TSUKUBA J. MATH.
Vol. 22 No. 2 (1998), 393–406

PROPER n-SHAPE AND THE FREUDENTHAL COMPACTIFICATION

By

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Abstract. The notions of n-shape for compact pairs and proper n-shape for locally compact spaces were introduced in [1] and [2], respectively. In this paper, strengthening n-shape of pairs, we define the notion of relative n-shape of compact pairs. By constructing a functor from the proper n-shape category to the relative n-shape category, we prove that for locally compact spaces X and Y with $\dim \leq n+1$, if $n\text{-Sh}_p(X) = n\text{-Sh}_p(Y)$ then $n\text{-Sh}(FX, EX) = n\text{-Sh}(FY, EY)$ rel. (EX, EY) and $n\text{-Sh}(CX, \{\infty\}) = n\text{-Sh}(CY, \{\infty\})$ rel. $(\{\infty\}, \{\infty\})$, where FX is the Freudenthal compactification of X with EX the ends, and $CX = X \cup \{\infty\}$ is the one-point compactification of X. As corollaries, (1) if X is connected SUV^n and $\dim X \leq n+1$, then $FX \in UV^n$, (2) if X, $Y \subset \mu_\infty^{n+1} = \mu^{n+1} \setminus \{*\}$ are Z-sets and $n\text{-Sh}_p(X) = n\text{-Sh}_p(Y)$, then $\mu_\infty^{n+1} \setminus X$ is homeomorphic $(\approx)$ to $\mu_\infty^{n+1} \setminus Y$.

1. Introduction

In this paper, spaces are separable metrizable and maps are continuous. We denote $\mathscr{L}\mathscr{K}$ the class of locally compact spaces, $\omega = \{1, 2, \ldots\}$ positive integers and $n \in \omega$. For a class $\mathscr{M}$ of spaces, $\mathscr{M}(n)$ denotes the subclass of $\mathscr{M}$ consisting of spaces with $\dim \leq n$. Let Σ be the subclass of $\mathscr{L}\mathscr{K}$ whose members have the metrizable Freudenthal compactifications.

In [5], Ball and Sher studied the relation of proper maps and the Freudenthal compactifications, defined the notion of proper shape and proved that for X, $Y \in \Sigma$, if $\text{Sh}_p(X) = \text{Sh}_p(Y)$ then $\text{Sh}(FX, EX) = \text{Sh}(FY, EY)$ rel. (EX, EY) [5,

Key words and phrases: relative n-shape, proper n-shape, the Freudenthal compactification, UV^n, SUV^n.
1991 Mathematics Subject Classification: 55P55, 54C56, 54D35, 57N60.
Received January 20, 1997
Revised August 4, 1997

Corollary 4.8]. On the other hand, the concept of n-shape for compact spaces was introduced by Chigogidze [8]. The notion of proper n-shape for locally compact spaces was introduced in [2] and proved that a locally compact connected space X has property SUV^n if and only if $n\text{-Sh}_p(X) = n\text{-Sh}_p(T)$ for some tree T, where SUV^n is a noncompact variant of UV^n (cf. [16]). In [2], the proper n-shape of locally compact spaces is defined by using embeddings of them into locally compact AR-spaces, so we can define the proper n-shape category $n\text{-SH}_p\ \mathscr{L}\mathscr{K}$. By $n\text{-SH}_p\ \mathscr{L}\mathscr{K}(n+1)(n\text{-SH}_p\ \Sigma(n+1))$, we denote the full-subcategory of $n\text{-SH}_p\ \mathscr{L}\mathscr{K}$ whose objects are in $\mathscr{L}\mathscr{K}(n+1)(\Sigma(n+1))$. The proper n-shape of locally compact spaces with $\dim \leq n+1$ is also defined by using embeddings of them into locally compact $(n+1)$-dimensional $LC^n \cap C^n$-spaces. We denote such a proper n-shape category by $n\text{-SH}'_p\ \mathscr{L}\mathscr{K}(n+1)$. It is not clear that $n\text{-SH}_p\ \mathscr{L}\mathscr{K}(n+1)$ and $n\text{-SH}'_p\ \mathscr{L}\mathscr{K}(n+1)$ are categorical isomorphic. However we can prove the following:

THEOREM 1. *There is a categorical embedding* $\Phi : n\text{-SH}_p\ \mathscr{L}\mathscr{K}(n+1) \to n\text{-SH}'_p\ \mathscr{L}\mathscr{K}(n+1)$ *such that* $\Phi(X) = X$ *for each* $X \in \mathscr{L}\mathscr{K}(n+1)$.

Let $\mathscr{K}^2(n+1)$ be the class of compact pairs with $\dim \leq n+1$ and $n\text{-SH}'_p\ \Sigma(n+1)$ be the full-subcategory of $n\text{-SH}'_p\ \mathscr{L}\mathscr{K}(n+1)$ whose objects are in $\Sigma(n+1)$. We also define the relative n-shape category $n\text{-SH}_{rel}\ \mathscr{K}^2(n+1)$, strengthening n-shape category of pairs [1], and prove the following.

THEOREM 2. *There is a functor* $\Psi : n\text{-SH}'_p\ \Sigma(n+1) \to n\text{-SH}_{rel}\ \mathscr{K}^2(n+1)$ *such that* $\Psi(X) = (FX, EX)$ *for each* $X \in \Sigma(n+1)$.

So we conclude the following corollaries.

COROLLARY 3. *There is a functor* $\Theta : n\text{-SH}_p\ \Sigma(n+1) \to n\text{-SH}_{rel}\ \mathscr{K}^2(n+1)$ *such that* $\Theta(X) = (FX, EX)$ *for each* $X \in \Sigma(n+1)$.

COROLLARY 4. *For* X, $Y \in \Sigma(n+1)$, *if* $n\text{-Sh}_p(X) = n\text{-Sh}_p(Y)$ *then* $n\text{-Sh}(FX, EX) = n\text{-Sh}(FY, EY)$ *rel.* (EX, EY).

COROLLARY 5. *If* X *is connected,* SUV^n *and* $\dim X \leq n+1$, *then* $FX \in UV^n$.

For each $X \in \mathscr{L}\mathscr{K}$, let $CX = X \cup \{\infty\}$ be the one-point compactification of X. Considering $(CX, \{\infty\})$ instead of (FX, EX), we have the following similarly to Theorem 2.

THEOREM 6. *There is a functor* $\Psi' : n\text{-SH}'_p\ \mathscr{L}\mathscr{K}(n+1) \to n\text{-SH}_{rel}\ \mathscr{K}^2(n+1)$ *such that* $\Psi(X) = (CX, \{\infty\})$ *for each* $X \in \mathscr{L}\mathscr{K}(n+1)$.

As a corollary, we have the following.

COROLLARY 7. *Let* $\mu_\infty^{n+1} = \mu^{n+1} \backslash \{*\}$, *where* $* \in \mu^{n+1}$, X, $Y \subset \mu_\infty^{n+1}$ *be Z-sets. If* $n\text{-Sh}_p(X) = n\text{-Sh}_p(Y)$ *then* $\mu_\infty^{n+1} \backslash X \approx \mu_\infty^{n+1} \backslash Y$.

2. The proper n-shape category

Let $Q = I^\omega$ be a Hilbert cube and μ^{n+1} an $(n+1)$-dimensional Menger compactum.

A map $f : X \to Y$ between spaces is called *n-invertible* if for any map $\alpha : Z \to Y$ from any space Z with $\dim Z \le n$ to Y, there exists a map $\tilde{\alpha} : Z \to X$ such that $f\tilde{\alpha} = \alpha$. It is easy to observe that if f and α are proper, then $\tilde{\alpha}$ is also proper.

THEOREM 2.1 ([10]). *There is an* $(n+1)$*-invertible* UV^n*-surjection* $f : \mu^{n+1} \to Q$ *such that the fibers are Z-sets and homeomorphic to* μ^{n+1}. □

Let μ_0^{n+1} be a Z-set in μ^{n+1} and homeomorphic to μ^{n+1} and $M = \mu^{n+1} \backslash \mu_0^{n+1}$. By the Z-set unknotting theorem [6], we may assume $f^{-1}(*) = \mu_0^{n+1}$ for some $* \in Q$. The following follows from Theorem 2.1.

PROPOSITION 2.2. *There is an* $(n+1)$*-invertible proper* UV^n*-surjection* $f : M \to Q \backslash \{*\}$ *such that the fibers are Z-sets and homeomorphic to* μ^{n+1}. □

Two proper maps f, $g : X \to Y$ are *properly n-homotopic* (written by $f \simeq_p^n g$) if, for any proper map $\alpha : Z \to X$ from any $Z \in \mathscr{L}\mathscr{K}(n)$ into X, the compositions $f\alpha$ and $g\alpha$ are properly homotopic ($f\alpha \simeq_p g\alpha$) in the usual sense. To see that f and g are properly n-homotopic, it suffices to verify the condition in case that α is an n-invertible proper surjection.

If there exist proper maps $f : X \to Y$ and $g : Y \to X$ such that $fg \simeq_p^n \mathrm{id}_Y$ and $gf \simeq_p^n \mathrm{id}_X$, then X and Y are said to be *properly n-homotopically equivalent* (written by $X \simeq_p^n Y$). If only the first relation is valid, then it is said that X *properly n-homotopically dominates* Y, or Y is *properly n-homotopically dominated by* X (written by $X \ge_p^n Y$, or $Y \le_p^n X$).

Suppose that X and Y are closed sets in $(n+1)$-dimensional locally compact $LC^n \cap C^n$-spaces M and N, respectively. Let $\Lambda = (\Lambda, \le)$ and $\Delta = (\Delta, \le)$ be directed sets. A net $f=\{f_\lambda\,|\,\lambda\in\Lambda\}$ of maps $f_\lambda : M\to N$ is called *a proper n-fundamental net* from X to Y in M and N if, for every closed neighborhood V of Y in N, there exist a closed neighborhood U of X in M and an index $\lambda_0\in\Lambda$ such that

$$f_\lambda|_U \simeq_p^n f_{\lambda_0}|_U \text{ in } V \text{ for all } \lambda\geq\lambda_0.$$

One should remark that each f_λ need not be proper but $f_\lambda|_U$ is proper for some closed neighborhood U of X in M (cf. [5, Lemma 3.2]). We denote that $f : X\to Y$ in (M,N).

Let $f=\{f_\lambda\,|\,\lambda\in\Lambda\}$, $g=\{g_\delta\,|\,\delta\in\Delta\} : X\to Y$ in (M,N) be proper n-fundamental nets. We say that f and g are *properly n-homotopic* (written by $f\simeq_p^n g$) if for each closed neighborhood V of Y there exist a closed neighborhood U of X and $\lambda_0\in\Lambda$, $\delta_0\in\Delta$ such that

$$f_\lambda|_U \simeq_p^n g_\delta|_U \text{ in } V \text{ for all } \lambda\geq\lambda_0 \text{ and } \delta\geq\delta_0.$$

The proper n-homotopy class of f is denoted by $[f]_p^n$.

By the same argument of [2], we can define the notion of proper n-shape for $(n+1)$-dimensional locally compact spaces by using embeddings of them into $(n+1)$-dimensional locally compact $LC^n\cap C^n$-spaces. *The proper n-shape category with* $\dim\leq n+1$, $n\text{-SH}_p'\ \mathscr{LK}(n+1)$, is the category whose objects are in $\mathscr{LK}(n+1)$ and whose morphisms are the proper n-homotopy classes of proper n-fundamental nets. If X, $Y\in\Sigma(n+1)$ are isomorphic in $n\text{-SH}_p'\ \mathscr{LK}(n+1)$, then we denote $n\text{-Sh}_p'(X)=n\text{-Sh}_p'(Y)$.

Proposition 2.3. *Let X, $Y\in\mathscr{LK}$, X be an LC^n-space, $f : X\to Y$ be a proper UV^n-surjection and $\mathscr{U}$ be an open covering of Y. Suppose that two proper maps $\phi : W_0\to X$ and $\psi : W\to Y$ such that $f\phi=\psi|_{W_0}$, where $W\in\mathscr{LK}(n+1)$ and W_0 is a closed subset of W. Then there exists a proper map $\gamma : W\to X$ such that $\gamma|_{W_0}=\phi$ and $f\gamma$ is $\mathscr{U}$-close to ψ.*

Proof. By [11, Theorem 16.11], Y is LC^n. Let $\mathscr{U}_1$ be a double star-refinement of $\mathscr{U}$. By [8, Proposition 2.1], there is an open refinement $\mathscr{U}'$ of $\mathscr{U}_1$ satisfying the following; for any two $\mathscr{U}'$-close proper maps g, $h : A\to Y$ from a closed subset A of a space B with $\dim\leq n+1$ such that g has a proper extension $G : B\to Y$ it follows that h also has a proper extension $H : B\to Y$ which is $\mathscr{U}_1$-close to G (cf. [7, Theorem 4.2(2)]). By [13, p. 156], there exists an open refinement $\mathscr{V}$ of $\mathscr{U}_1$ such that, for any simplicial polytope K with $\dim\leq n+1$, every partial realization of K in Y relative to $\mathscr{V}$ extends to a full realization of K in Y relative to $\mathscr{U}_1$. Let $\mathscr{W}$ be a canonical cover (cf. [13, p. 51]) of $W\setminus W_0$ with order $\leq n+1$ such that $\psi(\mathscr{W})$ is a refinement of $\mathscr{V}$. By the nerve replacement trick [13, p. 53] and the definition of $\mathscr{V}$, we have a proper map $\psi' : W^* \to Y$ such that $\psi' p$ is $\mathscr{U}_1$-close to ψ and $\psi'|_{W_0} = \psi|_{W_0}$, where $W^* = N(\mathscr{W}) \cup W_0$ and $p : W \to W^*$ is a canonical map with $p|_{W_0} = \mathrm{id}_{W_0}$. Since $\psi'|_{W_0} = f\phi$ and X is LC^n, ϕ extends to a proper map $\tilde{\phi} : W' \to X$, where W' is a closed neighborhood of W_0 in W^* and $W'\setminus W_0$ is a subpolyhedron of $N(\mathscr{W})$, such that $f\tilde{\phi}$ is $\mathscr{U}'$-close to $\psi'|_{W'}$. Then $f\tilde{\phi}$ has a proper extension $\tilde{\psi}' : W^* \to Y$ which is $\mathscr{U}_1$-close to ψ'. Note that $f\tilde{\phi}|_{W'\setminus W_0} = \tilde{\psi}'|_{W'\setminus W_0}$. By the lifting property [14, Lemma A] (cf. [6, Proposition 2.1.3]), we have a proper map $\gamma' : N(\mathscr{W}) \to X$ such that $\gamma'|_{W'\setminus W_0} = \tilde{\phi}|_{W'\setminus W_0}$ and $f\gamma'$ is $\mathscr{U}_1$-close to $\psi'|_{N(\mathscr{W})}$. Then $\gamma = (\gamma' \cup \phi)p$ is the desired proper map. □

By Proposition 2.2, we may assume that each $X \in \mathscr{L}\mathscr{K}(n+1)$ is embedded as a closed set into an AR-space $M_X \in \mathscr{L}\mathscr{K}$ and an $LC^n \cap C^n$-space $M'_X \in \mathscr{L}\mathscr{K}(n+1)$, and there is an $(n+1)$-invertible proper UV^n-surjection $\alpha_X : M'_X \to M_X$ such that $\alpha_X|_X = \mathrm{id}_X$.

LEMMA 2.4. *Let X, $Y \in \mathscr{L}\mathscr{K}(n+1)$. Then any proper n-fundamental net $f = \{f_\lambda \mid \lambda \in \Lambda\} : X \to Y$ in (M_X, M_Y) in the sense of* [2] *induces a proper n-fundamental net $f' = \{f'_\lambda \mid \lambda \in \Lambda\} : X \to Y$ in (M'_X, M'_Y) such that $f_\lambda \alpha_X = \alpha_Y f'_\lambda$ for each $\lambda \in \Lambda$.*

PROOF. Since α_Y is $(n+1)$-invertible, for each map $f_\lambda : M_X \to M_Y$ there is a map $f'_\lambda : M'_X \to M'_Y$ such that $f_\lambda \alpha_X = \alpha_Y f'_\lambda$. We show that $f' = \{f'_\lambda \mid \lambda \in \Lambda\} : X \to Y$ in (M'_X, M'_Y) is a proper n-fundamental net. Since α_Y is proper, for each closed neighborhood V' of Y in M'_Y there is a closed neighborhood V_1 of Y in M_Y such that $\alpha_Y^{-1}(\mathrm{int}\, V_1) \subset V'$. Note that $\alpha_Y^{-1}(\mathrm{int}\, V_1)$ is LC^n. Let $V \subset \mathrm{int}\, V_1$ be a closed neighborhood of Y in M_Y. Then there are a closed neighborhood U of X in M_X and an index $\lambda_0 \in \Lambda$ such that

$$f_\lambda|_U \simeq_p^n f_{\lambda_0}|_U \text{ in } V \text{ for all } \lambda \geq \lambda_0.$$

Let $U' = \alpha_X^{-1}(U)$ and fix $\lambda \geq \lambda_0$. Since $\alpha_Y f'_\lambda|_{U'} = f_\lambda \alpha_X|_{U'}$, we have $f'_\lambda(U') \subset \alpha_Y^{-1}(V) \subset \alpha_Y^{-1}(\mathrm{int}\, V_1) \subset V'$. Let $\beta' : Z \to U'$ be a proper map from $Z \in \mathscr{L}\mathscr{K}(n)$ to U'. Since $f_\lambda|_U \simeq_p^n f_{\lambda_0}|_U$ in V,

$$f_\lambda|_U \alpha_X \beta' \simeq_p f_{\lambda_0}|_U \alpha_X \beta' \text{ in } V,$$

i.e., there is a proper homotopy $H : Z \times I \to \mathrm{int}\, V_1$ such that $H_0 = f_{\lambda_0}|_U \alpha_X \beta'$ and $H_1 = f_\lambda|_U \alpha_X \beta'$. Let $h : Z \times \{0,1\} \to \alpha_Y^{-1}(\mathrm{int}\, V_1)$ be the map defined by $h|_{Z\times\{0\}} =$

$f'_{\lambda_0}|_{U'}\beta'$ and $h|_{Z\times\{1\}} = f'_\lambda|_{U'}\beta'$. Since $H|_{Z\times\{0,1\}} = \alpha_Y h$, by Proposition 2.3, there is a proper map $\tilde{h} : Z \times I \to \alpha_Y^{-1}(\mathrm{int}\, V)$ which is an extension of h. We conclude that $f'_\lambda|_{U'} \simeq_p^n f'_{\lambda_0}|_{U'}$ in $\alpha_Y^{-1}(\mathrm{int}\, V_1) \subset V'$. □

Lemma 2.5. *Let $f = \{f_\lambda \,|\, \lambda \in \Lambda\}$, $g = \{g_\delta \,|\, \delta \in \Delta\} : X \to Y$ in (M_X, M_Y) be proper n-fundamental nets and suppose f, g induce proper n-fundamental nets $f' = \{f'_\lambda \,|\, \lambda \in \Lambda\}$, $g' = \{g'_\delta \,|\, \delta \in \Delta\} : X \to Y$ in (M'_X, M'_Y) such that $f_\lambda \alpha_X = \alpha_Y f'_\lambda$ and $g_\delta \alpha_X = \alpha_Y g'_\delta$ for each $\lambda \in \Lambda$ and $\delta \in \Delta$. Then $f \simeq_p^n g$ if and only if $f' \simeq_p^n g'$.*

Proof. Suppose that $f \simeq_p^n g$. Since α_Y is proper, for each closed neighborhood V' of Y in M'_Y there is a closed neighborhood V of Y in M_Y such that $\alpha_Y^{-1}(V) \subset V'$. By the argument of Lemma 2.4, we may assume that $\alpha_Y^{-1}(V)$ is LC^n. Since $f \simeq_p^n g$, there are a closed neighborhood U of X in M_X and indices $\lambda_0 \in \Lambda$ and $\delta_0 \in \Delta$ such that

$$f_\lambda|_U \simeq_p^n g_\delta|_U \text{ in } V \text{ for all } \lambda \geq \lambda_0 \text{ and } \delta \geq \delta_0.$$

By replacing f_λ and f_{λ_0} to f_λ and g_δ in Lemma 2.4, we can conclude $f' \simeq_p^n g'$.

To prove the contrary, suppose that $f' \simeq_p^n g'$. Let V be a closed neighborhood of Y in M_Y. Since $f' \simeq_p^n g'$, for $V' = \alpha_Y^{-1}(V)$ there exist a closed a neighborhood U' of X in M'_X and indices $\lambda_0 \in \Lambda$, $\delta_0 \in \Delta$ such that

$$f'_\lambda|_{U'} \simeq_p^n g'_\delta|_{U'} \text{ in } V' \text{ for all } \lambda \geq \lambda_0 \text{ and } \delta \geq \delta_0.$$

Since α_X is proper, there is a closed neighborhood U of X in M_X such that $\alpha_X^{-1}(U) \subset U'$. Let $\beta : Z \to U$ be a proper map from $Z \in \mathscr{L}\mathscr{K}(n)$ to U. By the invertiblity of α_X, there is a proper map $\beta' : Z \to \alpha_X^{-1}(U)$ such that $\beta = \alpha_X \beta'$. Then

$$f_\lambda|_U\beta = f_\lambda|_U \alpha_X \beta' = \alpha_Y f'_\lambda|_{\alpha_X^{-1}(U)}\beta' \simeq_p \alpha_Y g'_\delta|_{\alpha_X^{-1}(U)}\beta' = g_\delta|_U \alpha_X \beta' = g_\delta|_U \beta \text{ in } V$$

for all $\lambda \geq \lambda_0$ and $\delta \geq \delta_0$, which implies $f \simeq_p^n g$. □

Theorem 2.6. *There is a categorical embedding $\Phi : n\text{-SH}_p\, \mathscr{L}\mathscr{K}(n+1) \to n\text{-SH}'_p\, \mathscr{L}\mathscr{K}(n+1)$ such that $\Phi(X) = X$ for each $X \in \mathscr{L}\mathscr{K}(n+1)$.*

Proof. For a proper n-fundamental net $f : X \to Y$ in (M_X, M_Y), we define $\Phi([f]_p^n) = [f']_p^n$, where $f' : X \to Y$ in (M'_X, M'_Y) is induced in Lemma 2.4. By Lemmas 2.4 and 2.5, we may only prove that Φ is functorial, that is,

$\Phi([g]_p^n[f]_p^n) = \Phi([f]_p^n)\Phi([g]_p^n)$ for each proper n-fundamental nets $f : X \to Y$ in (M_X, M_Y) and $g : Y \to Z$ in (M_Y, M_Z). Let $f' : X \to Y$ in (M'_X, M'_Y) and $g' : Y \to Z$ in (M'_Y, M'_Z) be proper n-fundamental nets induced from f and g. For each $\lambda \in \Lambda$ and $\delta \in \Delta$, since $f_\lambda \alpha_X = \alpha_Y f'_\lambda$ and $g_\delta \alpha_Y = \alpha_Z g'_\delta$,

$$g_\delta f_\lambda \alpha_X = g_\delta \alpha_Y f'_\lambda = \alpha_Z g'_\delta f'_\lambda,$$

which means that $g'f' = \{g'_\delta f'_\lambda \mid (\lambda, \delta) \in \Lambda \times \Delta\}$ is induced from $gf = \{g_\delta f_\lambda \mid (\lambda, \delta) \in \Lambda \times \Delta\}$. Therefore,

$$\Phi([g]_p^n[f]_p^n) = \Phi([gf]_p^n) = [g'f']_p^n = [g']_p^n[f']_p^n = \Phi([g]_p^n)\Phi([f]_p^n). \qquad \square$$

3. The Freudenthal compactification and compact pairs

In this section, we recall the Freudenthal compactification and study the relation of proper maps and compact pairs.

Suppose that X is rim-compact (i.e., any point has an arbitrary small neighborhoods with compact boundaries). *The Freudenthal compactification* of X, here denoted by FX, is defined as the least upper bound of all compactifications Y of X such that $\operatorname{ind}(Y \backslash X) = 0$. We call $EX = FX \backslash X$ *the space of ends* of X. It is known that FX is metrizable if and only if the space QX of quasi-components of X is compact, where EX is homeomorphic to a closed set of the Cantor set. Let Σ be the subclass of $\mathscr{L}\mathscr{K}$ consisting of spaces X such that QX is compact.

Let X, $Y \in \Sigma$. Then each proper map $f : X \to Y$ has the unique extension $Ff : (FX, EX) \to (FY, EY)$. If $g : X \to Y$ is a proper map and $f \simeq_p^0 g$, then $Ff|_{EX} = Fg|_{EX}$ (cf. [4, Lemmas 2.3 and 2.7]). Also, the assignment $f \to Ff$ is functorial, that is, $F(\mathrm{id}_X) = \mathrm{id}_{FX}$ and $F(fg) = (Ff)(Fg)$ for proper maps $f : X \to Y$ and $g : Y \to Z$. For details, refer to [5], [12].

Lemma 3.1. *Let f, $g : Z \to FY$ be maps from a compact space Z to FY and C a closed set in Z. Suppose that $f(Z \backslash C)$, $g(Z \backslash C) \subset Y$ and $f(z) = g(z) \in EY$ for each $z \in C$. If $f|_{Z\backslash C} \simeq_p g|_{Z\backslash C}$ in Y, then $f \simeq g$ rel. C in FY.*

Proof. Let $H : (Z \backslash C) \times I \to Y$ be a proper homotopy such that $H_0 = f|_{Z\backslash C}$ and $H_1 = g|_{Z\backslash C}$. Define the homotopy $H' : Z \times I \to FY$ by

$$H'(z,t) = \begin{cases} H(z,t) & \text{for } z \in Z \backslash C, \\ f(z) = g(z) & \text{for } z \in C. \end{cases}$$

We prove that H' is continuous. Let $\{(z_i, t_i)\}_{i\in\omega}$ be a sequence in $(Z\backslash C)\times I$ such that $(z_i, t_i) \to (z_0, t_0) \in C\times I$ as $i\to\infty$. Let V be a neighborhood of $f(z_0) = g(z_0)$ in FY. Since $\dim FY = 0$, there exist open sets V_0, V_1 in FY such that $V_0 \subset V$, $EY \subset V_0\cup V_1$ and $V_0\cap V_1 = \emptyset$. Then $L = Y\backslash(V_0\cup V_1) = FY\backslash(V_0\cup V_1)$ is compact in Y, so $H^{-1}(L)$ is compact in $(Z\backslash C)\times I$. Let $U = Z\backslash pH^{-1}(L)$, where $p: Z\times I\to Z$ is a projection. Then U is a neighborhood of C and $H(U\times I)\subset V_0\cup V_1$. Since $(z_i, t_i)\to(z_0, t_0)$, there exists $m_0\in\omega$ such that $\{z_m\}\times I\subset U\times I$ for each $m\geq m_0$. Note that $H(\{z_m\}\times I)\subset H(U\times I)\subset V_0\cup V_1$. By the continuity of f, $H(\{z_m\}\times 0) = f(z_m)\times V_0$. Then $H(\{z_m\}\times I)\subset V_0\subset V$ since $V_0\cap V_1 = \emptyset$. In particular, $H(z_m, t_m)\in V$ for each $m\geq m_0$, which implies that H' is continuous. □

Let f, $g: X\to Y$ be maps and $A\subset X$ a closed set. We denote $f\simeq^n g$ rel. A if $f\alpha\simeq g\alpha$ rel. $\alpha^{-1}(A)$ for any map $\alpha: Z\to X$ with $\dim Z\leq n$. To see $f\simeq^n g$ rel. A, it suffices to verify the condision in case that α is an n-invertible surjection. By using this notation, the following holds from Lemma 3.1.

Corollary 3.2. *Let f, $g: X\to Y$ be proper maps. Then $f\simeq^n_p g$ implies $Ff\simeq^n Fg$ rel. EX.* □

Remark. As is easily observed, the above is valid for maps between pairs.

We call (M, M_0) a *μ^{n+1}-manifold pair* if M and M_0 are μ^{n+1}-manifolds and M_0 is Z-set in M.

Lemma 3.3. *Let f, $g: U\to V$ be proper maps such that $f\simeq^n_p g$. Suppose that (FU, EU) and (FV, EV) are embedded in compact μ^{n+1}-manifold pairs (M, M_0) and (N, N_0), respectively, such that $EU = FU\cap M_0$ and $EV = FV\cap M_0$. If f has an extension $\tilde{f}: (M, M_0)\to(N, N_0)$ with $\tilde{f}^{-1}(EV) = EU$, then g has also an extension $\tilde{g}: (M, M_0)\to(N, N_0)$ such that $\tilde{g}^{-1}(EV) = EU$ and $\tilde{f}\simeq^n\tilde{g}$ rel. EU as maps between pairs.*

Proof. By the n-homotopy extension theorem for pairs [3, Theorem A.6 and its remark], we can extend g to a proper map $\bar{g}: (M\backslash EU, M_0\backslash EU)\to(N\backslash EV, N_0\backslash EV)$ such that $\bar{g}\simeq^n_p \tilde{f}|_{(M\backslash EU, M_0\backslash EU)}$. Since M_0 is Z-set in M, $F(M\backslash EU) = M$ by [15, Corollary 1]. Then we have $\tilde{g} = F(\bar{g}): (M, M_0)\to(N, N_0)$ which has the desired property by corollary 3.2. □

REMARK. In the above, $g \cup \tilde{f}|_{M_0} : U \cup M_0 \to V \cup N_0$ is a map which is n-homotopic to $\tilde{f}|_{U \cup M_0}$ rel. M_0. Then we can obtain $\tilde{g}$ satisfying $\tilde{g}|_{M_0} = \tilde{f}|_{M_0}$.

LEMMA 3.4. *Let f, $g : (M, M_0) \to (Y, Y_0)$ be maps from a compact pair (M, M_0) to an LC^n-pair (Y, Y_0) and $A \subset X$ be closed sets in M and $X_0 = X \cap M_0$. If $f|_{(X,X_0)} \simeq^n g|_{(X,X_0)}$ rel. A, then there exists a neighborhood pair (U, U_0) of (X, X_0) in (M, M_0) such that $f|_{(U,U_0)} \simeq^n g|_{(U,U_0)}$ rel. A.*

PROOF. Let $\alpha : (Z, Z_0) \to (M, M_0)$ be an n-invertible UV^n-surjection from an n-dimensional compact pair (Z, Z_0) to (M, M_0). By the assumption, there exists a homotopy $H : (\alpha^{-1}(X), \alpha^{-1}(X_0)) \times I \to (Y, Y_0)$ such that $H_0 = f\alpha|_{(\alpha^{-1}(X),\alpha^{-1}(X_0))}$, $H_1 = g\alpha|_{(\alpha^{-1}(X),\alpha^{-1}(X_0))}$ and $H_t|_{\alpha^{-1}(A)} = f\alpha|_{\alpha^{-1}(A)}$ for all $t \in I$. By [1, Lemma 2.1], there exists a neighborhood pair (W, W_0) of $(Z, Z_0) \times \{0, 1\} \cup (\alpha^{-1}(X), \alpha^{-1}(X_0)) \times I$ in $(Z, Z_0) \times I$ and an extension $H' : (W, W_0) \to (Y, Y_0)$ of H. Since α is proper, we can find a neighborhood pair (U, U_0) of (X, X_0) such that $(\alpha^{-1}(U), \alpha^{-1}(U_0)) \times I \subset (W, W_0)$, which implies that $f|_{(U,U_0)} \simeq^n g|_{(U,U_0)}$ rel. A. □

4. Proper n-shape and relative n-shape for compact pairs

Let $\mathscr{K}^2$ be the class of compact pairs. In this section, we define the relative n-shape category for compact pairs with dim $\leq n+1$, n-$\mathrm{SH}_{rel}\ \mathscr{K}^2(n+1)$, which is different from [1] and we construct a functor $\Psi : n\text{-}\mathrm{SH}'_p\ \Sigma(n+1) \to n\text{-}\mathrm{SH}_{rel}\ \mathscr{K}^2(n+1)$.

Let (X, X_0) be a Z-pair in (μ^{n+1}, μ_0^{n+1}) and let $(\boldsymbol{X}, \boldsymbol{X}_0) = \{(X_i, X_{0i}), p_i^{i+1}\}$ be an $LC^n(n+1)$-sequence associated with (X, X_0), where (X_i, X_{0i}) is a compact μ^{n+1}-manifold pair which is a closed neighborhood of (X, X_0), and bonding maps are inclusions [1, Proposition 2.2]. For convenience sake, we assume that $(X_1, X_{01}) = (\mu^{n+1}, \mu_0^{n+1})$. We call such an $LC^n(n+1)$-sequence *an inclusion sequence* associated with (X, X_0). Let (Y, Y_0) be a Z-pair in (μ^{n+1}, μ_0^{n+1}) and $(\boldsymbol{Y}, \boldsymbol{Y}_0) = \{(Y_i, Y_{0i}), q_i^{i+1}\}$ be an inclusion sequence associated with (Y, Y_0). An n-morphism $\boldsymbol{h} = (h, \{h_i\}) : (\boldsymbol{X}, \boldsymbol{X}_0) \to (\boldsymbol{Y}, \boldsymbol{Y}_0)$ is said to be *a relative n-morphism* if $h_i p_{h(i)}^{h(j)} \simeq^n q_i^j h_j$ rel. X_0 for any i, j with $j \geq i$, and we denote $\boldsymbol{h} : (\boldsymbol{X}, \boldsymbol{X}_0) \to (\boldsymbol{Y}, \boldsymbol{Y}_0)$ rel. X_0. Two relative n-morphisms $\boldsymbol{g}$, $\boldsymbol{h} : (\boldsymbol{X}, \boldsymbol{X}_0) \to (\boldsymbol{Y}, \boldsymbol{Y}_0)$ rel. X_0 are *relative n-homotopic* ($\boldsymbol{g} \simeq^n \boldsymbol{h}$ rel. X_0) if for each $i \in \omega$ there is an index $j \geq g(i)$, $h(i)$ such that $g_i|_{(X_j,X_{0j})} \simeq^n h_i|_{(X_j,X_{0j})}$ rel. X_0. By Lemma 3.4, $\boldsymbol{g} \simeq^n \boldsymbol{h}$ rel. X_0 if and only if $g_i|_X \simeq^n h_i|_X$ rel. X_0. The class of relative n-homotopy of the relative n-morphism $\boldsymbol{h}$ is denoted by $[\boldsymbol{h}]_{rel}^n$. *The relative n-shape category for compact pairs*

n-$\mathrm{SH}_{rel}\ \mathscr{K}^2(n+1)$ is defined as a category whose objects are in $\mathscr{K}^2(n+1)$ and whose morphisms are the relative n-homotopy classes of relative n-morphisms. If there exist two relative n-morphisms $\boldsymbol{f}:(X,X_0)\to(Y,Y_0)$ rel. X_0 and $\boldsymbol{g}:(Y,Y_0)\to(X,X_0)$ rel. Y_0 such that $\boldsymbol{gf}\simeq^n \boldsymbol{i}_{(X,X_0)}$ rel. X_0 and $\boldsymbol{fg}\simeq^n \boldsymbol{i}_{(Y,Y_0)}$ rel. Y_0, we denote n-$\mathrm{Sh}(X,X_0)=n$-$\mathrm{Sh}(Y,Y_0)$ rel. (X_0,Y_0). The relative n-shape for compact pairs is stronger that the n-shape for compact pairs in the sense of [1], that is, n-$\mathrm{Sh}(X,X_0)=n$-$\mathrm{Sh}(Y,Y_0)$ rel. (X_0,Y_0) implies n-$\mathrm{Sh}(X,X_0)=n$-$\mathrm{Sh}(Y,Y_0)$.

Suppose that FX and FY are Z-sets in μ^{n+1}, $EX=FX\cap\mu_0^{n+1}$, $EY=FY\cap\mu_0^{n+1}$ and $M=\mu^{n+1}\backslash\mu_0^{n+1}$, $(X,X_0)=\{(X_i,X_{0i}),p_i^{i+1}\}$ and $(Y,Y_0)=\{(Y_i,Y_{0i}),q_i^{i+1}\}$ are inclusion sequences associated with (FX,EX) and (FY,EY), respectively. For each proper n-fundamental net $f=\{f_\lambda\,|\,\lambda\in\Lambda\}:X\to Y$ in (M,M), we construct a relative n-morphism $\hat{\boldsymbol{f}}=(f,\{\hat{f}\}):(X,X_0)\to(Y,Y_0)$ rel. EX as follows:

Let $V_1\supset V_2\supset\cdots$ be closed neighborhoods of Y in M such that $Y=\bigcap_{i\in\omega}V_i$, $EV_i=EY$ and $FV_i\subset\operatorname{int}Y_i$. Choose closed neighborhoods $U_1^f\supset U_2^f\supset\cdots$ of X and indices $\lambda_1\le\lambda_2\le\cdots$ such that $X=\bigcap_{i\in\omega}U_i^f$, $EU_i^f=EX$ and

$$f_\lambda|_{U_i^f}\simeq_p^n f_{\lambda_i}|_{U_i^f}\text{ in }V_i\text{ for all }\lambda\ge\lambda_i.$$

Since (Y_1,Y_{01}) is an $LC^n\cap C^n$-pair, $F(f_{\lambda_1}|_{U_1^f}):(FU_1^f,EU_1^f)\to(FV_1,EV_1)\subset(Y_1,Y_{01})$ has an extension $\hat{f}_1:(X_1,X_{01})\to(Y_1,Y_{01})$ [1, Lemma 2.1]. Since $f_{\lambda_2}|_{U_2^f}\simeq_p^n f_{\lambda_1}|_{U_2^f}$ in V_1 and $f_{\lambda_1}|_{U_2^f}$ has an extension $\hat{f}_1$ with $EU_1^f=\hat{f}_1^{-1}(EV_1)$, by Lemma 3.3, $f_{\lambda_2}|_{U_2^f}$ has an extension $\tilde{f}_2:(X_1,X_{01})\to(Y_1,Y_{01})$ such that $\tilde{f}_2\simeq^n\hat{f}_1$ in (V_1,V_{01}) rel. $EU_2^f=EX$. Note that $\tilde{f}_2(FU_2^f)\subset FV_2\subset\operatorname{int}Y_2$. Then there exists a μ^{n+1}-manifold neighborhood pair $(\overline{U}_2^f,\overline{U}_{02}^f)$ of (FU_2^f,EU_2^f) such that $\tilde{f}_2(\overline{U}_2^f,\overline{U}_{02}^f)\subset(Y_2,Y_{02})$. Take $f(2)\in\omega$ such that $(X_{f(2)},X_{0f(2)})\subset(\overline{U}_2^f,\overline{U}_{02}^f)$ and let $\hat{f}_2=\tilde{f}_2|_{(X_{f(2)},X_{0f(2)})}:(X_{f(2)},X_{0f(2)})\to(Y_2,Y_{02})$. Observe that $\hat{f}_2\simeq^n\hat{f}_1|_{(X_{f(2)},X_{0f(2)})}$ in (Y_1,Y_{01}) rel. EX. Assume that we obtained an extension $\tilde{f}_i:(\overline{U}_i^f,\overline{U}_{0i}^f)\to(Y_i,Y_{0i})$ of $f_{\lambda_i}|_{U_i^f}$ for $i\ge2$ such that $(\overline{U}_i^f,\overline{U}_{0i}^f)$ is a μ^{n+1}-manifold neighborhood pair of (FU_i^f,EU_i^f) and $\tilde{f}_i\simeq\tilde{f}_{i-1}|_{(\overline{U}_i^f,\overline{U}_{0i}^f)}$ in (Y_i,Y_{0i}) rel. EX. Since $f_{\lambda_{i+1}}|_{U_{i+1}^f}\simeq_p^n f_{\lambda_i}|_{U_{i+1}^f}$ in V_i and $f_{\lambda_i}|_{U_{i+1}^f}$ has an extension $\tilde{f}_i$ with $EU_i^f=\tilde{f}_i^{-1}(EV_i)$, by Lemma 3.3, $f_{\lambda_{i+1}}|_{U_{i+1}^f}$ has an extension $\tilde{f}_{i+1}:(\overline{U}_i^f,\overline{U}_{0i}^f)\to(Y_i,Y_{0i})$ such that $\tilde{f}_{i+1}\simeq^n\tilde{f}_i$ in (V_i,V_{0i}) rel. EX. Note that $\tilde{f}_{i+1}(FU_{i+1}^f)\subset FV_{i+1}\subset\operatorname{int}Y_{i+1}$. Then there exists a μ^{n+1}-manifold neighborhood pair $(\overline{U}_{i+1}^f,\overline{U}_{0i+1}^f)$ of (FU_{i+1}^f,EU_{i+1}^f) such that $\tilde{f}_{i+1}(\overline{U}_{i+1}^f,\overline{U}_{0i+1}^f)\subset(Y_{i+1},Y_{0i+1})$. Take $f(i+1)\in\omega$ such that $(X_{f(i+1)},X_{0f(i+1)})\subset(\overline{U}_{i+1}^f,\overline{U}_{0i+1}^f)$ and let $\hat{f}_{i+1}=\tilde{f}_{i+1}|_{(X_{i+1},X_{0i+1})}:(X_{f(i+1)},X_{0f(i+1)})\to(Y_{i+1},Y_{0i+1})$. Observe that $\hat{f}_{i+1}\simeq{}^n\hat{f}_i|_{(X_{f(i+1)},X_{0f(i+1)})}$ in (Y_i,Y_{0i})

rel. EX. By the induction, we have a sequence $\hat{f} = (f, \{\hat{f}_i\}) : (X, X_0) \to (Y, Y_0)$. It is easy to see that $\hat{f}$ is a relative n-morphism. In fact, for each $j \geq i$,

$$q_i^j \hat{f}_j = \hat{f}_j \simeq^n \hat{f}_{j-1}|_{(X_{f(j)}, X_{0f(j)})} \text{ in } (Y_{j-1}, Y_{0j-1}) \text{ rel. } EX$$

$$\simeq^n \hat{f}_{j-2}|_{(X_{f(j)}, X_{0f(j)})} \text{ in } (Y_{j-2}, Y_{0j-2}) \text{ rel. } EX$$

$$\cdots$$

$$\simeq^n \hat{f}_i|_{(X_{f(j)}, X_{0f(j)})} = \hat{f}_{f(j)} p_{f(i)}^{f(j)} \text{ in } (Y_i, Y_{0i}) \text{ rel. } EX$$

Then above construction of $\hat{f}$ is denoted by $f \Rightarrow \{V_i, U_i^f, \lambda_i, \tilde{f}_i, \bar{U}_i^f, f(i)\}(\hat{f})$.

Theorem 4.1. *There is a functor* $\Psi : n\text{-SH}'_p\ \Sigma(n+1) \to n\text{-SH}_{rel}\ \mathscr{K}^2(n+1)$ *such that* $\Psi(X) = (FX, EX)$ *for each* $X \in \Sigma(n+1)$.

Proof. Let $f = \{f_\lambda \mid \lambda \in \Lambda\} : X \to Y$ in (M, M) be a proper n-fundamental net. By the construction $f \Rightarrow \{V_i, U_i^f, \lambda_i, \tilde{f}_i, \bar{U}_i^f, f(i)\}(\hat{f})$, we have a relative n-morphism $\hat{f} : (X, X_0) \to (Y, Y_0)$ rel. EX. We define $\Psi([f]_p^n) = [\hat{f}]_{rel}^n$.

First, we prove that Ψ is well-defined, that is, if $f \simeq_p^n g$ for another proper n-fundamental net $g = \{g_\delta \mid \delta \in \Delta\} : X \to Y$ in (M, M), then $\hat{f} \simeq^n \hat{g}$ rel. EX. By the construction $g \Rightarrow \{V_i', U_i^g, \delta_i, \tilde{g}_i, \bar{U}_i^g, g(i)\}(\hat{g})$, we have a relative n-morphism $\hat{g} = (g, \{\hat{g}_i\}) : (X, X_0) \to (Y, Y_0)$ rel. EX such that $\hat{g}_i = \tilde{g}_i|_{(X_{g(i)}, X_{0g(i)})}$ and $\tilde{g}_i|_{U_i^g} = g_{\delta_i}|_{U_i^g}$. Since $f \simeq_p^n g$, there exist a closed neighborhood W of X in M, $\lambda_0 \in \Lambda$ and $\delta_0 \in \Delta$ such that

$$f_\lambda|_W \simeq_p^n g_\delta|_W \text{ in } V_i \cup V_i' \text{ for all } \lambda \geq \lambda_0 \text{ and } \delta \geq \delta_0.$$

Let $W' = W \cap U_i^f \cap U_i^g$. For each $\lambda \geq \lambda_0$, λ_i and $\delta \geq \delta_0$, δ_i, since

$$f_\lambda|_{W'} \simeq_p^n f_{\lambda_i}|_{W'} \text{ in } V_i \text{ and } g_\delta|_{W'} \simeq_p^n g_{\delta_i}|_{W'} \text{ in } V_i',$$

we have

$$f_{\lambda_i}|_{W'} \simeq_p^n f_\lambda|_{W'} \simeq_p^n g_\delta|_{W'} \simeq_p^n g_{\delta_i}|_{W'} \text{ in } V_i \cup V_i'.$$

Since $\hat{f}_i$ and $\hat{g}_i$ are extensions of $f_{\lambda_i}|_{W'}$ and $g_{\delta_i}|_{W'}$, by Corollary 3.2,

$$\hat{f}_i|_{FX} \simeq^n \hat{g}_i|_{FX} \text{ rel. } EX \text{ in } Y_i,$$

which implies $\hat{f} \simeq^n \hat{g}$ rel. EX. Therefore, Ψ is well-defined.

Next, we show that Ψ is functorial. Let $f : X \to Y$ in (M, M) and $g : Y \to Z$ in (M, M) be proper n-fundamental nets and $(Z, Z_0) = \{(Z_i, Z_{0i}), r_i^{i+1}\}$ be an inclusion sequence associated with (FZ, EZ). By the constructions $f \Rightarrow \{V_i, U_i^f, \lambda_i, \tilde{f}_i, \bar{U}_i^f, f(i)\}(\hat{f})$ and $g \Rightarrow \{W_i, U_i^f, \delta_i, \tilde{g}_i, \bar{V}_i^g, g(i)\}(\hat{g})$, we have relative n-morphisms $\hat{f} : (X, X_0) \to (Y, Y_0)$ rel. EX and $\hat{g} : (Y, Y_0) \to (Z, Z_0)$ rel. EY. Observe that $V_{g(i)} \subset Y_{g(i)} \subset \bar{V}_i^g$. Since $gf = \{g_\delta f_\lambda \mid (\lambda, \delta) \in \Lambda \times \Delta\} : X \to Z$ in (M, M) and $\hat{g}\hat{f} = (fg, \{\hat{g}_i \hat{f}_{g(i)}\}) : (X, X_0) \to (Z, Z_0)$, it is easy to verify $gf \Rightarrow \{W_i, U_{g(i)}^f, (\delta_i, \lambda_{g(i)}), \tilde{g}_i \tilde{f}_{g(i)}, \bar{U}_{g(i)}^f, fg(i)\}(\hat{g}\hat{f})$, hence

$$\Psi([gf]_p^n) = [\hat{g}\hat{f}]_{rel}^n = [\hat{g}]_{rel}^n [\hat{f}]_{rel}^n = \Psi([g]_p^n)\Psi([f]_p^n). \qquad \square$$

Combining Theorems 2.6 and 4.1, we have the following.

Corollary 4.2. *There is a functor* $\Theta : n\text{-SH}_p\ \Sigma(n+1) \to n\text{-SH}_{rel}\ \mathscr{K}^2(n+1)$ *such that* $\Theta(X) = (FX, EX)$ *for each* $X \in \Sigma(n+1)$. $\square$

As a direct consequence of the above, we have

Corollary 4.3. *For* $X, Y \in \Sigma(n+1)$, *if* $n\text{-Sh}_p(X) = n\text{-Sh}_p(Y)$ *then* n-Sh $(FX, EX) = n\text{-Sh}(FY, EY)$ *rel.* (EX, EY), *hence* $n\text{-Sh}(FX, EX) = n\text{-Sh}(FY, EY)$ *in the sense of* [1] *and* $EX \approx EY$. $\square$

Corollary 4.4. *If* X *is connected* SUV^n *and* $\dim X \leq n+1$, *then* $FX \in UV^n$.

Proof. By [2, Theorem 3.3], there exists a tree T such that $n\text{-Sh}_p(X) = n\text{-Sh}_p(T)$. By Corollary 4.3, $n\text{-Sh}(FX, EX) = n\text{-Sh}(FT, ET)$ rel. (EX, ET). In particular, $n\text{-Sh}(FX) = n\text{-Sh}(FT)$. Since FT is contractible, $n\text{-Sh}(FX) = n\text{-Sh}(1)$, that is, $FX \in UV^n$ (cf. [9, Proposition 3.1]). $\square$

5. Proper n-shape and the one-point compactification

In this section, we consider the relation of proper n-shape and the one-point compactification.

Let $CX = X \cup \{\infty\}$ be *the one-point compactification* of $X \in \mathscr{LK}$. It is known that CX is metrizable. For a proper map $f : X \to Y$ between $X, Y \in \mathscr{LK}$, there is an unique extension $Cf : (CX, \{\infty\}) \to (CY, \{\infty\})$. If $f, g : X \to Y$ are proper maps and $f \simeq_p g$, then $Cf \simeq Cg$ rel. $\{\infty\}$. It is easy to see that the arguments of Section 3 are valid for the pairs $(CX, \{\infty\})$ and the maps Cf. By replacing FX by CX and EX by $\{\infty\}$ in the same proof of Theorem 4.1, we can obtain the following.

THEOREM 5.1. *There is a functor* $\Psi' : n\text{-SH}'_p\ \mathscr{L}\mathscr{K}(n+1) \to n\text{-SH}_{rel}\ \mathscr{K}^2(n+1)$ *such that* $\Psi'(X) = (CX, \{\infty\})$ *for each* $X \in \mathscr{L}\mathscr{K}(n+1)$. □

Combining Theorems 2.6 and 5.1, we have the following.

COROLLARY 5.2. *There is a functor* $\Theta' : n\text{-SH}_p\ \mathscr{L}\mathscr{K}(n+1) \to n\text{-SH}_{rel}\ \mathscr{K}^2(n+1)$ *such that* $\Theta'(X) = (CX, \{\infty\})$ *for each* $X \in \mathscr{L}\mathscr{K}(n+1)$. □

As a direct consequence of the above, we have

COROLLARY 5.3. *For* $X,\ Y \in \mathscr{L}\mathscr{K}(n+1)$, *if* $n\text{-Sh}_p(X) = n\text{-Sh}_p(Y)$ *then* $n\text{-Sh}(CX, \{\infty\}) = n\text{-Sh}(CY, \{\infty\})$ *rel.* $(\{\infty\}, \{\infty\})$, *hence pointed compacta* $(CX, \{\infty\})$ *and* $(CY, \{\infty\})$ *have the same pointed n-shape.* □

COROLLARY 5.4. *Let* $\mu_\infty^{n+1} = \mu^{n+1} \backslash \{*\}$, *where* $* \in \mu^{n+1}$, $X,\ Y \subset \mu_\infty^{n+1}$ *be Z-sets. If* $n\text{-Sh}_p(X) = n\text{-Sh}_p(Y)$ *then* $\mu_\infty^{n+1} \backslash X \approx \mu_\infty^{n+1} \backslash Y$.

PROOF. We can assume that $CX,\ CY \subset \mu^{n+1}$ as Z-sets with $\{*\} = \{\infty\}$. By Corollary 5.3, we have $n\text{-Sh}(CX, \{\infty\}) = n\text{-Sh}(CY, \{\infty\})$ rel. $(\{\infty\}, \{\infty\})$. In particular, $n\text{-Sh}(CX) = n\text{-Sh}(CY)$. By the complement theorem [8], $\mu_\infty^{n+1} \backslash X = \mu^{n+1} \backslash CX \approx \mu^{n+1} \backslash CY = \mu_\infty^{n+1} \backslash Y$. □

Acknowledgment

The author withes to thank Professor K. Sakai and the referee for their helpful comments.

References

[1] Y. Akaike, The n-shape of compact pairs and weak proper n-homotopy, Grasnik Mat. **31**(51) (1996), 295–306.
[2] ———, Proper n-shape and property SUV^n, Bull. Polish Acad. Sci., Math. **45** (1997), in press.
[3] ——— and K. Sakai, The complement theorem in n-shape theory for compact pairs, Grasnik Mat. **31**(51) (1996), 307–319.
[4] B. J. Ball, Proper shape retracts, Fund. Math. **89** (1975), 177–189.
[5] ——— and R. B. Sher, A theory of proper shape for locally compact metric spaces, Fund. Math. **86** (1974), 163–192.
[6] M. Bestvina, Characterizing k-dimensional universal Menger compacta, Mem. Amer. Math. Soc., No.380, Amer. Math. Soc., Providence, RI, 1988.
[7] T. A. Chapman, Lectures on Hilbert cube manifilds, CBMS 28, Amer. Math. Soc., Providence, 1976.

[8] A. Chigogidze, Compacta lying in the n-dimensional Menger compactum and having homeomorphic complements in it, Mat. Sb. **133** (1987), 481–496. (English transl., Math. USSR Sbornik, **61** (1988), No.2 471–484.)

[9] ———, The theory of n-shape, Uspekhi Mat. Nauk, **44:5** (1989), 117–140. (English transl., Russian Math. Surveys **44:5** (1989), 145–174.)

[10] ———, UV^n-equivalence and n-equivalence, Topology Appl. **45** (1992), 283–291.

[11] R. Daverman, Decompositions of Manifolds, Academic Press, New York, 1986.

[12] R. F. Dickman and R. A. McCoy, The Freudenthal compactification, Dissertationes Math. (Rozprawy Mat.) 262 (Warszawa 1988).

[13] S. T. Hu, Theory of Retracts, Wayne St. Univ. Press, Detroit, 1965.

[14] R. C. Lacher, Cell-like mappings and their generalizations, Bull. Amer. Math. Soc. **83** (1977), 495–552.

[15] K. Nowiński, Closed mappings and the Freudenthal compactification, Fund. Math. **76** (1972), 71–83.

[16] R. B. Sher, Property SUV^∞ and proper shape theory, Trans. Amer. Math. Soc. **190** (1974), 345–356.

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