

EINSTEIN-WEYL STRUCTURES ON ALMOST CONTACT METRIC MANIFOLDS

Dedicated to Professor Shūkichi Tanno on his sixtieth birthday

By

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1. Introduction

Pedersen and Swann [7] and Higa [2] studied the existence of an Einstein-Weyl structure on principal circle bundles over Einstein Kaehler manifolds of positive scalar curvature and obtained many examples of Einstein-Weyl spaces. In a Riemannian submersion $\pi : M \rightarrow N$ with totally geodesic fibers of dimension one over an Einstein manifold N , we studied the relation between an Einstein-Weyl structure and a Sasakian structure of M (cf. [5]).

On the other hand, in [4], we investigated some geometric structures of a Riemannian submersion $\pi : M \rightarrow N$, where M is a CR-submanifold of a locally conformal Kaehler manifold L . Let M be a leaf of the canonical foliation \mathcal{M} given by the Lee form $\omega = 0$ of a locally conformal Kaehler manifold L . Then M admits an almost contact metric structure. We obtained a necessary and sufficient condition for the manifold M to admits a Sasakian structure.

In this paper, we shall study the existence of an Einstein-Weyl structure on an almost contact metric manifold. Let M be a complete and simply connected Sasakian manifold with constant ϕ -sectional curvature k . We show that if $k \geq 1$, then M admits an Einstein-Weyl structure.

Next, we assume that all local Kaehler metrics $g' = e^{-r}g$ of a locally conformal Kaehler manifold L have the same constant nonnegative holomorphic sectional curvature ρ . We show that if the induced almost contact metric structure of a leaf M of the canonical foliation \mathcal{M} of L is Sasakian, then M admits an Einstein-Weyl structure. Finally, we discuss the existence of an Einstein-Weyl structure on a leaf of the canonical foliation of a complete and simply connected generalized Hopf manifold.

2. Preliminaries

Firstly, we give the definition of an Einstein-Weyl space. Let (M, g) be a Riemannian manifold. Let D be a torsion-free affine connection on M . A manifold M is said to have an *Einstein-Weyl structure* if there exist a 1-form μ and a function Λ on M such that

$$(1) \quad Dg = \mu \otimes g \quad \text{and} \quad {}^D\text{Ric}(X, Y) + {}^D\text{Ric}(Y, X) = \Lambda g(X, Y),$$

where ${}^D\text{Ric}$ is the Ricci tensor of D . Since D is not a metric connection, the Ricci tensor is not necessarily symmetric. The Einstein-Weyl equation is conformally invariant. Let ∇ be the Levi-Civita connection of g . We define a vector field E by $g(X, E) = \mu(X)$. Then, since $Dg = \mu \otimes g$, we have

$$(2) \quad D_X Y = \nabla_X Y - \frac{1}{2}\mu(X)Y - \frac{1}{2}\mu(Y)X + \frac{1}{2}g(X, Y)E.$$

Let ${}^D R$ and R be the curvature tensor of D and ∇ respectively. Then, we have

$$(3) \quad \begin{aligned} {}^D R(X, Y)Z &= R(X, Y)Z - \frac{1}{2} \left\{ \left[(\nabla_X \mu)Z + \frac{1}{2}\mu(X)\mu(Z) \right] Y \right. \\ &\quad - \left[(\nabla_Y \mu)Z + \frac{1}{2}\mu(Y)\mu(Z) \right] X + ((\nabla_X \mu)Y)Z - ((\nabla_Y \mu)X)Z \\ &\quad \left. - g(Y, Z) \left(\nabla_X E + \frac{1}{2}\mu(X)E \right) + g(X, Z) \left(\nabla_Y E + \frac{1}{2}\mu(Y)E \right) \right\} \\ &\quad - \frac{1}{4}|\mu|^2(g(Y, Z)X - g(X, Z)Y), \end{aligned}$$

where X, Y and Z are any vector fields on M .

Next, we give the definition of an almost contact metric manifold. A Riemannian manifold (M, g) is said to be an *almost contact metric manifold* if there exist a tensor field ϕ of type $(1, 1)$, a unit vector field V and a 1-form η such that

$$(4) \quad \begin{aligned} \eta(V) &= 1, \quad \phi^2 X = -X + \eta(X)V, \\ g(\phi X, \phi Y) &= g(X, Y) - \eta(X)\eta(Y), \end{aligned}$$

for any vector fields X, Y on M (cf. [1]).

For an almost contact metric structure (ϕ, V, η, g) on M we put $\Phi(X, Y) = g(X, \phi Y)$. An almost contact metric structure is said to be:

Contact metric if $d\eta = \Phi$.

K-contact if $d\eta = \Phi$ and V is a Killing vector field with respect to g .

Sasakian if $d\eta = \Phi$ and $N_\phi + 2d\eta \otimes V = 0$, where $N_\phi(X, Y) = [\phi X, \phi Y] - \phi[\phi X, Y] - \phi[X, \phi Y] + \phi^2[X, Y]$.

If the Ricci tensor Ric of a contact metric manifold M is of the form

$$(5) \quad Ric(X, Y) = \beta g(X, Y) + \gamma \eta(X)\eta(Y),$$

β and γ being constant, then M is called *η -Einstein manifold*.

Next, let L be an almost Hermitian manifold with metric g , complex structure J and the fundamental 2-form Ω . The manifold L is said to be a *locally conformal Kaehler manifold* if every $x \in L$ has an open neighborhood U with a differentiable function $r : U \rightarrow \mathbb{R}$ such that $g'_U = e^{-r}g|_U$ is a Kaehler metric on U . If we can take $U = L$, the manifold is *globally conformal Kaehler*. The locally conformal Kaehler manifold L is characterized by

$$(6) \quad N_J = 0, \quad d\Omega = \omega \wedge \Omega, \quad d\omega = 0,$$

where N_J is the Nijenhuis tensor of J and ω is a globally defined 1-form on L . We call ω the *Lee form*. Since for $\dim L = 2$ we have $d\Omega = 0$, we may suppose $\dim L \geq 4$. Next we define a *Lee vector field* B by

$$(7) \quad g(X, B) = \omega(X).$$

The *Weyl connection* ${}^W\nabla$ is the linear connection defined by

$$(8) \quad {}^W\nabla_X Y := \nabla_X Y - \frac{1}{2}\omega(X)Y - \frac{1}{2}\omega(Y)X + \frac{1}{2}g(X, Y)B,$$

where ∇ is the Levi-Civita connection of g . ${}^W\nabla$ is the Levi-Civita connection of a local Kaehler metric g' . It is shown in [9] that an almost Hermitian manifold L is a locally conformal Kaehler if and only if there is a closed 1-form ω on L such that

$$(9) \quad {}^W\nabla_X J = 0.$$

Let L be a locally conformal Kaehler manifold. Let R^L be the curvature tensor field of the connection ∇ and WR the curvature tensor field of the Weyl connection ${}^W\nabla$. Since $d\omega = 0$, we obtain $(\nabla_X \omega)Y - (\nabla_Y \omega)X = 0$. Thus, we have

$$(10) \quad \begin{aligned} {}^WR(X, Y)Z &= R^L(X, Y)Z - \frac{1}{2} \left\{ \left[(\nabla_X \omega)Z + \frac{1}{2}\omega(X)\omega(Z) \right] Y \right. \\ &\quad \left. - \left[(\nabla_Y \omega)Z + \frac{1}{2}\omega(Y)\omega(Z) \right] X - g(Y, Z) \left(\nabla_X B + \frac{1}{2}\omega(X)B \right) \right. \\ &\quad \left. + g(X, Z) \left(\nabla_Y B + \frac{1}{2}\omega(Y)B \right) \right\} - \frac{1}{4}|\omega|^2(g(Y, Z)X - g(X, Z)Y), \end{aligned}$$

where X, Y and Z are any vector fields on L (cf. [11]).

Let \mathcal{M} be the foliation given by $\omega = 0$ of a locally conformal Kaehler manifold (L, J, g) . \mathcal{M} is called the *canonical foliation*.

A locally conformal Kaehler manifold (L, J, g) is said to be a *generalized Hopf manifold* if the Lee form is parallel, that is $\nabla\omega = 0$ ($\omega \neq 0$).

Let M be a submanifold of a Riemannian manifold L . We denote by the same g the Riemannian metric tensor field induced on M from that of L . Let ∇^M denote covariant differentiation of M . Then the Gauss formula for M is written as

$$(11) \quad \nabla_X Y = \nabla_X^M Y + \sigma(X, Y)$$

for any tangent vector fields X, Y on M where σ denotes the second fundamental form of M in L .

Let R^M be the Riemannian curvature tensor field of M . Then we have the equation of Gauss

$$(12) \quad R^L(W, Z, X, Y) = R^M(W, Z, X, Y) + g(\sigma(X, Z), \sigma(Y, W)) - g(\sigma(Y, Z), \sigma(X, W)).$$

3. Einstein-Weyl structures

Let (M, ϕ, V, η, g) be an almost contact metric manifold of dimension $2n + 1$. Let ∇^M be the Levi-Civita connection of g and Ric^M be the Ricci tensor of ∇^M . Then we obtain the following result.

THEOREM 1. *Let (M, ϕ, V, η, g) be an almost contact metric manifold satisfying $\nabla_X^M V = -\phi(X)$. If $Ric^M(X, Y) = \beta g(X, Y) + \gamma\eta(X)\eta(Y)$, where β and γ are constant, and $\gamma \leq 0$, then M admits an Einstein-Weyl structure.*

PROOF. We define 1-form μ by $\mu = f\eta$, where f is a function on M . Let E be the dual vector field of μ . We define the connection D by

$$(13) \quad D_X Y = \nabla_X^M Y - \frac{1}{2}\mu(X)Y - \frac{1}{2}\mu(Y)X + \frac{1}{2}g(X, Y)E.$$

Then D is a torsion-free connection and $Dg = \mu \otimes g$. Let ${}^D Ric$ be the Ricci tensor of D . From (3), we obtain

$$(14) \quad \begin{aligned} {}^D Ric(X, Y) &= Ric^M(X, Y) + n(\nabla_X^M \mu)Y - \frac{1}{2}(\nabla_Y^M \mu)X \\ &\quad + \frac{1}{4}(2n - 1)\mu(X)\mu(Y) + g(X, Y)\left(\frac{1}{2}\operatorname{div} E - \frac{1}{4}(2n - 1)|\mu|^2\right) \text{ (cf. [7]).} \end{aligned}$$

Let $V, X_1, \phi X_1, \dots, X_n, \phi X_n$ be an orthonormal basis of $T_p M$. Since $\nabla_X^M V = -\phi(X)$, for $j = 1, \dots, n$, we get $\nabla_V^M V = 0$ and $g(\nabla_{X_j}^M X_j, V) = -g(\nabla_{X_j}^M V, X_j) = 0$. Thus we have $(\nabla_{X_j}^M \mu)V = X_j(f)$, $(\nabla_V^M \mu)(X_j) = 0$, $(\nabla_V^M \mu)V = V(f)$, $(\nabla_{X_i}^M \mu)X_j = 0$ for all i, j , $(\nabla_{X_i}^M \mu)(\phi X_j) = 0$ for $i \neq j$ and $(\nabla_{X_j}^M \mu)(\phi X_j) + (\nabla_{\phi X_j}^M \mu)X_j = 0$. From these, we have

$$(15) \quad {}^D Ric(X_j, V) + {}^D Ric(V, X_j) = \frac{1}{2}(2n-1)X_j(f),$$

$$(16) \quad {}^D Ric(\phi X_j, V) + {}^D Ric(V, \phi X_j) = \frac{1}{2}(2n-1)\phi X_j(f),$$

$$(17) \quad 2 \cdot {}^D Ric(V, V) = 2(\beta + \gamma) + \frac{1}{2}(2n-1)(2V(f) + f^2) + \operatorname{div} E - \frac{1}{2}(2n-1)f^2,$$

$$(18) \quad 2 \cdot {}^D Ric(X_j, X_j) = 2\beta + \operatorname{div} E - \frac{1}{2}(2n-1)f^2,$$

and

$$(19) \quad 2 \cdot {}^D Ric(\phi X_j, \phi X_j) = 2\beta + \operatorname{div} E - \frac{1}{2}(2n-1)f^2.$$

We set $f^2 = -(4/(2n-1))\gamma$. Then $V(f) = 0$, $X_j(f) = \phi X_j(f) = 0$ for $j = 1, \dots, n$ and $\operatorname{div} E = V \cdot \mu(V) = V(f) = 0$. Thus, by using equations (15)–(19), we obtain

$$(20) \quad {}^D Ric(X, Y) + {}^D Ric(Y, X) = \Lambda g(X, Y),$$

where X, Y are tangent vectors of M and $\Lambda = 2(\beta + \gamma)$. Therefore M admits an Einstein-Weyl structure. \blacksquare

REMARK 1. K -contact manifold and Sasakian manifold satisfy the condition $\nabla_X^M V = -\phi(X)$ (cf. [1]). Let $\pi : M^{2n+1} \rightarrow N^{2n}$ be a Riemannian submersion with totally geodesic fibers of dimension one over an Einstein manifold N^{2n} such that $Ric^N(\tilde{X}, \tilde{Y}) = c\tilde{g}(\tilde{X}, \tilde{Y})$. Moreover, we assume that $(M^{2n+1}, \phi, V, \eta, g)$ is a standard Sasakian manifold. Let $V, X_1, \phi X_1, \dots, X_n, \phi X_n$ be an orthonormal basis of $T_p M$. Then we get $Ric^M(X_j, X_j) = Ric^M(\phi X_j, \phi X_j) = c - 2$, $Ric^M(V, V) = 2n$, $Ric^M(X_j, V) = Ric^M(\phi X_j, V) = 0$, $Ric^M(X_i, \phi X_j) = 0$ for all i, j and $Ric^M(X_i, X_j) = Ric^M(\phi X_i, \phi X_j) = 0$ for $i \neq j$. (cf. [5]). Therefore M is an η -Einstein manifold such that $Ric^M(X, Y) = (c - 2)g(X, Y) + (2n - c + 2)\eta(X)\eta(Y)$. If the scalar curvature \tilde{s} of N is $\tilde{s} \geq 4n(n + 1)$, then $2n - c + 2 = (1/(2n))(4n^2 + 4n - \tilde{s}) \leq 0$. Thus, as a corollary of Theorem 1, we obtain Theorem 2 in [5].

Next, let (M, ϕ, V, η, g) be a Sasakian manifold of constant ϕ -sectional curvature k and $\dim M = 2n + 1$. The curvature of M is

$$\begin{aligned}
(21) \quad R^M(X, Y, Z, W) &= \frac{k+3}{4} \{g(X, Z)g(Y, W) - g(X, W)g(Y, Z)\} \\
&\quad - \frac{k-1}{4} \{\eta(X)\eta(Z)g(Y, W) - \eta(Y)\eta(Z)g(X, W) \\
&\quad + g(X, Z)\eta(Y)g(V, W) - g(Y, Z)\eta(X)g(V, W) \\
&\quad + g(\phi Y, Z)g(\phi X, W) - g(\phi X, Z)g(\phi Y, W) \\
&\quad + 2g(X, \phi Y)g(\phi Z, W)\} \quad (\text{cf. [12]}).
\end{aligned}$$

From this equation, we get

$$(22) \quad Ric^M(X, Y) = \beta g(X, Y) + \gamma \eta(X)\eta(Y),$$

where $\beta = ((n+1)/2)k + ((3n-1)/2)$ and $\gamma = -((n+1)/2)(k-1)$.

Let S^{2n+1} be the unit sphere in C^{n+1} and τ be real number such that $\tau > 0$. Let (J, h) be the flat Kaehler structure on C^{n+1} and U be the unit normal vector field of S^{2n+1} in C^{n+1} . We define tensor field ϕ and 1-form $\bar{\eta}$ on S^{2n+1} by $J(X) = \phi(X) - \bar{\eta}(X)U$. And we put $V = (1/\tau)JU$, $\eta = \tau\bar{\eta}$ and $g = \tau h' + \tau(\tau-1)\bar{\eta} \otimes \bar{\eta}$, where h' is the induced metric on S^{2n+1} by h . Then (ϕ, V, η, g) is a Sasakian structure with constant ϕ -sectional curvature $k = (4/\tau) - 3$ and we denote S^{2n+1} with this structure by $S^{2n+1}(k)$ (cf. [8]). Thus we have (22) and $\nabla_X^S V = -\phi(X)$, where ∇^S is the Levi-Civita connection of g . From a Theorem of Tanno [8] and Theorem 1, we obtain the following result.

THEOREM 2. *Let M be a complete and simply connected Sasakian manifold with constant ϕ -sectional curvature k . If $k \geq 1$, then M admits an Einstein-Weyl structure.*

REMARK 2. Let (ϕ, V, η, g) be a Sasakian structure of the sphere $(S^{2n+1}(k), g)$. In the Hopf fibration $\pi : (S^{2n+1}(k), g) \rightarrow (P_n(C)(k+3), \tilde{g})$, for constant $a \neq 0$, we define a Riemannian metric g_a by $g_a = \pi^*\tilde{g} + a^2\eta \otimes \eta$ (cf. [2]). We set $\tilde{V} = (1/a)V$ and $\tilde{\eta} = a\eta$. Then $d\tilde{\eta} = a\tilde{\Phi}$, where $\tilde{\Phi}(X, Y) = g_a(X, \phi Y)$. Let A be the integrability tensor of the Riemannian submersion $\pi_a : (S^{2n+1}(k), g_a) \rightarrow (P_n(C)(k+3), \tilde{g})$ with totally geodesic fibers. Let R^S and R^P denote the curvature tensors of $(S^{2n+1}(k), g_a)$ and $(P_n(C)(k+3), \tilde{g})$ respectively. We recall the following curvature identity.

$$\begin{aligned}
(23) \quad R^S(W, Z, X, Y) &= R^P(\tilde{W}, \tilde{Z}, \tilde{X}, \tilde{Y}) - g_a(A_Y Z, A_X W) \\
&\quad + g_a(A_X Z, A_Y W) + 2g_a(A_X Y, A_Z W),
\end{aligned}$$

where \tilde{X} , \tilde{Y} , \tilde{Z} and \tilde{W} are tangent vector fields on $P_n(\mathcal{C})(k+3)$ and X , Y , Z and W are the horizontal lifts of \tilde{X} , \tilde{Y} , \tilde{Z} and \tilde{W} respectively (cf. [6]). For a function f on $S^{2n+1}(k)$, we put $\mu = f\eta$. We define the connection \bar{D} , by

$$\bar{D}_X Y = \nabla_X^a Y - \frac{1}{2}\mu(X)Y - \frac{1}{2}\mu(Y)X + \frac{1}{2}g_a(X, Y)E,$$

where ∇^a is the Levi-Civita connection of g_a and E is the dual vector field of μ . Let $\bar{D}Ric^a$ be the Ricci tensor of \bar{D} . Since $A_X Y = (1/2)\mathcal{V}[X, Y]$ (cf. [6]) and \tilde{V} is a unit vertical vector field, we get $A_X Y = -d\tilde{\eta}(X, Y)\tilde{V} = -ag_a(X, \phi Y)\tilde{V}$, where X , Y are horizontal vector fields. Using $|\mu|^2 = (f^2/a^2)$, for an orthonormal basis $\tilde{V}, X_1, \phi X_1, \dots, X_n, \phi X_n$ of $T_p S^{2n+1}(k)$, we get following.

$$2 \cdot \bar{D}Ric^a(\tilde{V}, \tilde{V}) = 4na^2 + \frac{1}{2}(2n-1)\left(\frac{2}{a}\tilde{V}(f) + \frac{f^2}{a^2}\right) + \operatorname{div} E - \frac{1}{2}(2n-1)\frac{f^2}{a^2},$$

$$2 \cdot \bar{D}Ric^a(X_j, X_j) = (n+1)(k+3) - 4a^2 + \operatorname{div} E - \frac{1}{2}(2n-1)\frac{f^2}{a^2}$$

and

$$\bar{D}Ric^a(X_j, \tilde{V}) + \bar{D}Ric^a(\tilde{V}, X_j) = \frac{1}{2}(2n-1)X_j\left(\frac{f}{a}\right).$$

For ϕX_j , we also have same equations.

Let constant a be $(k+3)/4 \geq a^2$ and set $f^2 = (2(n+1)a^2/(2n-1)) \times (k+3-4a^2)$. Then we have

$$\bar{D}Ric^a(X, Y) + \bar{D}Ric^a(Y, X) = \Lambda g_a(X, Y),$$

where $\Lambda = 4na^2$. Therefore, for $k > -3$, $(S^{2n+1}(k), g_a, \mu)$ admits an Einstein-Weyl structure but not Sasakian for $a \neq 1$.

4. Foliations of locally conformal Kaehler manifolds

As an application of Theorem 1 and Theorem 2, we consider the canonical foliation of a locally conformal Kaehler manifold. Let (L, J, g) be a locally conformal Kaehler manifold of dimension $2n+2$ and ω the Lee form and \mathcal{M} the canonical foliation given by $\omega = 0$. Let M be a leaf of the canonical foliation \mathcal{M} , that is M is an orientable real hypersurface of L . Let B be the Lee vector field. We set $C = (B/|\omega|)$, $V = JC$, $\eta(X) = g(X, V)$ and $JX = \phi X - \eta(X)C$. It is known that every orientable real hypersurface of an almost Hermitian manifold has an almost contact metric structure (ϕ, V, η, g) (cf. [1], [10]).

We show the following theorem.

THEOREM 3. *Let (L, g) be a locally conformal Kaehler manifold and assume that all its local Kaehler metrics $g' = e^{-r}g$ have the same constant nonnegative holomorphic sectional curvature ρ . Let M be a leaf of the canonical foliation \mathcal{M} . If the induced almost contact metric structure of M is Sasakian, then M admits an Einstein-Weyl structure.*

PROOF. It is known that M admits a Sasakian structure if and only if

$$(24) \quad \sigma(X, Y) = - \left(\left(\frac{1}{2} |\omega| - 1 \right) g(X, Y) + \alpha \eta(X) \eta(Y) \right) C,$$

where α is a function on M (cf. [4]). Let $C, V, X_1, JX_1, \dots, X_n, JX_n$ be an orthonormal basis of $T_p L$ such that $V, X_1, \phi X_1, \dots, X_n, \phi X_n$ ($\phi X_i = JX_i$) form an orthonormal basis of $T_p M$. We denote the Ricci tensor of ∇ of L by Ric^L . Using (12), for $X, Y \in T_p M$, the Ricci tensor Ric^M of ∇^M of M is

$$(25) \quad Ric^M(X, Y) = Ric^L(X, Y) - R^L(C, Y, C, X) - \{g(\sigma(V, Y), \sigma(X, V)) \\ - g(\sigma(X, Y), \sigma(V, V)) + \sum_{i=1}^n [g(\sigma(X_i, Y), \sigma(X, X_i)) \\ - g(\sigma(X, Y), \sigma(X_i, X_i)) + g(\sigma(\phi X_i, Y), \sigma(X, \phi X_i)) \\ - g(\sigma(X, Y), \sigma(\phi X_i, \phi X_i))]\}.$$

Since the local Kaehler metrics g' have the same constant nonnegative holomorphic sectional curvature ρ , the curvature tensor of g' is given by

$$(26) \quad R'(X, Y, Z, W) = \frac{\rho}{4} \{g'(X, Z)g'(Y, W) - g'(X, W)g'(Y, Z) \\ + g'(X, JZ)g'(Y, JW) - g'(X, JW)g'(Y, JZ) \\ + 2g'(X, JY)g'(Z, JW)\} \quad (\text{cf. [12]}).$$

We put ${}^W R(X, Y, Z, W) = g({}^W R(Z, W)Y, X)$. Since $R'(X, Y, Z, W) = e^{-r} {}^W R(X, Y, Z, W)$, we get

$$(27) \quad {}^W R(X, Y, Z, W) = \frac{\rho}{4} e^{-r} \{g(X, Z)g(Y, W) - g(X, W)g(Y, Z) \\ + g(X, JZ)g(Y, JW) - g(X, JW)g(Y, JZ) \\ + 2g(X, JY)g(Z, JW)\}.$$

From this equation, we get

$$(28) \quad {}^w Ric(X, Y) = \frac{\rho}{4} e^{-r} (2n + 4) g(X, Y).$$

From (10), for any vector fields X, Y tangent to M , we have

$$(29) \quad R^L(C, Y, C, X) = \frac{\rho}{4} e^{-r} (g(X, Y) + 3\eta(X)\eta(Y)) - \frac{1}{2} g(((\nabla_X \omega) Y) C, C) \\ - \frac{1}{2} g(X, Y) g(\nabla_C B, C)$$

and

$$(30) \quad Ric^L(X, Y) = \frac{\rho}{4} e^{-r} (2n + 4) g(X, Y) - n(\nabla_X \omega) Y - \left(\frac{1}{2} \operatorname{div} B - \frac{1}{2} n|\omega|^2 \right) g(X, Y).$$

From (11) and (24), we have

$$(31) \quad (\nabla_X \omega) Y = -\omega(\nabla_X Y) = \left(\left(\frac{1}{2} |\omega| - 1 \right) g(X, Y) + \alpha \eta(X) \eta(Y) \right) |\omega|$$

and

$$(32) \quad \operatorname{div} B = g(\nabla_C B, C) + g(\nabla_V B, V) + \sum_{i=1}^n g(\nabla_{X_i} B, X_i) + \sum_{i=1}^n g(\nabla_{JX_i} B, JX_i) \\ = g(\nabla_C B, C) + (2n + 1) |\omega| \left(\frac{1}{2} |\omega| - 1 \right) + \alpha |\omega|.$$

By using (29), (30), (31) and (32), we have

$$(33) \quad Ric^L(X, Y) - R^L(C, Y, C, X) \\ = \frac{\rho}{4} e^{-r} \{ (2n + 3) g(X, Y) - 3\eta(X) \eta(Y) \} \\ + \left(-\frac{n}{2} |\omega|^2 + 2n |\omega| - \frac{1}{2} \alpha |\omega| \right) g(X, Y) - \left(n |\omega| - \frac{1}{2} |\omega| \right) \alpha \eta(X) \eta(Y).$$

From (24), we obtain

$$(34) \quad g(\sigma(V, Y), \sigma(X, V)) + \sum_{i=1}^n [g(\sigma(X_i, X), \sigma(X_i, Y)) + g(\sigma(\phi X_i, X), \sigma(\phi X_i, Y))] \\ = \left(\frac{1}{2} |\omega| - 1 \right)^2 g(X, Y) + (\alpha^2 + \alpha |\omega| - 2\alpha) \eta(X) \eta(Y).$$

Thus, since $\sigma(V, V) = -((1/2)|\omega| - 1 + \alpha)C$ and $\sigma(X_i, X_i) = -((1/2)|\omega| - 1)C$, we have

$$\begin{aligned}
 (35) \quad & -\{g(\sigma(V, Y), \sigma(X, V)) - g(\sigma(X, Y), \sigma(V, V)) + \sum_{i=1}^n [g(\sigma(X_i, Y), \sigma(X, X_i)) \\
 & - g(\sigma(X, Y), \sigma(X_i, X_i)) + g(\sigma(\phi X_i, Y), \sigma(X, \phi X_i)) - g(\sigma(X, Y), \sigma(\phi X_i, \phi X_i))]\} \\
 & = -\left(\frac{1}{2}|\omega| - 1\right)^2 g(X, Y) - (\alpha^2 + \alpha|\omega| - 2\alpha)\eta(X)\eta(Y) \\
 & + (2n+1)\left\{\left(\frac{1}{2}|\omega| - 1\right)^2 g(X, Y) + \left(\frac{1}{2}|\omega| - 1\right)\alpha\eta(X)\eta(Y)\right\} \\
 & + \alpha^2\eta(X)\eta(Y) + \left(\frac{1}{2}|\omega| - 1\right)\alpha g(X, Y) \\
 & = \left(\frac{n}{2}|\omega|^2 - 2n|\omega| + \frac{1}{2}\alpha|\omega| + 2n - \alpha\right)g(X, Y) \\
 & + \left(n|\omega| - \frac{1}{2}|\omega| - 2n + 1\right)\alpha\eta(X)\eta(Y).
 \end{aligned}$$

Using (33) and (35), from (25), we have

$$(36) \quad Ric^M(X, Y) = \beta g(X, Y) + \gamma\eta(X)\eta(Y),$$

where $\beta = 2n - \alpha + (\rho/4)e^{-r}(2n+3)$ and $\gamma = \alpha(1-2n) - (3/4)\rho e^{-r}$. It is known that if all local Kaehler metrics $g' = e^{-r}g$ of L have the same constant holomorphic sectional curvature ρ , then $\rho = 0$ or L is a globally conformal Kaehler manifold (cf. [11]). Since $B \in TM^\perp$, for tangent vector field X on M , $X(r) = dr(X) = \omega(X) = 0$. Hence r is constant on M . Since M is a Sasakian manifold, β and γ are constant on M and $\beta + \gamma = 2n$ (cf. [12]). Thus, $\alpha = (\rho/4)e^{-r} = \text{constant}$ on M and $\beta = 2n + (\rho/2)e^{-r}(n+1)$, $\gamma = -(\rho/2)e^{-r}(n+1) \leq 0$. Therefore, from Theorem 1, M admits an Einstein-Weyl structure. ■

EXAMPLE. Let H_λ^{n+1} be a Hopf manifold. H_λ^{n+1} is isometric to $S^1 \times S^{2n+1}$, where S^{2n+1} is the unit sphere in C^{n+1} with constant curvature 1. Let ω be the Lee form of H_λ^{n+1} . The sphere S^{2n+1} is a leaf of the canonical foliation given by the Lee form $\omega = 0$. Since the local Kaehler metric of H_λ^{n+1} is flat and S^{2n+1} admits a Sasakian structure, S^{2n+1} admits an Einstein-Weyl structure.

Next, we shall consider generalized Hopf manifolds.

Let (M, ϕ, V, η, g) be a Sasakian manifold and $L = M \times \mathbb{R}$. We set

$$(37) \quad \begin{aligned} J\left(X, x \frac{\partial}{\partial s}\right) &= \left(\phi X + xV, -\eta(X) \frac{\partial}{\partial s}\right), \\ h\left(\left(X, x \frac{\partial}{\partial s}\right), \left(Y, y \frac{\partial}{\partial s}\right)\right) &= g(X, Y) + xy, \end{aligned}$$

where X, Y are vector fields on M and x, y functions on L . Then (L, J, h) is a generalized Hopf manifold with Lee form $\omega = 2ds$ and Lee vector field $B = 2(\partial/\partial s)$ (cf. [10]). Therefore $S^{2n+1}(k) \times \mathbb{R}$ is a generalized Hopf manifold.

Let (L, J, h) be a generalized Hopf manifold. Then $|\omega|$ is constant. We set

$$(38) \quad c = \frac{|\omega|}{2}, \quad u = \frac{\omega}{|\omega|}, \quad v = -u \circ J, \quad C = \frac{B}{|\omega|}, \quad \tilde{V} = JC.$$

Then we have $dv = c(\Omega + 2u \wedge v)$.

Let \mathcal{M} be the canonical foliation of a generalized Hopf manifold (L, J, h) and M be a leaf of \mathcal{M} . Then M is a totally geodesic submanifold of L . We set

$$(39) \quad V = \frac{1}{c} \tilde{V}|_M, \quad \eta = cv, \quad g = c^2 h|_M, \quad \phi = J + \eta \otimes C|_M.$$

Then (M, ϕ, V, η, g) admits a Sasakian structure (cf. [10]).

From a Theorem of Vaisman [10] and Theorem 2, we have following.

THEOREM 4. *Let L be a complete and simply connected generalized Hopf manifold and every leaf M of the canonical foliation \mathcal{M} be of constant ϕ -sectional curvature k . If $k \geq 1$, then M admits an Einstein-Weyl structure.*

REMARK 3. In a generalized Hopf manifold (L, h) , if all the local Kaehler metrics $g^t = e^{-t}h$ have the same constant holomorphic sectional curvature ρ , then every leaf M of the canonical foliation is of constant ϕ -sectional curvature $k = (1/c^2)\rho e^{-t} + 1$, where $c = (|\omega|/2)$. But converse is not true.

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