \end{pmatrix} \mathbf{w}_j.$$

The least eigenvalue of the matrix in RHS is $(1 - k^{-1}|\sin k|)/2$ which implies

$$\int_0^1 |\alpha_j \sin kx + \beta_j \cos kx|^2 \, dx \geq \frac{1}{2}\left(1 - \frac{|\sin k|}{k}\right)(|\alpha_j|^2 + |\beta_j|^2). \tag{3.5}$$

Next, we consider the equation $H_\theta \psi = 0$. By (2.1), the solution of this equation is given by

$$\psi(x) = \alpha_j + \beta_j(x - j), \quad x \in (j, j+1).$$

By the same argument as above, we have

$$\int_0^1 |\alpha_j + \beta_j x|^2 \, dx \geq \frac{4 - \sqrt{13}}{6}(|\alpha_j|^2 + |\beta_j|^2). \tag{3.6}$$

(3.5) and (3.6) imply that, if $\sum_{j\in \boldsymbol{Z}}(|\alpha_j|^2 + |\beta_j|^2) = \infty$, then $k^2 \notin \sigma_p(H_\theta)$.

In what follows, we mimic the argument in [11]. Let

$$E(n) := \{\theta \in \boldsymbol{R}/\boldsymbol{Z} \mid V_\theta(m - q_n) = V_\theta(m) = V_\theta(m + q_n), 1 \leq m \leq q_n\},$$

$$M := \{\theta \in \boldsymbol{R}/\boldsymbol{Z} \mid \sigma_p(H_\theta) = \varnothing\},$$

where q_n is determined by (3.2). Suppose there exists an integer r such that

$$V_\theta(j - r) = V_\theta(j) = V_\theta(j + r),$$

for $1 \leq j \leq r$. Then we have

$$\max(|\mathbf{w}_{-r}|, |\mathbf{w}_r|, |\mathbf{w}_{2r}|) \geq |\mathbf{w}_0|/2, \tag{3.7}$$

where $|\cdot|$ is the Euclidean norm in $\boldsymbol{C}^2$ ([7], Lemma 1) and hence

$$\limsup_{n\to\infty} E(n) \subset M.$$

On the other hand, by the same argument as in Kaminaga ([11], Section 2, 3), we can show

$$\text{(a)} \quad \mu(M) = 0 \text{ or } 1,$$

$$\text{(b)} \quad \mu\left(\limsup_{n\to\infty} E(n)\right) > 0,$$

if $\limsup_{n\to\infty} a_n \geq 4$ or $A = \Phi([1-\alpha, 1))$. μ denotes the Lebesgue measure on $\boldsymbol{R}/\boldsymbol{Z}$. Hence we have

$$\mu(M) = 1,$$

under that condition which completes the proof. □

4. The Spectrum of H_θ

From now on, we consider the case where $A = \Phi([1-\alpha, 1))$ and $\alpha \in (0, 1)$ is irrational. We prove the following property which corresponds to Lemma 3 in [4].

Proposition 4.1. *Let $A = \Phi([1-\alpha, 1))$ and $\alpha \in (0, 1)$ is irrational. Then the set $\sigma(H_\theta)$ is independent of $\theta \in \boldsymbol{R}/\boldsymbol{Z}$.*

Proof. Let τ_n be the n-shift operator on $L^2(\boldsymbol{R})$:

$$(\tau_n f)(x) := f(x+n), \quad n \in \boldsymbol{Z}.$$

It is easy to see $H_{\theta_1 - \Phi(n\alpha)} = \tau_n^{-1} H_{\theta_1} \tau_n$ for any $\theta_1 \in \boldsymbol{R}/\boldsymbol{Z}$, $n \in \boldsymbol{Z}$, thereby unitarity of τ_n gives

$$\sigma(H_{\theta_1 - \Phi(n\alpha)}) = \sigma(H_{\theta_1}). \tag{4.1}$$

Since α is irrational, for any $\theta_2 \in \boldsymbol{R}/\boldsymbol{Z}$ there exists a sequence of integers $\{n_k\}_{k=1}^{\infty}$ such that

$$0 \leq \theta_1 - \Phi(n_k\alpha) - \theta_2 \to 0, \quad \text{as } k \to \infty.$$

In what follows, we prove $H_{\theta_1 - \Phi(n_k\alpha)} \to H_{\theta_2}$ as $k \to \infty$ in the strong resolvent sense. Then by Theorem VIII.24 in [19] and the fact that $\sigma(H_{\theta_1 - \Phi(n_k\alpha)})$ is independent of k, we have

$$\sigma(H_{\theta_2}) \subset \sigma(H_{\theta_1 - \Phi(n_k\alpha)}) = \sigma(H_{\theta_1}),$$

from which the assertion follows.

Because $V_\theta(j)$ is right continuous and piecewise constant with respect to θ, it follows that for each $j \in \boldsymbol{Z}$, there exists K_j such that

$$V_{\theta_1-\Phi(n_k\alpha)}(j) = V_{\theta_2}(j), \tag{4.2}$$

if $k \geq K_j$. The resolvent of H_θ has the following representation ([1], Theorem III.2.1.3)

$$\begin{aligned}(H_\theta - E)^{-1}\psi(x) = &\int_{\boldsymbol{R}} G_E(x-y)\psi(y)\,dy \\ &- \sum_{i,j\in\boldsymbol{Z}} (T_{E,\theta}^{-1})_{ij} G_E(x-i)\int_{\boldsymbol{R}} G_E(y-j)\psi(y)\,dy,\end{aligned} \tag{4.3}$$

where $\psi \in L^2(\boldsymbol{R})$, $\operatorname{Im} E \neq 0$, and

$$G_E(x) = \frac{1}{2\sqrt{-E}} e^{-\sqrt{-E}|x|},$$

is Green's function of the free Laplacian. The branch of $\sqrt{-E}$ is chosen such that $Re(\sqrt{-E}) > 0$. $T_{E,\theta}$ is the bounded operator on $l^2(\boldsymbol{Z})$ whose (i,j)-component is

$$(T_{E,\theta})_{ij} = \frac{\delta_{ij}}{V_\theta(j)} + G_E(i-j), \quad i,j \in \boldsymbol{Z}. \tag{4.4}$$

δ_{ij} is Kronecker's delta. Since the operator $B : L^2(\boldsymbol{R}) \to l^2(\boldsymbol{Z})$ defined by

$$(B\psi)(n) := \int_{\boldsymbol{R}} G_E(x-n)\psi(x)\,dx, \quad n \in \boldsymbol{Z}, \psi \in L^2(\boldsymbol{R}),$$

is bounded, it suffices to show $T_{E,\theta_1-\Phi(n_k\alpha)} \to T_{E,\theta_2}$ in the strong resolvent sense. By (4.4), we have

$$\|(T_{E,\theta_1-\Phi(n_k\alpha)} - T_{E,\theta_2})u\|_{l^2(\boldsymbol{Z})}^2 = \sum_{j\in\boldsymbol{Z}} \left| \left(\frac{1}{V_{\theta_1-\Phi(n_k\alpha)}(j)} - \frac{1}{V_{\theta_2}(j)} \right) u(j) \right|^2,$$

for any $u \in l^2(\boldsymbol{Z})$. $\|\cdot\|_{l^2(\boldsymbol{Z})}$ is the $l^2(\boldsymbol{Z})$-norm. By (4.2), the last expression goes to zero as $k \to \infty$. Since $T_{E,\theta_1-\Phi(n_k\alpha)}$ and T_{E,θ_2} are bounded and symmetric, and since $T_{E,\theta}^{-1}$ is uniformly bounded with respect to θ if $|Im\, E|$ is sufficiently large, it follows that $T_{E,\theta_1-\Phi(n_k\alpha)} \to T_{E,\theta_2}$ in the strong resolvent sense. □

By Proposition 4.1, we have only to consider the case $\theta = 0$. For the sake of simplicity, we write H, $h(\lambda)$, instead of H_0, $h_0(\lambda)$ respectively. We introduce the following function to state Theorem 4.1 which is the main result of this paper.

$$g_\lambda(x) = \begin{cases} 2\cos\sqrt{x} + \lambda\frac{\sin\sqrt{x}}{\sqrt{x}}, & (\text{if } x > 0), \\ 2 + \lambda, & (\text{if } x = 0). \end{cases}$$

THEOREM 4.1. *Let $A = \Phi([1-\alpha, 1))$ and $\alpha \in (0,1)$ is irrational. Then*

$$\sigma(H) = \{E \in [0,\infty) \mid g_{\lambda_2}(E) \in \sigma(h(\lambda)), \lambda = g_{\lambda_2}(E) - g_{\lambda_1}(E)\}.$$

The following corollary may be regarded as an analogue of the result of Kirsch-Martinelli in (1.6).

COROLLARY 4.1.

(1) $\{n^2\pi^2 \mid n = 1, 2, \ldots, \} \subset \sigma(H)$;

(2) $\sigma(H) \subset \Sigma(\min\{\lambda_1, \lambda_2\})$, *in particular*, $0 \notin \sigma(H)$.

LEMMA 4.1 (Kato [12], Theorem 4.10). *Let $\mathscr{H}$ be a Hilbert space and $\mathscr{B}(\mathscr{H})$ the set of all bounded operators on $\mathscr{H}$. Let T be self-adjoint and $K \in \mathscr{B}(\mathscr{H})$ symmetric. Then $J = T + K$ is self-adjoint and $dist(\sigma(J), \sigma(T)) \le \|K\|$, that is,*

$$\sup_{t\in\sigma(J)} dist(t, \sigma(T)) \le \|K\|, \quad \sup_{t\in\sigma(T)} dist(t, \sigma(J)) \le \|K\|.$$

PROOF OF COROLLARY 4.1.

(1) Using Theorem 4.1 and $\sigma(h(0)) = [-2, 2]$, we have

$$g_{\lambda_2}(n^2\pi^2) = 2\cdot(-1)^n \in \sigma(h(0)) = \sigma(h(g_{\lambda_2}(n^2\pi^2) - g_{\lambda_1}(n^2\pi^2))).$$

Hence we obtain the inclusion in (1).

(2) At first, assume $\lambda_1 > \lambda_2 > 0$ and $E \ne 0$. It is easy to show $\Sigma(\lambda_1) \subset \Sigma(\lambda_2)$. Let

$$U_+ := \bigcup_{n=1}^{\infty} \{E > 0 \mid (2n-1)\pi \le \sqrt{E} \le 2n\pi\},$$

$$U_- := \bigcup_{n=0}^{\infty} \{E > 0 \mid 2n\pi < \sqrt{E} < (2n+1)\pi\}.$$

Suppose $E \in \sigma(H) \cap U_+$. By Theorem 4.1, $g_{\lambda_2}(E) \in \sigma(h(\lambda))$, where $\lambda = g_{\lambda_2}(E) - g_{\lambda_1}(E) = (\lambda_2 - \lambda_1)\sin\sqrt{E}/\sqrt{E} \ge 0$. Then by Lemma 4.1, $\sigma(h(\lambda)) \subset [-2, 2+\lambda] = [-2, -2+\lambda) \cup ([-2,2]+\lambda)$ (we define $C + \lambda := \{a + \lambda \mid a \in C\}$, for a set C $(\subset \boldsymbol{R})$, and $\lambda \in \boldsymbol{R}$). It follows that $g_{\lambda_2}(E) \in [-2, -2+\lambda) \cup ([-2,2]+\lambda)$ and hence at least one of the following two holds:

$$\text{(a)} \quad g_{\lambda_1}(E) < -2 \le g_{\lambda_2}(E), \quad \text{or} \quad \text{(b)} \quad g_{\lambda_1}(E) \in [-2, 2].$$

$E \in U_+$ implies $g_{\lambda_2}(E) \le 2$ and hence $E \in \Sigma(\lambda_2)$. $\sigma(H) \cap U_- \subset \Sigma(\lambda_2)$ follows similarly. If $0 \in \sigma(H)$, we would have

$$g_{\lambda_2}(0) = 2 + \lambda_2 \in \sigma(h(\lambda_2 - \lambda_1)) \subset [-2 + \lambda_2 - \lambda_1, 2],$$

which is impossible since $\lambda_2 > 0$. Therefore we proved Corollary 4.1 (2) when $\lambda_1 > \lambda_2$. The proof of the other case is similar. □

We recall some notations and definitions used in [4] to state a series of lemmas, which are necessary to prove Theorem 4.1. Let $T(n, e)$ be the transfer matrix associated to $h(\lambda)$:

$$T(n, e) := \begin{pmatrix} e - \lambda v(n) & -1 \\ 1 & 0 \end{pmatrix}, \quad n \in \boldsymbol{Z}, e \in \boldsymbol{R}.$$

Moreover, let

$$M(n, e) := T(q_n, e) \cdots T(2, e) T(1, e), \quad n \ge 1,$$

$$M(0, e) := T(0, e) = \begin{pmatrix} e & -1 \\ 1 & 0 \end{pmatrix},$$

$$M(-1, e) := \begin{pmatrix} 1 & -\lambda \\ 0 & 1 \end{pmatrix},$$

$$\xi(n, e) := \operatorname{tr} M(n, e), \quad n \ge -1.$$

We collect some properties of these defined above which we use later. These facts are proved in [4]. When $\alpha = (\sqrt{5} - 1)/2$, they are proved in [23].

LEMMA 4.2 ([4], Lemma 1, Proposition 1).
(1) $v(n) = [(n+1)\alpha] - [n\alpha]$, $[x] := \max\{n \in \boldsymbol{Z} \mid n \le x\}$.
(2) $v(q_n + k) = v(k)$, $1 \le k \le q_{n+1} - 2$, $n \ge 1$.
(3) $v(-n) = v(n-1)$, $n \ge 2$.
(4) (*recursive equation*)

$$M(n+1, e) = M(n-1, e) M(n, e)^{a_{n+1}}, \quad n \ge 0. \tag{4.5}$$

We recall that $a_n \in \boldsymbol{N}$ is defined in the fraction expansion of α (section 3).
(5) *For each 2×2 matrix with det $M = 1$, and $a \in \boldsymbol{N}$,*

$$M^a = S_{a-1}(\xi) M - S_{a-2}(\xi) I, \tag{4.6}$$

where $\xi := tr\ M$, *I is the identity matrix, and* $S_a(x)$ *is the Chebyshev polynomial generated by the following recursive equations.*

$$S_{-1}(x) = 0, \quad S_0(x) = 1, \quad S_1(x) = x, \tag{4.7}$$
$$S_a(x) = S_{a-1}(x)x - S_{a-2}(x).$$

THEOREM 4.2 ([4], Theorem).

$$\sigma(h(\lambda)) = B_\infty,$$

$$\textit{where} \quad B_\infty := \{e \in \boldsymbol{R} \mid \textit{the sequence } \{\xi(n,e)\}_{n=-1}^{\infty} \textit{ is bounded}\}.$$

REMARK 4.1. *Bellissard et al.* [4] *also proved* $\sigma(h(\lambda))$ *has zero Lebesgue measure.*

The following lemma is a detailed presentation of Proposition 2 in [4]. For its proof, we need some notations. We recall that if $|x| < 2$,

$$S_a(x) = \frac{\sin(a+1)\theta}{\sin\theta}, \quad a \geq 1,$$

where $\theta = \arccos(x/2) \in (0, \pi)$. If $S_a(x) = \sin(a+1)\theta/\sin\theta = 0$ for some $|x| < 2$, then $S_{a+1}(x) = (-1)^n$, where n is uniquely determined by $(a+1)\theta = n\pi$. In this case, we write $n = f(x,a) = ((a+1)/\pi)\arccos(x/2) \in \boldsymbol{N}$.

LEMMA 4.3. *Let* $\{M_n\}_{n=-1}^{\infty}$ *be a sequence of* 2×2 *matrices such that det* $M_n = 1$ *and satisfies the following recursive equation*

$$M_{n+1} = M_{n-1} M_n^{a_{n+1}}, \quad n \geq 0. \tag{4.8}$$

Let $\xi_n := tr\ M_n$ $(n \geq -1)$. *Then the sequence* $\{\xi_n\}_{n=-1}^{\infty}$ *is uniquely determined if we specify the initial values* ξ_{-1}, ξ_0, ξ_1, *and* $tr(M_0 M_{-1})$, *and does not depend on any particular form of these matrices.*

REMARK 4.2. *The same result also holds if* $\{M'_n\}_{n=-1}^{\infty}$ *satisfies the following recursive equation instead of* (4.8).

$$M'_{n+1} = M_n'^{a_{n+1}} M'_{n-1}. \tag{4.9}$$

For the sake of completeness, we give the proof of Lemma 4.3.

PROOF. Fix n $(n \geq 1)$. We aim to represent ξ_n in terms of ξ_k for $k \leq n-1$. We consider two complementary cases:

case (1) there exists k $(k = 1, \dots, n)$ such that $S_{a_k-1}(\xi_{k-1}) \neq 0$,

case (2) $S_{a_k-1}(\xi_{k-1}) = 0$, for all $k = 1, \dots, n$.

CASE (1). We have the following equation whose proof is given below.

$$\xi_{n+1} = (-1)^{\sum_{l=K_n+1}^{n} g(l)} S_{a_{n+1}-1}(\xi_n) b_{K_n} - S_{a_{n+1}-2}(\xi_n)\xi_{n-1}, \tag{4.10}$$

where

$$g(l) := f(\xi_{l-1}, a_l - 1) \in \boldsymbol{N},$$

$$b_k := \frac{S_{a_k}(\xi_{k-1})}{S_{a_k-1}(\xi_{k-1})}\xi_k - \frac{1}{S_{a_k-1}(\xi_{k-1})}\xi_{k-2}, \quad k \geq 1,$$

$$K_n := \max\{k \in \boldsymbol{N} \mid 1 \leq k \leq n, S_{a_k-1}(\xi_{k-1}) \neq 0\}.$$

When $K_n = n$, we define $\sum_{l=K_n+1}^{n} g(l) := 0$.

CASE (2). We have the following equation.

$$\xi_{n+1} = (-1)^{\sum_{l=1}^{n} g(l)} S_{a_{n+1}-1}(\xi_n)\ \mathrm{tr}(M_0 M_{-1}) - S_{a_{n+1}-2}(\xi_n)\xi_{n-1}. \tag{4.11}$$

The statement of Lemma 4.3 clearly follows from (4.10), (4.11).

PROOF OF (4.10). If $K_n = n$, the derivation of (4.10) is given in [4]. If $K_n \leq n-1$, then $S_{a_k-1}(\xi_{k-1}) = 0$, for $k = K_n+1$, $K_n+2, \dots, n$. For such k, by (4.6), (4.7),

$$M_{k-1}^{a_k} = -S_{a_k-2}(\xi_{k-1})I = S_{a_k}(\xi_{k-1})I = (-1)^{f(\xi_{k-1}, a_k-1)} I.$$

Multiplying M_{k-2} from the left and using (4.8), we have

$$M_k = (-1)^{g(k)} M_{k-2}, \quad k = K_n+1, K_n+2, \dots, n. \tag{4.12}$$

CASE (1a). If n and K_n+1 have the same parity, then

$$\begin{aligned} M_n &= (-1)^{\sum_{l=0}^{(n-(K_n+1))/2} g(K_n+1+2l)} M_{K_n-1}, \\ M_{n-1} &= (-1)^{\sum_{l=0}^{(n-1-(K_n+2))/2} g(K_n+2+2l)} M_{K_n}. \end{aligned} \tag{4.13}$$

CASE (1b). If n and K_n+1 have the opposite parity, then

$$
\begin{aligned}
M_n &= (-1)^{\sum_{l=0}^{(n-(K_n+2))/2} g(K_n+2+2l)} M_{K_n}, \\
M_{n-1} &= (-1)^{\sum_{l=0}^{(n-1-(K_n+1))/2} g(K_n+1+2l)} M_{K_n-1}.
\end{aligned} \tag{4.14}
$$

In Case (1a), by using (4.8), (4.6), and (4.13), we have

$$
\begin{aligned}
M_{n+1} &= M_{n-1} M_n^{a_{n+1}} \\
&= S_{a_{n+1}-1}(\xi_n) M_{n-1} M_n - S_{a_{n+1}-2}(\xi_n) M_{n-1} \\
&= (-1)^{\sum_{l=K_n+1}^{n} g(l)} S_{a_{n+1}-1}(\xi_n) M_{K_n} M_{K_n-1} \\
&\quad - S_{a_{n+1}-2}(\xi_n) M_{n-1}.
\end{aligned} \tag{4.15}
$$

On the other hand,

$$
\begin{aligned}
M_{K_n} &= M_{K_n-2} M_{K_n-1}^{a_{K_n}} \\
&= S_{a_{K_n}-1}(\xi_{K_n-1}) M_{K_n-2} M_{K_n-1} - S_{a_{K_n}-2}(\xi_{K_n-1}) M_{K_n-2},
\end{aligned} \tag{4.16}
$$

$$
\begin{aligned}
M_{K_n} M_{K_n-1} &= M_{K_n-2} M_{K_n-1}^{a_{K_n}+1} \\
&= S_{a_{K_n}}(\xi_{K_n-1}) M_{K_n-2} M_{K_n-1} - S_{a_{K_n}-1}(\xi_{K_n-1}) M_{K_n-2}.
\end{aligned} \tag{4.17}
$$

Combining (4.16) with (4.17), we have

$$
\begin{aligned}
M_{K_n} M_{K_n-1} &= S_{a_{K_n}}(\xi_{K_n-1}) \left\{ \frac{M_{K_n} + S_{a_{K_n}-2}(\xi_{K_n-1}) M_{K_n-2}}{S_{a_{K_n}-1}(\xi_{K_n-1})} \right\} \\
&\quad - S_{a_{K_n}-1}(\xi_{K_n-1}) M_{K_n-2}. \\
&= \frac{S_{a_{K_n}}(\xi_{K_n-1})}{S_{a_{K_n}-1}(\xi_{K_n-1})} M_{K_n} - \frac{1}{S_{a_{K_n}-1}(\xi_{K_n-1})} M_{K_n-2}.
\end{aligned} \tag{4.18}
$$

We used $S_{a_{K_n}-1}(\xi_{K_n-1}) \neq 0$, and the fact that the quantity $S_a(x)S_{a-2}(x) - S_{a-1}^2(x)$ is constant independent of a: $S_a(x)S_{a-2}(x) - S_{a-1}^2(x) = -1$, which follows from (4.7). Combining (4.18) with (4.15), we have

$$
\begin{aligned}
M_{n+1} &= (-1)^{\sum_{l=K_n+1}^{n} g(l)} S_{a_{n+1}-1}(\xi_n) \\
&\quad \times \left(\frac{S_{a_{K_n}}(\xi_{K_n-1})}{S_{a_{K_n}-1}(\xi_{K_n-1})} M_{K_n} - \frac{1}{S_{a_{K_n}-1}(\xi_{K_n-1})} M_{K_n-2} \right) \\
&\quad - S_{a_{n+1}-2}(\xi_n) M_{n-1}.
\end{aligned}
$$

By taking trace, we obtain (4.10). The proof of that in Case (1b) is similar, except that we use the cyclicity of trace.

PROOF OF (4.11). (4.12) implies

$$M_n = (-1)^{\sum_{l=1}^{(n+1)/2} g(2l-1)} M_{-1}, \quad M_{n-1} = (-1)^{\sum_{l=1}^{(n-1)/2} g(2l)} M_0, \tag{4.19}$$

if n is odd, and

$$M_n = (-1)^{\sum_{l=1}^{n/2} g(2l)} M_0, \quad M_{n-1} = (-1)^{\sum_{l=1}^{n/2} g(2l-1)} M_{-1}, \tag{4.20}$$

if n is even. We combine (4.19) or (4.20) with (4.15), depending on the parity of n. Then

$$M_{n+1} = (-1)^{\sum_{l=1}^{n} g(l)} S_{a_{n+1}-1}(\xi_n) M_0 M_{-1} - S_{a_{n+1}-2}(\xi_n) M_{n-1},$$

if n is odd. If n is even, $M_0 M_{-1}$ should be replaced by $M_{-1} M_0$. Taking trace, we obtain (4.11). □

LEMMA 4.4 ([4], Proposition 4). *$\{\xi(n,e)\}_{n=-1}^{\infty}$ is unbounded if and only if there exists $N_1 \geq 0$ such that*

$$|\xi(N_1 - 1, e)| \leq 2, \quad |\xi(N_1, e)| > 2, \quad |\xi(N_1 + 1, e)| > 2.$$

Moreover, N_1 in uniquely determined, $|\xi(n+2)| > |\xi(n+1)|\,|\xi(n)|/2$ for all $n \geq N_1$, and $\xi(n) > 2C^{q_n}$, $n \geq N_1$ for some $C > 1$.

We define the following sets to state the next lemma.

$$\rho_n := \{e \in \boldsymbol{R} \mid |\xi(n,e)| > 2\}, \quad n \geq -1.$$

LEMMA 4.5

$$(1) \quad B_\infty^c = \bigcup_{n \geq N} (\rho_n \cap \rho_{n+1}), \quad \textit{for any } N \geq 0,$$

$$(2) \quad \rho_n \cap \rho_{n+1} = \bigcap_{k \geq n} \rho_k, \qquad \textit{for any } n \geq 0.$$

PROOF. By Lemma 4.4,

$$B_\infty^c = (\rho_0 \cap \rho_1) \cup \left(\bigcup_{n=1}^{\infty} (\sigma_{n-1} \cap \rho_n \cap \rho_{n+1}) \right), \tag{4.21}$$

where $\sigma_n := \rho_n^c$ (complement is taken in $\boldsymbol{R}$). Clearly, the equality

$$\rho_n \cap \rho_{n+1} = (\rho_0 \cap \rho_1) \cup \left(\bigcup_{k=1}^{n} (\sigma_{k-1} \cap \rho_k \cap \rho_{k+1} \cdots \cap \rho_{n+1}) \right) \quad \text{(disjoint union)}$$

holds for any $n \geq 0$. By Lemma 4.4, $|\xi(n,e)| > 2$ holds for all $n \geq N_1$ which implies

$$\rho_n \cap \rho_{n+1} = (\rho_0 \cap \rho_1) \cup \left(\bigcup_{k=1}^{n} (\sigma_{k-1} \cap \rho_k \cap \rho_{k+1}) \right).$$

By this equality, $\rho_n \cap \rho_{n+1}$ is monotonically increasing with respect to n: $\rho_n \cap \rho_{n+1} \subset \rho_{n+1} \cap \rho_{n+2}$ which gives the second statement of Lemma 4.5. Moreover, we have

$$\bigcup_{n \geq N} (\rho_n \cap \rho_{n+1}) = (\rho_0 \cap \rho_1) \cup \left(\bigcup_{n=1}^{\infty} (\sigma_{n-1} \cap \rho_n \cap \rho_{n+1}) \right),$$

for any $N \geq 0$ which, combined with (4.21), gives the first statement. □

Remark 4.3. *The proof of Lemma* 4.5 *requires the conclusion of Lemma* 4.4 *only. The proof of Lemma* 4.4 *requires* $|\xi_{-1}| \leq 2$, (4.8), (4.10) (*in the case of* $K_n = n$), *and the fact that the following quantity is constant with respect to* n:

$$I_n := \xi_{n+1}^2 + \xi_n^2 + [tr(M_n M_{n+1})]^2 - \xi_{n+1}\xi_n \, tr(M_n M_{n+1}),$$

which follows from (4.8). *Therefore, the conclusion of Lemma* 4.4 *and* 4.5 *also holds for any sequence* $\{M_n\}_{n=-1}^{\infty}$ *of* 2×2 *matrices which satisfies* $\det M_n = 1$, $|\xi_{-1}| \leq 2$, *and* (4.8) *or* (4.9).

We consider the Kronig-Penney analogue of these facts stated above. Let

$$\begin{aligned}
W(n,E) &:= R(q_n,E) \cdots R(2,E)R(1,E), \quad n \geq 1, \\
W(0,E) &:= R(0,E), \\
W(-1,E) &:= R_{\lambda_1}(E) R_{\lambda_2}(E)^{-1} \\
&= \begin{cases} \begin{pmatrix} 1 & \frac{\lambda_1 - \lambda_2}{\sqrt{E}} \\ 0 & 1 \end{pmatrix}, & (\text{if } E > 0), \\ \begin{pmatrix} 1 & 0 \\ \lambda_1 - \lambda_2 & 1 \end{pmatrix}, & (\text{if } E = 0), \end{cases} \\
r(n,E) &:= \operatorname{tr} W(n,E).
\end{aligned}$$

Then $W(n,E)$ satisfies the same recursion equation as $M(n,E)$:

$$W(n+1,E) = W(n-1,E)W(n,E)^{a_{n+1}}, \quad n \geq 0. \tag{4.22}$$

We define the following sets.

$$\tilde{B}_\infty := \{E \in [0,\infty) \mid \{r(n,E)\}_{n=-1}^{\infty} \text{ is bounded}\},$$

$$\tilde{\rho}_n := \{E \in [0,\infty) \mid |r(n,E)| > 2\}.$$

In the next lemma, we compare the initial values of the sequence $\{\xi(n,e)\}_{n=-1}^{\infty}$ with that of $\{r(n,E)\}_{n=-1}^{\infty}$ by which we have the relationship between $\sigma(h(\lambda))$ and $\sigma(H)$.

LEMMA 4.6.

$$\tilde{B}_\infty = \{E \in [0,\infty) \mid g_{\lambda_2}(E) \in \sigma(h(\lambda)), \lambda = g_{\lambda_2}(E) - g_{\lambda_1}(E)\},$$

$$\tilde{B}_\infty^c \cap [0,\infty) = \bigcup_{n \geq N} (\tilde{\rho}_n \cap \tilde{\rho}_{n+1}), \quad \textit{for any } N \geq 0,$$

$$\textit{and} \quad \tilde{\rho}_n \cap \tilde{\rho}_{n+1} = \bigcap_{k \geq n} \tilde{\rho}_k \quad \textit{for any } n \geq 0.$$

PROOF. By (4.22), the sequence $\{W(n,E)\}_{n=-1}^{\infty}$ satisfies the assumption of Lemma 4.3. Moreover, by Remark 4.3, the conclusion of Lemma 4.4, and 4.5 are also valid for $r(n,E)$, $\tilde{B}_\infty$, and $\tilde{\rho}_n$ instead of $\xi(n,e)$, B_∞, and ρ_n, respectively. Therefore we have the second and the third equality. Lemma 4.3 implies that the sequence $\{r(n,E)\}_{n=-1}^{\infty}$ is determined by the following initial conditions.

$$r(-1,E) = 2, \quad r(0,E) = g_{\lambda_2}(E),$$

$$r(1,E) = S_{a_1-1}(g_{\lambda_2}(E))g_{\lambda_1}(E) - 2S_{a_1-2}(g_{\lambda_2}(E)),$$

$$tr(W(0,E)W(-1,E)) = g_{\lambda_1}(E).$$

The sequence $\{\xi(n,e)\}_{n=-1}^{\infty}$ is determined by

$$\xi(-1,e) = 2, \quad \xi(0,e) = e,$$

$$\xi(1,e) = S_{a_1-1}(e)(e-\lambda) - 2S_{a_1-2}(e),$$

$$tr(M(0,e)M(-1,e)) = e - \lambda.$$

Let $E \in X := \{E \in [0,\infty) \mid g_{\lambda_2}(E) \in \sigma(h(\lambda)), \lambda = g_{\lambda_2}(E) - g_{\lambda_1}(E)\}$. Then $e :=$

$g_{\lambda_2}(E) \in \sigma(h(\lambda))$ for $\lambda := g_{\lambda_2}(E) - g_{\lambda_1}(E)$ and hence by Theorem 4.2, the sequence $\{\xi(n,e)\}_{n=-1}^{\infty}$ is bounded. The initial condition of $\{\xi(n,e)\}_{n=-1}^{\infty}$ is

$$\xi(-1,e) = 2, \quad \xi(0,e) = e = g_{\lambda_2}(E),$$

$$\begin{aligned}\xi(1,e) &= S_{a_1-1}(e)(e-\lambda) - 2S_{a_1-2}(e)\\ &= S_{a_1-1}(g_{\lambda_2}(E))g_{\lambda_1}(E) - 2S_{a_1-2}(g_{\lambda_2}(E)),\end{aligned}$$

$$tr(M(0,e)M(-1,e)) = e - \lambda = g_{\lambda_1}(E).$$

Therefore by Lemma 4.3, $r(n,E) = \xi(n,e)$ and the sequence $\{r(n,E)\}_{n=-1}^{\infty}$ is bounded which gives $E \in \tilde{\boldsymbol{B}}_{\infty}$. The proof of converse is similar. □

Due to Lemma 4.6, it suffices to show $\sigma(H) = \tilde{\boldsymbol{B}}_{\infty}$ to prove Theorem 4.1. In order to do that, we consider a periodic approximation of the operator H:

$$H^{(n)} := -\frac{d^2}{dx^2} + \sum_{j \in \boldsymbol{Z}} V^{(n)}(j)\delta(x-j),$$

where $V^{(n)}(j)$ is the periodic sequence given by

$$V^{(n)}(j) := \lambda_1 \chi_{A_n}(\Phi(\alpha_n j)) + \lambda_2 \chi_{A_n^c}(\Phi(\alpha_n j)), \quad A_n := [1-\alpha_n, 1),$$

and $\alpha_n := p_n/q_n$ is the principal convergent of α given in (3.1), (3.2).

To discuss the spectrum of the periodic Kronig-Penney model in general situation, let

$$H_{per} := -\frac{d^2}{dx^2} + \sum_{j \in \boldsymbol{Z}} f(j)\delta(x-j),$$

where $\{f(j)\}_{j \in \boldsymbol{Z}}$ is a real-valued, positive, and periodic sequence with period $L \in \boldsymbol{N}$. Let $R^L(j,E)$ be the transfer matrix of H_{per} : $R^L(j,E) := R_{f(j)}(E)$ (*i.e.*, V in (2.2) is replaced by $f(j)$) and define

$$M_L(E) := R^L(L,E)R^L(L-1,E)\cdots R^L(2,E)R^L(1,E).$$

Lemma 4.7

$$\sigma(H_{per}) = \{E \in \boldsymbol{R} \mid |tr\ M_L(E)| \le 2\}.$$

Proof. We consider the direct integral decomposition of H_{per} (*e.g.*, [21])

$$H_{per} = \int_{[0,2\pi)}^{\oplus} H_{per}(\theta)\frac{d\theta}{2\pi},$$

where

$$H_{per}(\theta) := -\frac{d^2}{dx^2}, \quad \text{on } \mathscr{D}_\theta,$$

$$\begin{aligned}\mathscr{D}_\theta := \{g \in H^1(1/2, L+1/2) \cap H^2((1/2, L+1/2)\backslash\{1,2,\dots,L\}) \mid \\ g(L+1/2) = e^{i\theta}g(1/2), g'(L+1/2) = e^{i\theta}g'(1/2), \\ g'(j+0) - g'(j-0) = f(j)g(j),\ j = 1,2,\dots,L\}.\end{aligned}$$

$H_{per}(\theta)$ have purely discrete spectrum for each $\theta \in [0, 2\pi)$ (*e.g.*, [2]) and we have

$$\sigma(H_{per}) = \bigcup_{\theta \in [0,2\pi)} \sigma(H_{per}(\theta)). \tag{4.23}$$

Let $E \in \sigma(H_{per})$. Then $E \in \sigma(H_{per}(\theta))$ for some $\theta \in [0, 2\pi)$. Let $\varphi \in \mathscr{D}_\theta$ satisfy the equation $H_{per}(\theta)\varphi = E\varphi$. φ has the form as given in (2.1) on $(j, j+1)$ $(j = 0, 1, \dots, L)$ with $\mathbf{w}_j := (\alpha_j, \beta_j) \in \boldsymbol{C}^2$ which satisfies $\mathbf{w}_L = M_L(E)\mathbf{w}_0$. Since $\varphi \in \mathscr{D}_\theta$, $\mathbf{w}_L = e^{i\theta}\mathbf{w}_0$. Therefore $E \in \sigma(H_{per}(\theta))$ if and only if the corresponding matrix $M_L(E)$ has eigenvalue $e^{i\theta}$. Since $\det M_L(E) = 1$, eigenvalues of $M_L(E)$ are equal to λ, λ^{-1} for some $\lambda \in \boldsymbol{C}$, and hence $\lambda, \lambda^{-1} = e^{\pm i\theta}$. Therefore $|\mathrm{tr}\ M_L(E)| = |2\cos\theta| \le 2$. Since $\mathrm{tr}\ M_L(E) \in \boldsymbol{R}$, the converse is easy to prove. □

LEMMA 4.8. (1) *If n is even,*

$$V^{(n)}(j) = V(j), \quad \textit{for } -q_n - 1 \le j \le q_n. \tag{4.24}$$

(2) *If n is odd,*

$$V^{(n)}(j) = V(j), \quad \textit{for } -q_n + 1 \le j \le q_n - 2. \tag{4.25}$$

PROOF. Since $V(j) = (\lambda_1 - \lambda_2)v(j) + \lambda_2$, it suffices to show $v(j) = v^{(n)}(j)$, where $v^{(n)}(j) := [(j+1)\alpha_n] - [j\alpha_n]$. If n is even, $\alpha_n := p_n/q_n < \alpha$ ([16], p. 8). A sufficient condition to have $[(j+1)\alpha_n] = [(j+1)\alpha]$ is

$$(j+1)(\alpha - \alpha_n) = \frac{j+1}{q_n}(q_n\alpha - p_n) \le \|(j+1)\alpha\|.$$

Therefore by (3.4), the condition $1 \le j \le q_n - 1$ is sufficient to have $v(j) =$

$v^{(n)}(j)$. By direct computation, we always have $v^{(n)}(0) = v(0) = 0$ and $v^{(n)}(-1) = v(-1) = 1$. When $j = q_n$,

$$\begin{aligned} v(q_n) &= [(q_n + 1)\alpha] - [q_n\alpha] \\ &= [q_n\alpha - p_n + \alpha] - [q_n\alpha - p_n]. \end{aligned}$$

Because $0 < q_n\alpha - p_n = \|q_n\alpha\|$ and $\alpha \in (0, 1)$, we have $v(q_n) = 0$. Direct computation gives $v^{(n)}(q_n) = 0$. Therefore by Lemma 4.2 (3), we obtain (4.24). The proof of (4.25) is similar. □

In what follows, we let

$$\rho_{\boldsymbol{R}}(T) := \rho(T) \cap \boldsymbol{R},$$

$$\rho_+(T) := \rho(T) \cap [0, \infty),$$

where $\rho(T)$ is the resolvent set of an operator T.

Lemma 4.9. *$H^{(n)}$ converges to H in the strong resolvent sense. Moreover, $\rho_+(H^{(n)}) = \tilde{\rho}_n$.*

Proof. The first statement follows easily by using Lemma 4.8 and the resolvent formula (4.3). Let $R^{(n)}(j, E) := R_{V^{(n)}(j)}(E)$ be the transfer matrix of $H^{(n)}$. When n is even, by Lemma 4.7, 4.8, we have

$$\begin{aligned} \tilde{\rho}_n &= \{E \geq 0 \mid |tr(R(q_n, E)R(q_n - 1, E) \cdots R(1, E))| > 2\} \\ &= \{E \geq 0 \mid |tr(R^{(n)}(q_n, E)R^{(n)}(q_n - 1, E) \cdots R^{(n)}(1, E))| > 2\} \\ &= \rho_+(H_n). \end{aligned}$$

When n is odd, we use the following equality [23]

$$v(-q_{n-1} + l) = v(l), \quad 1 \leq l \leq q_n, \tag{4.26}$$

which holds if n is sufficiently large. To prove (4.26) for $1 \leq l \leq q_n - 2$, we proceed similarly as in the proof of Lemma 4.8. For $l = q_n - 1$, q_n, we compute $v(-q_{n-1} + l)$ and $v(l)$ explicitly, and see that they are equal. Then $\rho_+(H^{(n)}) = \tilde{\rho}_n$ follows from (4.26), the periodicity of $v^{(n)}(l)$, and the cyclicity of trace. □

Lemma 4.10 ([19], Theorem VIII.24). *Let L, L_m be self-adjoint operators on a Hilbert space, and assume that L_m converges to L in the strong resolvent sense. Then*

$$Int\left(\bigcap_{m=1}^{\infty} \rho_{\boldsymbol{R}}(L_m)\right) \subset \rho_{\boldsymbol{R}}(L),$$

where Int C is the interior of C in $\boldsymbol{R}$.

LEMMA 4.11 ([23], Lemma 1). *Let B be a* 2×2 *matrix with det* $B = 1$. *Then*

$$\max\{|tr\ B|\,|B\mathbf{u}|, |B^2\mathbf{u}|\} \geq |\mathbf{u}|/2,$$

for any $\mathbf{u} \in \boldsymbol{C}^2$.

PROOF OF THEOREM 4.1. By Lemma 4.6, it is sufficient to show $\sigma(H) = \tilde{\boldsymbol{B}}_\infty$. Assume $E \in \tilde{\boldsymbol{B}}_\infty \backslash \sigma(H)$ and let

$$f(x) := \begin{cases} \chi_{(-1,0)}(x) \cos \sqrt{E}x, & (\text{if } E > 0), \\ \chi_{(-1,0)}(x), & (\text{if } E = 0). \end{cases}$$

By assumption, the equation $(H - E)\psi = f$ has an L^2-solution. In particular, $((H - E)\psi)(x) = 0$ holds for $x \notin (-1, 0)$ which implies

$$\psi(x) = \begin{cases} \alpha_j \sin \sqrt{E}(x - j) + \beta_j \cos \sqrt{E}(x - j), & (\text{if } E > 0), \\ \alpha_j + \beta_j(x - j), & (\text{if } E = 0), \end{cases}$$

for $x \in (j, j + 1)$ $(j \neq -1)$. Since $E \in \tilde{\boldsymbol{B}}_\infty$, there exists a constant $C > 0$ such that

$$|r(n, E)| \leq C, \quad \text{for all } n \geq -1.$$

By (2.3) and Lemma 4.2 (2), we have $\mathbf{w}_{q_n} = W(n, E)\mathbf{w}_0$, $\mathbf{w}_{2q_n} = W(n, E)^2\mathbf{w}_0$ which, combined with Lemma 4.11, imply

$$\max\{C|\mathbf{w}_{q_n}|, |\mathbf{w}_{2q_n}|\} \geq |\mathbf{w}_0|/2. \tag{4.27}$$

Let $L(n, E) := [R(1, E)R(2, E) \cdots R(q_n, E)]^{-1}$. By Lemma 4.2 (3),

$$L(n, E) = R(-q_n - 1, E)^{-1} R(-q_n, E)^{-1} \cdots R(-3, E)^{-1} R(-2, E)^{-1},$$

so that we have $\mathbf{w}_{-q_n-2} = L(n, E)\mathbf{w}_{-2}$, $\mathbf{w}_{-2q_n-2} = L(n, E)^2\mathbf{w}_{-2}$. On the other hand, $\operatorname{tr} L(n, E) = \operatorname{tr} K(n, E)$, where $K(n, E)$ is given by

$$K(n, E) := R(1, E)R(2, E) \cdots R(q_n, E), \quad n \geq 1,$$

$$K(0, E) := R(0, E),$$

$$K(-1, E) := R_{\lambda_2}^{-1}(E) R_{\lambda_1}(E).$$

Then $\det K(n,E) = 1$ and $K(n,E)$ satisfies the following recursive equation.

$$K(n+1,E) = K(n,E)^{a_{n+1}} K(n-1,E), \quad n \geq 0. \tag{4.28}$$

By Remark 4.2, and (4.28), we see that $s(n,E) := \operatorname{tr} K(n,E)$ satisfies the same equations (4.10), (4.11), and initial conditions as $r(n,E)$. Therefore $\operatorname{tr} L(n,E) = s(n,E) = r(n,E)$ so that $|\operatorname{tr} L(n,E)| \leq C$. By using Lemma 4.11 again, we have

$$\max\{C|\mathbf{w}_{-q_n-2}|, |\mathbf{w}_{-2q_n-2}|\} \geq |\mathbf{w}_{-2}|/2. \tag{4.29}$$

Therefore since $\psi \in L^2(\boldsymbol{R})$, (4.27) and (4.29) implies $\mathbf{w}_0 = \mathbf{w}_{-2} = 0$ and thus $\psi(x) = 0$ for $x \notin (-1,0)$. Moreover, from the continuity of ψ, we have $\psi(0) = \psi(-1) = 0$ and $\psi'(0) = \psi'(-1) = 0$. Since $\psi(j) = 0$, $j \in \boldsymbol{Z}$,

$$-\frac{d^2}{dx^2}\psi = E\psi + f \in L^2(\boldsymbol{R}),$$

where the derivative in LHS is taken in the sense of distribution. Therefore $\psi \in H^2(\boldsymbol{R})$. If $E > 0$, by the definition of H,

$$\begin{aligned} \|f\|_2^2 &= ((H-E)\psi, f) \\ &= -\int_{-1}^{0} \psi''(x) \cos\sqrt{E}x\, dx - E\int_{-1}^{0} \psi(x) \cos\sqrt{E}x\, dx. \end{aligned} \tag{4.30}$$

Take $\psi_n(x) \in C_0^\infty(\boldsymbol{R})$ such that $\psi_n \to \psi$ as $n \to \infty$ in $H^2(\boldsymbol{R})$. Then

$$\int_{-1}^{0} \psi''(x) \cos\sqrt{E}x\, dx = \lim_{n\to\infty} \int_{-1}^{0} \psi_n''(x) \cos\sqrt{E}x\, dx.$$

By integration by parts,

$$\begin{aligned} &\int_{-1}^{0} \psi_n''(x) \cos\sqrt{E}x\, dx \\ &\quad = [\psi_n'(x) \cos\sqrt{E}x]_{-1}^{0} + \sqrt{E}\int_{-1}^{0} \psi_n'(x) \sin\sqrt{E}x\, dx \\ &\quad = [\psi_n'(x) \cos\sqrt{E}x]_{-1}^{0} + \sqrt{E}[\psi_n(x) \sin\sqrt{E}x]_{-1}^{0} - E\int_{-1}^{0} \psi_n(x) \cos\sqrt{E}x\, dx. \end{aligned}$$

By Sobolev's embedding theorem (*e.g.*, [20], Theorem IX.24), $\psi_n^{(l)}(j) \to \psi^{(l)}(j) = 0$ as $n \to \infty$ for $l = 0,1$, and $j = -1,0$. Thus

$$\int_{-1}^{0} \psi''(x) \cos \sqrt{E}x \, dx = -E \int_{-1}^{0} \psi(x) \cos \sqrt{E}x \, dx,$$

which, together with (4.30), concludes $f = 0$ and hence contradicts the assumption. If $E = 0$,

$$\begin{aligned} \|f\|_2^2 &= (H\psi, f) \\ &= -\int_{-1}^{0} \psi''(x) \, dx \\ &= -\lim_{n\to\infty} \int_{-1}^{0} \psi_n''(x) \, dx \\ &= -\lim_{n\to\infty} (\psi_n'(0) - \psi_n'(-1)) \\ &= 0. \end{aligned}$$

Therefore we obtain $\tilde{B}_\infty \subset \sigma(H)$. Conversely, take $E \in \tilde{B}_\infty^c$, $E > 0$. Lemma 4.6 implies $E \in \tilde{\rho}_n \cap \tilde{\rho}_{n+1}$ for some $n \geq 0$ which can be taken sufficiently large, and then Lemma 4.9 implies $E \in \rho_+(H^{(n)}) \cap \rho_+(H^{(n+1)})$. Since $E \neq 0$, we have $E \in \mathrm{Int}(\rho_+(H^{(n)}) \cap \rho_+(H^{(n+1)}))$. Therefore

$$\begin{aligned} E &\in \mathrm{Int}(\tilde{\rho}_n \cap \tilde{\rho}_{n+1}) \\ &= \mathrm{Int}\left(\bigcap_{k\geq n} \tilde{\rho}_k \right) \\ &= \mathrm{Int}\left(\bigcap_{k\geq n} \rho_+(H^{(k)}) \right) \\ &\subset \rho_{\boldsymbol{R}}(H), \end{aligned}$$

where we used Lemma 4.6 and 4.10. If $0 \in \tilde{B}_\infty^c$, then $0 \in \tilde{\rho}_n \cap \tilde{\rho}_{n+1}$ for some $n \geq 0$, and $[0, \delta) \subset \rho_+(H^{(n)}) \cap \rho_+(H^{(n+1)}) = \bigcap_{k\geq n} \rho_+(H^{(k)})$ for some $\delta > 0$. Since $(-\infty, 0) \subset \rho_{\boldsymbol{R}}(H^{(k)})$ for all $k \geq 0$, we have $(-\delta, \delta) \subset \bigcap_{k\geq n} \rho_{\boldsymbol{R}}(H^{(k)})$ and conclude

$$0 \in \mathrm{Int} \bigcap_{k\geq n} \rho_{\boldsymbol{R}}(H^{(k)}) \subset \rho_{\boldsymbol{R}}(H).$$

Thus Theorem 4.1 is proved. □

REMARK 4.4. *The inequalities* (4.27), (4.29), *together with the argument in the proof of Theorem* 3.2 *also show the absence of point spectrum of* H.

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