TSUKUBA J. MATH.
Vol. 24 No. 1 (2000), 127–132

THE TYPE NUMBER ON REAL HYPERSURFACES IN A QUATERNION SPACE FORM

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0. Introduction

Let $M_n(c)$ be a $4n$-dimensional quaternion space form with the metric g of constant quaternion sectional curvature $8c$. The standard models of quaternion space forms are the quaternion projective space $P_n(Q), (c > 0)$, the quaternion space $Q, (c = 0)$ and the quaternion hyperbolic space $H_n(Q), (c < 0)$. Let M be a connected real hypersurface in $M_n(c)$ with the induced metric.

In particular in [9], J. S. Pak characterized real hypersurfaces in $P_n(Q)$ in terms of the second fundamental form.

When we give a Riemannian manifold and its submanifold, the rank of determined second fundamental form is called the *type number*.

B. Y. Chen and T. Nagano ([2]) investigated totally geodesic submanifolds in Riemannian symmetric spaces, and as one of their results the following holds

THEOREM A ([2]). *Spheres and hyperbolic spaces are only simply connected irreducible symmetric spaces admitting a totally geodesic hypersurface.*

Then it will be an interesting problem to study the type number t of real hypersurfaces in simply connected irreducible symmetric spaces excepted for spheres and hyperbolic spaces.

As a partial answer, it is known that there exists a point such that $t(p) \geq 2$ in any real hypersurface in complex space form with nonzero constant holomorphic sectional curvature and complex dimension ≥ 3 (cf. [8], [10]). Naturally we can consider the following question.

Does $M_n(c)$ satisfy the similar fact?

We answer this question affirmatively, i.e., we shall prove the following

Received January 12, 1999
Revised June 5, 2000

MAIN THEOREM. *Let M be a connected real hypersurfaces in $M_n(c)$ $(c \neq 0, n \geq 2)$. Then there exists a point p in M such that $t(p) \geq 2$.*

1. Preliminaries

A quaternion Kähler manifold is a Riemannian manifold $(\overline{M}, g)$ on which there exists a 3-dimensional vector bundle $\overline{V}$ of tensors of type $(1,1)$ satisfying the following properties:

(1) In any open set W in M, there is a local base $\{J_i(i=1,2,3)\}$ of $\overline{V}$ such that

$$J_i^2 = -I, \tag{1.1}$$

$$J_i J_{i+1} = J_{i+2} = -J_{i+1} J_i \quad (i \bmod 3), \tag{1.2}$$

where I denotes the identity endmorophism.

Such a local base $\{J_i(i=1,2,3)\}$ is called a *canonical local base* of the bundle $\overline{V}$ in W.

(2) There is a Riemannian metric g on $\overline{M}$ such that

$$g(J_i X, Y) + g(X, J_i Y) = 0, \tag{1.3}$$

for any $X, Y \in \mathfrak{X}(W)$, where $\mathfrak{X}(W)$ is the set of all vector fields on W.

(3) The Levi-Civita connection D on $\overline{M}$ satisfies following conditions: If $\{J_i(i=1,2,3)\}$ is a canonical local base of $\overline{V}$ in W, then there exists three local 1-forms p_i $(i=1,2,3)$ on $\overline{M}$ such that

$$D_X J_i = p_{i+2}(X) J_{i+1} - p_{i+1}(X) J_{i+2} \quad (i \bmod 3), \tag{1.4}$$

for all $X \in \mathfrak{X}(\overline{M})$.

Let $Q(X)$ be the 4-plane spanned by vectors $X, J_1 X$, $J_2 X$ and $J_3 X$, for any $X \in T_x\overline{M}$, $x \in \overline{M}$. If the sectional curvature of any section for $Q(X)$ depends only on X, we call it Q-sectional curvature.

A quaternion space form of Q-sectional curvature $8c$ is connected quaternion Kahler manifold with constant Q-sectional curvature $8c$, which denotes by $M_n(c)$.

Let M be a real hypersurface in $M_n(c)$ $(n \geq 2, c \neq 0)$. In a neighborhood of each point, we choose a unit normal vector field N in $M_n(c)$. The Levi-Civita connection D in $M_n(c)$ and ∇ in M are related by the following formulas for any $X, Y \in \mathfrak{X}(M)$:

$$D_X Y = \nabla_X Y + \langle AX, Y \rangle N, \tag{1.5}$$

$$D_X N = -AX, \tag{1.6}$$

where $\langle\,,\rangle$ denotes the Riemannian metric on M induced from the metric g on $M_n(c)$ and A is the shape operator of M.

It is known that M has an almost contact metric structure induced from the quaternion structure J_i on $M_n(c)$, i.e., we define a tensor ϕ_i of type $(1,1)$, a vector field ξ_i and a 1-form η_i on M by the following,

$$\langle \phi_i X, Y\rangle = g(J_i X, Y), \quad \langle \xi_i, X\rangle = \eta_i(X) = g(J_i X, N). \tag{1.7}$$

Then from (1.1) we have

$$\langle \phi_i X, Y\rangle + \langle X, \phi_i Y\rangle = 0, \quad \langle \phi_i X, \phi_i Y\rangle = \langle X, Y\rangle - \eta_i(X)\eta_i(Y), \tag{1.8}$$

$$\phi_i \xi_{i+1} = \xi_{i+2} = -\phi_{i+1}\xi_i \quad (i \bmod 3). \tag{1.9}$$

From (1.3), we obtain

$$\phi_i^2 = -I + \eta_i \otimes \xi_i, \quad \eta_i(\xi_i) = 1, \quad \phi_i \xi_i = 0, \tag{1.10}$$

$$\eta_i(\xi_{i+1}) = \eta_i(\xi_{i+2}) = 0 \quad (i \bmod 3), \tag{1.11}$$

$$\phi_i = \phi_{i+1}\phi_{i+2} - \eta_{i+2} \otimes \xi_{i+1} = -\phi_{i+2}\phi_{i+1} + \eta_{i+1} \otimes \xi_{i+2} \quad (i \bmod 3). \tag{1.12}$$

Furthermore from (1.2) and (1.7), we get

$$\begin{aligned}(\nabla_X \phi_i)Y = {} & p_{i+1}(X)\phi_{i+2}Y - p_{i+2}(X)\phi_{i+1}Y \\ & + \eta_i(Y)AX - \langle AX, Y\rangle \xi_i \quad (i \bmod 3).\end{aligned} \tag{1.13}$$

In terms of (1.4) we have the following Codazzi equation

$$(\nabla_X A)Y - (\nabla_Y A)X = c\sum_{i=1}^{3}(\eta_i(X)\phi_i Y - \eta_i(Y)\phi_i X - 2\langle \phi_i X, Y\rangle \xi_i).$$

2. Formulas

We assume that the rank of A is not larger than m on an open set W, then there exists an open set W_0 such that t takes the constant m. Then the Codazzi equation gives

$$\begin{aligned}-A(\nabla_X Y - \nabla_Y X) &= (\nabla_X A)Y - (\nabla_Y A)X \\ &= c\sum_{i=1}^{3}(\eta_i(X)\phi_i Y - \eta_i(Y)\phi_i X - 2\langle \phi_i X, Y)\xi_i\rangle),\end{aligned} \tag{2.1}$$

for any vector fields $X, Y \in \ker A|_{W_0}$.

Taking the inner product of (2.1) with $Z \in \ker A|_{W_0}$, from (1.7) and $c \neq 0$, we have

$$0 = \sum_{i=1}^{3}(\eta_i(X)\langle\phi_i Y, Z\rangle + \eta_i(Y)\langle\phi_i Z, X\rangle - 2\eta_i(Z)\langle\phi_i X, Y\rangle). \tag{2.2}$$

Putting $Z = X$ in (2.2), we obtain

$$\sum_{i=1}^{3}\eta_i(X)\langle\phi_i Y, X\rangle = 0. \tag{2.3}$$

3. Proof of the Main theorem

Since Theorem A, we get $m \geq 1$. Suppose that $m = 1$. Let λ be the nonzero principal curvature with principal subspace T_λ. Choose a local orthonormal frame field $U, e_1, \ldots, e_{4n-2}$ on M such that $e_1, \ldots, e_{4n-2}$ is in $\ker A|_{W_0}$ and $U \in T_\lambda$. We use the following convention on the range of indices otherwise stated: $r, s, \ldots = 1, \ldots, 4n-2$.

Putting $Z = e_r$ in (2.2), we get

$$\sum_{i=1}^{3}(\eta_i(X)\langle\phi_i Y, e_r\rangle - \eta_i(Y)\langle\phi_i X, e_r\rangle - 2\langle\phi_i X, Y\rangle\eta_i(e_r)) = 0. \tag{3.1}$$

LEMMA. *There exists a number i such that $\eta_i(U) \neq 0$.*

PROOF. We assume that

$$\eta_i(U) = 0, \tag{3.2}$$

for any number i. Then multiplying (3.1) by $\langle\phi_i U, e_r\rangle$ and summing up for r, since (1.8)$\sim$(1.12) and (3.2) we have

$$\begin{aligned} &-\eta_{i+1}(X)\langle\phi_{i+2}Y, U\rangle + \eta_{i+1}(Y)\langle\phi_{i+2}X, U\rangle \\ &\quad + \eta_{i+2}(X)\langle\phi_{i+1}Y, U\rangle - \eta_{i+2}(Y)\langle\phi_{i+1}X, U\rangle = 0 \quad (i \bmod 3). \end{aligned}$$

Putting $X = e_r$ in above equation and summing up for r, from (1.9)$\sim$(1.11) and (3.2) we obtain

$$\langle\phi_i U, Y\rangle = 0,$$

together with equation $\langle\phi_i U, U\rangle = 0$, we get

$$\phi_i U = 0. \tag{3.3}$$

Putting $X = U$ and $Y = \xi_i$ in (1.13) and taking the inner product with U, then using (1.10), (3.2) and (3.3) we get $\lambda = 0$, which is a contradiction. □

On the other hand, (2.3) implies

$$\sum_{i=1}^{3} \eta_i(X)\langle \phi_i e_r, X\rangle = 0. \tag{3.4}$$

Multiplying (3.1) by $\langle \phi_i U, e_r\rangle$ and summing up for r, from (1.9), (1.10), (1.12) and equation $\sum_r \langle \phi_i U, e_r\rangle e_r = \phi_i U$, we get

$$\eta_i(U)\sum_{j=1}^{3}\eta_j^2(X) + \eta_{i+1}(X)\langle U, \phi_{i+2}X\rangle - \eta_{i+2}(X)\langle U, \phi_{i+1}X\rangle = 0 \quad (i \bmod 3).$$

Putting $X = e_r$ in above equation and summing up for r, by (1.9) we have

$$\eta_i(U)\left(\sum_{j=1}^{3}\eta_j^2\left(\sum e_r\right) - 2\right) = 0.$$

According to Lemma, above equation implies

$$\sum_{j=1}^{3}\eta_j^2\left(\sum e_r\right) = 2. \tag{3.5}$$

Multiplying (3.4) by $\eta_i(e_r)$ and summing up for r, then using (1.9), (1.10) and Lemma we have

$$\sum_{j=1}^{3}\eta_j(X)\langle U, \phi_j X\rangle = 0. \tag{3.6}$$

Again multiplying (3.4) by $\langle \phi_i X, e_r\rangle$ and summing up for r and since (1.8), (1.12) and (3.6) we obtain

$$\eta_i(X)\left(\|X\|^2 - \sum_{j=1}^{3}\eta_j^2(X)\right) = 0. \tag{3.7}$$

Suppose that $\eta_i(X) = 0$ for any number i. Then we observe $\eta_i(\xi_i) = \eta_i(U) = 1$. This implies $\xi_i = U$ for any number i, which is a contradiction. Thus by (3.7) we get

$$\sum_{j=1}^{3}\eta_j^2(X) = \|X\|^2.$$

Putting $X = e_r$ in above equation and summing up for r, we have

$$\sum_{j=1}^{3} \eta_j^2\left(\sum e_r\right) = 4n - 2,$$

which contradicts (3.5).

It completes the proof of Main Theorem.

REMARK (ADDED IN PROOF). J. E. D'Atri [3], J. Berndt [1] and A. Martinez [6] gave some examples of real hypersurfaces in $M_n(c), c \neq 0$. In case $M_n(c)$ is $H_n(Q)$, the type number of these examples is maximum. In case $M_n(c)$ is $P_2(Q)$, there is an example of $t \equiv 4$ in the above. However, we don't know an example of real hypersurface in $M_n(c)$, $c \neq 0$ such that $t \equiv 2$.

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