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ON A CLASS OF EVEN-DIMENSIONAL MANIFOLDS STRUCTURED BY A $\mathscr{T}$-PARALLEL CONNECTION

By

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Abstract. Geometrical and structural properties are proved for a class of even-dimensional manifolds which are equiped with a $\mathscr{T}$-parallel connection.

1. Introduction

Riemannian manifolds (M, g) structured by a $\mathscr{T}$-parallel connection have been defined in [12]. We recall that if M is such a manifold carrying a globally defined vector field $\mathscr{T}(\mathscr{T}^a)$ and θ_b^a (resp. e_a) are the connection forms (resp. the vectors of an orthonormal basis), the connection forms satisfy

$$\theta_b^a = \langle \mathscr{T}, e_b \wedge e_a \rangle, \tag{1}$$

where $\wedge$ is the wedge product. The equations (1) imply $\nabla_{\mathscr{T}} e_a = 0$ and this agrees with the definition of a $\mathscr{T}$-parallel connection.

In the present paper we assume that M is of even dimension $2m$. In Section 3 we prove that M is a space-form with the following properties:

(i) M carries a locally conformal symplectic form Ω having $\mathscr{T}^\flat$ $(= \alpha)$ as covector of Lee;

(ii) $\mathscr{T}$ is closed torse forming

$$\nabla \mathscr{T} = (c + t)\, dp - \alpha \otimes \mathscr{T},$$

where dp is the soldering form of M, c is a constant, $t = \|\mathscr{T}\|^2/2$, and $d\alpha = 0$;

(iii) $\mathscr{T}$ defines a relative conformal transformation of Ω [14] (see also [7]), i.e.

$$d(\mathscr{L}_{\mathscr{T}} \Omega) = 4(c + f)\alpha \wedge \Omega,$$

where f is the principal scalar field on M;

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(iv) the components $\mathscr{T}^a$ $(a = 1, \dots, 2m)$ of $\mathscr{T}$ are eigenfunctions of the Laplacian Δ and have all as eigenvalue f.

In Section 4 we consider the tangent bundle TM of the manifold M discussed in Section 3. Let $V(v^a)$ be the Liouville vector field [3] on TM and ψ the associated Finslerian 2-form [3]; the following properties are proved

(i) the complete lift Ω^c [18] of Ω defines a conformal symplectic structure on TM and $\mathscr{T}$ defines as for Ω a relative conformal transformation of Ω^c [14] [7];

(ii)

$$d(\mathscr{L}_{\mathscr{T}}\Omega^c) = 2(c+1)\alpha \wedge \Omega^c,$$

and since $\mathscr{L}_V\Omega^c = \Omega^c$, and $\mathscr{L}_V\psi = \psi$, both Ω^c and ψ are homogeneous and of class 1;

(iii) if X is a skew-symmetric Killing vector field [15] having $\mathscr{T}$ as generative, then Ω^c is invariant by X, i.e. $\mathscr{L}_X\Omega^c = 0$, and X defines also an infinitesimal conformal transformation of the canonical symplectic form $II = f\psi$, i.e.

$$\mathscr{L}_X II = -g(X, \mathscr{T})II;$$

(iv) the vertical lift X^V of X defines a relative conformal transformation of the Finslerian form ψ, i.e.

$$d(\mathscr{L}_{X^V}\psi) = (dg(X, \mathscr{T}) + g(X, \mathscr{T})X^\flat) \wedge \psi.$$

2. Preliminaries

Let (M, g) be a Riemannian C^∞-manifold and let ∇ be the covariant differential operator with respect to the metric tensor g. We assume that M is oriented and ∇ is the Levi-Civita connection of g. Let $\Gamma TM = \Xi(M)$ be the set of sections of the tangent bundle, and

$$\flat : TM \xrightarrow{\flat} T^*M \quad \text{and} \quad \sharp : TM \xleftarrow{\sharp} T^*M$$

the classical isomorphisms defined by g (i.e. $^\flat$ is the index lowering operator, and $^\sharp$ is the index raising operator).

Following [11], we denote by

$$A^q(M, TM) = \Gamma \operatorname{Hom}(\Lambda^q TM, TM),$$

the set of vector valued q-forms ($q < \dim M$), and we write for the covariant derivative operator with respect to ∇

$$d^\nabla : A^q(M, TM) \to A^{q+1}(M, TM). \tag{2}$$

It should be noticed that in general $d^{\nabla^2} = d^{\nabla} \circ d^{\nabla} \neq 0$, unlike $d^2 = d \circ d = 0$. If $p \in M$ then the vector valued 1-form $dp \in A^1(M, TM)$ is the canonical vector valued 1-form of M, and is also called the soldering form of M [2]. Since ∇ is symmetric one has that $d^{\nabla}(dp) = 0$. A vector field Z which satisfies

$$d^{\nabla}(\nabla Z) = \nabla^2 Z = \pi \wedge dp \in A^2(M, TM), \quad \pi \in \Lambda^1 M, \tag{3}$$

is defined to be an exterior concurrent vector field [13] (see also [10]). The 1-form π in (3) is called the concurrence form and is defined by

$$\pi = \lambda Z^{\flat}, \quad \lambda \in \Lambda^0 M. \tag{4}$$

Let $\mathcal{O} = \{e_a \mid a = 1, \dots 2m\}$ be a local field of orthonormal frames over M and let $\mathcal{O}^* = \mathrm{covect}\{\omega^a\}$ be its associated coframe. Then E. Cartan's structure equations can be written in indexless manner as

$$\nabla e = \theta \otimes e, \tag{5}$$

$$d\omega = -\theta \wedge \omega, \tag{6}$$

$$d\theta = -\theta \wedge \theta + \Theta. \tag{7}$$

In the above equations θ (resp Θ) are the local connection forms in the tangent bundle TM (resp. the curvature 2-forms on M).

3. Manifolds structured by a $\mathcal{T}$-parallel connection

Let (M, g) be a $2m$-dimensional oriented Riemannian C^{∞}-manifold and

$$\mathcal{T} = \mathcal{T}^a e_a, \quad \mathcal{T}^{\flat} = \alpha = \sum \mathcal{T}^a \omega^a \tag{8}$$

be a globally defined vector field and its dual form respectively. Let θ_b^a $(a, b \in \{1, \dots 2m\})$ be the local connection forms in the tangent bundle TM. Then, by reference to [12], (M, g) is structured by a $\mathcal{T}$-parallel connection if the connection forms θ satisfy

$$\theta_b^a = \langle \mathcal{T}, e_b \wedge e_a \rangle, \tag{9}$$

where $\wedge$ means the wedge product of vector fields. Making use of Cartan's structure equations (5), we find by (8) and (9) that

$$\theta_b^a = \mathcal{T}^b \omega^a - \mathcal{T}^a \omega^b, \tag{10}$$

and in consequence of (10), the equations (5) take the form

$$\nabla e_a = \mathcal{T}^a \, dp - \omega^a \otimes \mathcal{T}. \tag{11}$$

Since one has that $\theta_b^a(\mathscr{T}) = 0$, then following [6] one may say that the connection forms θ_b^a are relations of integral invariance for $\mathscr{T}$.

From (11) it also follows that

$$\nabla_{\mathscr{T}} e_a = 0, \tag{12}$$

which expresses that all the vectors of the $\mathcal{O}$-basis $\mathcal{O} = \{e_a\}$ are $\mathscr{T}$-parallel and this legitimates our definition regarding the structure of M. Further, making use of E. Cartan's structure equations (6) one derives that

$$d\omega^a = \alpha \wedge \omega^a, \tag{13}$$

where we have set $\alpha = \mathscr{T}^\flat$. Hence, by (13) it follows that all the pfaffians ω^a of the covector basis $\mathcal{O}^*$ are exterior recurrent forms [1]. Consequently, the pfaffian α can be seen to be in fact a closed form, i.e.

$$d\alpha = 0. \tag{14}$$

Since

$$\alpha = \mathscr{T}^\flat = \sum \mathscr{T}^a \omega^a, \tag{15}$$

one has by (11) $d\mathscr{T}^a \wedge \omega^a = 0$, and by reference to [9], one may write

$$d\mathscr{T}^a = f\omega^a, \quad f \in \Lambda^0 M, \tag{16}$$

and call f the distinguished scalar on M. By (16) and (14) it can now be seen that α is also an exact form, and that one may set

$$\alpha = -\frac{df}{f}. \tag{17}$$

Further, taking the covariant differential of $\mathscr{T}$, one finds by (11) and (16) that

$$\nabla \mathscr{T} = (f + 2t)\, dp - \alpha \otimes \mathscr{T}, \tag{18}$$

where we have set

$$2t = \|\mathscr{T}\|^2. \tag{19}$$

Hence, according to [17] (see also [16] [15] [9]), equation (18) expresses that $\mathscr{T}$ is a torse forming vector field, which in addition, by (11), has the property to be closed; by (19) one may also write

$$dt = f\alpha. \tag{20}$$

Further, operating on (11) by the exterior covariant operator d^∇, one gets

$$d^\nabla(\nabla e_a) = \nabla^2 e_a = 2(f+t)\omega^a \wedge dp. \tag{21}$$

This reveals that all the constituents of the vector basis $\{e_a\}$ are exterior concurrent vector fields [13] with $2(f+t)$ as exterior concurrent scalar. Under these conditions it suffices to make use of the general formula

$$\nabla^2 Z = Z^a \Theta_a^b \otimes e_b, \tag{22}$$

where $Z \in \Xi(M)$ and Θ_a^b are the curvature 2-forms on M, to derive

$$\Theta_a^b = 2(f+t)\omega^a \wedge \omega^b. \tag{23}$$

It is well known that the equation (23) shows that the manifold M under consideration is a space form of curvature

$$\kappa = -2(f+t)$$

(see also [9]), and we agree to set

$$f + t = c = \text{const.}. \tag{24}$$

In another perspective, we agree to call the 2-form Ω of rank $2m$ given by

$$\Omega = \sum \omega^i \wedge \omega^{i^*}, \quad i = 1, \dots m, \; i^* = i + m, \tag{25}$$

the fundamental almost symplectic form of M. Taking the exterior derivative of Ω, and in view of (13), one finds that

$$d\Omega = 2\alpha \wedge \Omega. \tag{26}$$

This affirms the fact that M is endowed with a locally conformal symplectic structure having α as covector of Lee. Then, as is known [5], calling the mapping $Z \to -i_Z\Omega = {}^\flat Z$ the symplectic isomorphism, one has

$${}^\flat\mathcal{T} = \sum (\mathcal{T}^{i^*}\omega^i - \mathcal{T}^i\omega^{i^*}), \tag{27}$$

and by (16) one finds that

$$d({}^\flat\mathcal{T}) = 2f\Omega. \tag{28}$$

Taking now the Lie derivative of Ω with respect to the Lee vector field $\mathcal{T}$, yields

$$\mathcal{L}_{\mathcal{T}}\Omega = 2c\Omega + 2\alpha \wedge {}^\flat\mathcal{T}, \tag{29}$$

and by exterior differentiation one gets

$$d(\mathscr{L}_{\mathscr{T}}\Omega) = 4(f + c)\alpha \wedge \Omega. \tag{30}$$

Hence, following a known definition [14] (see also [7]), the above equation means that $\mathscr{T}$ defines a relative conformal transformation of Ω.

Recall now that if $\tau \in \Lambda^0 M$ is any scalar field, then the Laplacian of τ is expressed by

$$\Delta\tau = \delta\, df = -\text{div}\, df = -\text{div}\, \nabla\mathscr{T},$$

where $\nabla\tau$ is the gradient of τ. Coming back to the case under discussion, then with the help of (16) one derives that

$$\nabla\mathscr{T}^a = f\mathscr{T}^a. \tag{31}$$

This shows that $\mathscr{T}^a$ is an eigenfunction of Δ corresponding to the eigenvalue f. Hence one may say that the vector field $\mathscr{T}$ forms an eigenspace E^{2m} of eigenvalue f.

THEOREM 3.1. *Let M be a $2m$-dimensional Riemannian manifold structured by a $\mathscr{T}$-parallel connection and let $\mathscr{T}(\mathscr{T}^a)$ be the vector field which defines this connection and $\mathscr{T}^\flat$ the dual form of $\mathscr{T}$. Any such manifold is a space-form and is endowed with a locally conformal symplectic form Ω having $\mathscr{T}^\flat$ as covector of Lee, i.e.*

$$d\Omega = 2\mathscr{T}^\flat \wedge \Omega,$$

and $\mathscr{T}$ defines a relative conformal transformation of Ω, i.e.

$$d(\mathscr{L}_{\mathscr{T}}\Omega) = 4(c + f)\mathscr{T}^\flat \wedge \Omega,$$

where c is a constant and f is the distinguished scalar on M. The vector field $\mathscr{T}$ is closed torse forming and its components $\mathscr{T}^a$ form an eigenspace E^{2m} of eigenvalue f.

4. Geometry of the tangent bundle

Let now TM be the tangent bundle of the manifold M discussed in Section 3. Denote as usual by $V(v^a)$ $(a \in \{1, \dots 2m\})$ the Liouville vector field (or the canonical vector field [3]). Under these conditions, one may consider the set $\mathscr{B}^* = \{\omega^a, dv^a\}$ as an adapted cobasis in TM. Following [3] one denotes by i_v the vertical derivation (i_v is a derivation of degree 0 on ΛTM), i.e.

$$i_v\lambda = 0, \quad i_v\, dv^a = \omega^a, \quad i_v\omega^a = 0. \tag{32}$$

Next, the complete lift of Ω is, as is known from [18], expressed by

$$\Omega^c = \sum (dv^i \wedge \omega^{i^*} + \omega^i \wedge dv^{i^*}). \tag{33}$$

Then, on behalf of (13), the exterior differential of Ω^c is given by

$$d\Omega^c = \alpha \wedge \Omega^c. \tag{34}$$

Hence, the complete lift Ω^c of Ω defines on TM a conformal symplectic structure, as Ω does on M. Moreover, similarly as for Ω, one can derive that

$$d(\mathscr{L}_{\mathscr{T}}\Omega^c) = 2(c+1)\alpha \wedge \Omega^c, \tag{35}$$

which proves that $\mathscr{T}$ defines a relative conformal transformation of Ω^c.

Next, as is known [4], the Liouville vector field V is expressed by

$$V = \sum V^a \frac{\partial}{\partial v^a}, \tag{36}$$

and the basic 1-form

$$\mu = \sum V^a \omega^a \tag{37}$$

is called the Liouville 1-form. By (33) one has that

$$i_V \Omega^c = \sum (V^i \omega^{i^*} - V^{i^*} \omega^i), \tag{38}$$

and by (34) and (38) one gets

$$\mathscr{L}_V \Omega^c = \Omega^c. \tag{39}$$

Equation (39) shows that Ω^c is a homogeneous 2-form of class 1 [4] on TM.

Further, taking the exterior differential of the Liouville form μ, one derives that

$$d\mu = \alpha \wedge \mu + \psi, \tag{40}$$

where we have set

$$\psi = \sum dv^a \wedge \omega^a. \tag{41}$$

Then, since one first calculates that

$$i_V \psi = \mu, \quad \alpha(V) = 0, \tag{42}$$

one finally gets that

$$\mathscr{L}_V \psi = \psi, \tag{43}$$

which shows that, as Ω^c, the form ψ is also a homogeneous 2-form of class 1.

Moreover, by (32) one has that

$$i_v \psi = 0, \tag{44}$$

which together with (43) proves that ψ is a Finslerian form [3].

In another order of ideas, we recall that the vertical lift Z^V [18] of any vector field Z on M with components Z^a is expressed by

$$Z^V = \begin{pmatrix} 0 \\ Z^a \end{pmatrix} = Z^a \frac{\partial}{\partial v^a} \tag{45}$$

Therefore, in the case under consideration, one has

$$\mathscr{T}^V = \sum \mathscr{T}^a \frac{\partial}{\partial v^a}, \quad a \in \{1, \dots 2m\}, \tag{46}$$

and by (41) and (32), one finds that

$$i_v \psi = 0. \tag{47}$$

But, by (40) and (17), one has

$$i_{\mathscr{T}^V} \psi = \alpha, \tag{48}$$

and one derives

$$\mathscr{L}_{\mathscr{T}^V} \psi = 0, \tag{49}$$

which shows that ψ is invariant by $\mathscr{T}^V$.

Next, setting

$$\Pi \doteq f\psi, \tag{50}$$

it follows from (17) and (32) that

$$d\Pi = 0. \tag{51}$$

Therefore, the exact symplectic 2-form Π can be viewed as the canonical symplectic form of the manifold TM. Since, as is known from [18], the Killing property for vector fields is invariant by complete liftings, we will now consider a skew-symmetric Killing vector field X [12] on M having $\mathscr{T}$ as generative. Hence, one must write

$$\nabla X = X \wedge \mathscr{T}, \tag{52}$$

where $\wedge$ denotes the wedge product of vector fields. Since by (11) one has that

$$\nabla X = \sum dX^a \otimes e_a + g(X, \mathcal{T})\, dp - X^\flat \otimes \mathcal{T}, \tag{53}$$

one gets from (52)

$$dX^a + g(X, \mathcal{T})\omega^a = X^a \alpha, \quad (\alpha = \mathcal{T}^\flat). \tag{54}$$

Then, since

$$X^\flat = \sum X^a \omega^a,$$

it follows from (13) that

$$dX^\flat = 2\alpha \wedge X^\flat, \tag{55}$$

which is in agreement with Rosca's lemma [15] concerning skew-symmetric Killing en conformal skew-symmetric Killing vector fields.

Next, since a problem of current interest consists of infinitesimal transformations due to the Lie derivaties, one finds in a first step

$$i_X \Omega^c = \sum (X^i dv^{i^*} - X^{i^*} dv^i). \tag{56}$$

Hence, taking the Lie derivative of the complete 2-form Ω^c, one deduces that

$$\mathcal{L}_X \Omega^c = 0, \tag{57}$$

and this reveals that Ω^c is invariant by X. We also notice that taking the Lie bracket $[\mathcal{T}, X]$ one gets by (53) and (18)

$$[\mathcal{T}, X] = -fX, \tag{58}$$

and this shows that $\mathcal{T}$ defines an infinitesimal conformal transformation of X. Further, by (17), (41), (45) and (51), one calculates that

$$\mathcal{L}_X II = -g(X, \mathcal{T}) II, \tag{59}$$

and this affirms that X defines an infinitesimal conformal transformation of the canonical symplectic form on TM. Finally, let

$$X^V = \sum X^a \frac{\partial}{\partial v^a}$$

be the vertical lift of X. By (41) one has that

$$i_{X^V} \psi = \sum X^a \omega^a, \tag{60}$$

and, taking the Lie derivative with respect to X^V, one derives consecutively that

$$L_{X^V}\psi = g(X, \mathscr{T})\psi + 3\alpha \wedge X^\flat, \tag{61}$$

and

$$d(L_{X^V}\psi) = (dg(X, \mathscr{T}) + g(X, \mathscr{T})X^\flat) \wedge \psi. \tag{62}$$

Hence, (62) shows that the vertical lift X^V of the Killing vector field X defines a relative conformal transformation of the Finslerian form ψ.

THEOREM 4.1. *Let TM be the tangent bundle manifold, having as basis the $2m$-dimensional space-form manifold $M(\Omega, \mathscr{T}, \mathscr{T}^\flat = \alpha)$ discussed in Section 3. The complete lift Ω^c of the conformal symplectic form Ω defines also on TM a conformal symplectic structure and the structure vector field $\mathscr{T}$ defines also a relative conformal transformation of Ω^c, i.e.*

$$d(\mathscr{L}_{\mathscr{T}}\Omega^c) = 2(c+1)\alpha \wedge \Omega^c.$$

In addition, if V (resp. ψ) means the Liouville vector field on TM (resp. the Finslerian form), one has

$$\mathscr{L}_V\Omega^c = \Omega^c, \quad \text{and} \quad \mathscr{L}_V\psi = \psi,$$

which shows that both Ω^c and ψ are homogeneous and of class 1. If X is a skew-symmetric Killing vector field having $\mathscr{T}$ as generative, then Ω^c is invariant by X, i.e.

$$\mathscr{L}_X\Omega^c = 0,$$

and X defines also an infinitesimal conformal transformation of the canonical symplectic form $\Pi = f\psi$ on TM. Finally, the vertical lift X^V of X defines a relative conformal transformation of the Finslerian form ψ.

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