TSUKUBA J. MATH.
Vol. 26 No. 1 (2002), 49–60

1-TYPE MINIMAL SURFACES IN COMPLEX GRASSMANN MANIFOLDS AND ITS GAUSS MAP

By

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Abstract. In this paper we establish an isometric imbedding of a complex Grassmann manifold $G(m,n)$ into a Euclidean space. Then we use this isometric imbedding to study 1-type minimal surfaces in $G(m,n)$ and its Gauss map, and obtain some results.

1. Introduction

A submanifold M (connected but not necessary compact) of an Euclidean N-space E^N is said to be of finite type if each component of its position vector ϕ can be written as a finite sum of eigenfunctions of the Laplacian Δ of M, that is,

$$\phi = \phi_0 + \phi_1 + \cdots + \phi_k,$$

where ϕ_0 is a constant vector (called the center of mass of M) and $\Delta\phi_t = \lambda_t\phi_t$, $t = 1,2,\dots,k$. If in particular all eigenvalues $\{\lambda_1,\lambda_2,\dots,\lambda_k\}$ are mutually different, then M is said to be of k-type (c.f. [9] for details).

In terms of finite-type submanifolds, a well-known theorem of Takahashi [6] says that a compact submanifold M of Euclidean space is of 1-type if and only if M lies in some hypersphere as a minimal submanifold, and such a submanifold is always mass-symmetric, i.e., the center of mass of M is the center of the hypersphere.

As is well-known, the complex projective space CP^n can be isometrically imbedded into a Euclidean space [1], and this isometric imbedding is basic to studying spectral geometry of submanifolds in CP^n. For related results one is referred to see [10, 11]. In this paper, we shall establish an isometric imbedding of a complex Grassmann manifold $G(m,n)$ into a Euclidean space. Then, we use this isometric imbedding to study 1-type minimal surfaces in $G(m,n)$. We also study

Received July 31, 2000.
Revised May 25, 2001.

the Gauss map for minimal surfaces in $G(m,n)$ and obtain some equivalent conditions for their Gauss maps to be harmonic.

2. An Isometric Imbedding of $G(m,n)$ into Euclidean Space

We equip the complex n-space C^n with the standard Hermitian inner product. The space of unitary bases can be identified with the unitary group $U(n)$. Let Z_A $(A = 1, \ldots, n)$ be regarded as the projection from $U(n)$ to C^n by mapping a matrix Z to its Ath column vector Z_A. Writing

$$dZ_A = \sum_B \omega_{AB} Z_B, \tag{2.1}$$

then ω_{AB} are the components of the Maurer-Cartan form on $U(n)$, and they satisfy

$$\omega_{AB} + \bar{\omega}_{BA} = 0. \tag{2.2}$$

Here and in the following we use the following ranges of indices:

$$1 \le A, B, \ldots \le n; \quad 1 \le \alpha, \beta, \ldots \le m; \quad m+1 \le i, j, \ldots \le n.$$

Taking the exterior derivative of (2.1), we get

$$d\omega_{AB} = \sum_C \omega_{AC} \wedge \omega_{CB}. \tag{2.3}$$

All elements of the complex Grassmann manifold $G(m,n)$ can be defined by the multivector $Z_1 \wedge \cdots \wedge Z_m$ for any $Z \in U(n)$ up to a factor. The vectors Z_α and their orthogonal vectors Z_i are defined up to a transformation of $U(m)$ and $U(n-m)$, respectively, so that $G(m,n)$ becomes a symmetric space $U(n)/(U(m) \times U(n-m)) = \{[Z] \mid Z \in U(n)\}$ [2]. In particular, the form

$$ds^2 = \sum_{\alpha, i} \omega_{\alpha i} \bar{\omega}_{\alpha i} \tag{2.4}$$

is a positive Hermitian form on the Lie subspace for $U(n)/U(m) \times U(n-m)$, and defines the canonical Kaehler metric on $G(m,n)$. Let $HM(n) = \{F \in gl(n,C) : \bar{F}^t = F\}$ be the set of all Hermitian $n \times n$-matrices. We define an inner product $\langle \cdot, \cdot \rangle$ on the vector space $HM(n)$ (and its complexification $gl(n,C)$) by

$$\langle F_1, F_2 \rangle = \frac{1}{2} \operatorname{tr} F_1 \bar{F}_2^t, \quad F_1, F_2 \in HM(n) \ \ (\text{or } gl(n,C)).$$

Then $HM(n)$ becomes the Euclidean space R^{n^2}. We can define a map f from

$G(m,n)$ to $HM(n)$ by

$$f([Z]) = Z\begin{pmatrix} I_m & 0 \\ 0 & 0 \end{pmatrix}\bar{Z}^t = \sum_\alpha Z_\alpha \bar{Z}_\alpha^t$$

for $Z \in U(n)$, where I_m is the identity $m \times m$-matrix.

Lemma 2.1. *The map $f : G(m,n) \to HM(n)$ defined as above is an isometric imbedding, and that $f(G(m,n)) = \{F \in HM(n) : F^2 = F, \mathrm{tr}\, F = m\}$.*

Proof. It is obvious that $f(G(m,n)) \subset \{F \in HM(n) : F^2 = F, \mathrm{tr}\, F = m\}$. Let $F \in HM(n)$ such that $F^2 = F$ and $\mathrm{tr}\, F = m$. Then the eigenvalues of F are 1 or 0, and consequently there exists $P \in U(n)$ such that

$$F = P\begin{pmatrix} I_m & 0 \\ 0 & 0 \end{pmatrix}\bar{P}^t. \tag{2.5}$$

This implies that $f(G(m,n)) \supset \{F \in HM(n) : F^2 = F, \mathrm{tr}\, F = m\}$. By virtue of (2.1) and (2.2) we have

$$df = \sum_{\alpha, i}(\omega_{\alpha i} Z_i \bar{Z}_\alpha^t + \bar{\omega}_{\alpha i} Z_\alpha \bar{Z}_i^t), \tag{2.6}$$

from which we see that the metric on $G(m,n)$ induced from that on $HM(n)$ by f is

$$\langle df, df\rangle = \frac{1}{2}\,\mathrm{tr}(df \cdot df) = \sum_{\alpha, i} \omega_{\alpha i}\bar{\omega}_{\alpha i}.$$

This together with (2.4) implies that f is isometric. So the lemma is proved.

Now we identify $G(m,n)$ with $f(G(m,n)) \subset HM(n)$. In this point of view, we have the following lemma which can be verified similarly as in [1].

Lemma 2.2. *Let $F \in G(m,n)$, then*

$$T_F(G(m,n)) = \{X \in HM(n) : XF + FX = X\},$$

$$T_F^\perp(G(m,n)) = \{N \in HM(n) : NF = FN\},$$

$$\bar{\sigma}(X, Y) = (XY + YX)(I - 2F), \quad X, Y \in T_F(G(m,n)),$$

where $\bar{\sigma}$ is the second fundamental form of $G(m,n)$ in $HM(n)$, and $I = I_n$.

For any $Z \in U(n)$, $\{\sqrt{2} Z_A \bar{Z}_B^t\}$ gives a unitary basis of $(gl(n.C), \langle\cdot\,,\cdot\rangle) \cong C^{n^2}$, and $\{Z_i\bar{Z}_\alpha^t, Z_\alpha\bar{Z}_i^t\}$ spans the complexification of $T_F G(m,n)$, where $F = [Z] \in$

$G(m,n)$. Set $E_{\alpha i} = Z_i \bar{Z}_\alpha^t + Z_\alpha \bar{Z}_i^t$ and $E_{\alpha^* i^*} = \sqrt{-1}(Z_i \bar{Z}_\alpha^t - Z_\alpha \bar{Z}_i^t)$, then $\{E_{\alpha i}, E_{\alpha^* i^*}\}$ forms an orthonormal basis of $T_F(G(m,n))$. The complex structure J on $G(m,n)$ is given by

$$JE_{\alpha i} = E_{\alpha^* i^*}, \quad JE_{\alpha^* i^*} = -E_{\alpha i}.$$

By use of Lemma 2.2, a direct computation shows that the mean curvature vector field $\bar{H}$ of $G(m,n)$ in $HM(n)$ is given by, at $F \in G(m,n)$,

$$\bar{H}_F = \frac{1}{2m(n-m)} \sum_{\alpha, i} (\bar{\sigma}(E_{\alpha i}, E_{\alpha i}) + \bar{\sigma}(E_{\alpha^* i^*}, E_{\alpha^* i^*})) = \frac{2n}{m(n-m)} \left(\frac{m}{n} I - F \right). \quad (2.7)$$

The following lemma can be shown similarly as in [1].

LEMMA 2.3. (a) *For any* $X \in T_F(G(m,n))$, $JX = \sqrt{-1}(I - 2F)X$.

(b) $\bar{\sigma}(JX, JY) = \bar{\sigma}(X, Y)$, $\bar{\nabla}\bar{\sigma} = 0$, *where* $X, Y \in T_F(G(m,n))$, *and* $\bar{\nabla}$ *is the Levi-Civita connection of* $G(m,n)$.

(c) $G(m,n)$ *is a minimal submanifold of a hypersphere in* $HM(n)$, *whose centre is* $\frac{m}{n} I$ *and whose radius is* $\sqrt{m(n-m)/2n}$.

3. 1-Type Minimal Surfaces in $G(m,n)$

Let M be a Riemann surface with the metric $ds_M^2 = \varphi\bar{\varphi}$, where φ is the complex-valued 1-form defined on M. The structure equations on M are given by

$$d\varphi = -\sqrt{-1}\rho \wedge \varphi, \quad d\rho = -\frac{\sqrt{-1}}{2} K\varphi \wedge \bar{\varphi}, \quad (3.1)$$

where ρ is the real-valued connection 1-form of ds_M^2, and K is the Gaussian curvature of ds_M^2. Let $\phi : M \to G(m,n)$ be a smooth map, and put

$$\phi^*(\omega_{\alpha i}) = a_{\alpha i}\varphi + b_{\alpha i}\bar{\varphi}. \quad (3.2)$$

Taking the exterior derivative of (3.2) and making use of (2.1) and (2.3), it is easy to see that there exist locally defined complex-valued functions $p_{\alpha i}$, $q_{\alpha i}$ and $r_{\alpha i}$ such that (see, for instance, [2])

$$p_{\alpha i}\varphi + q_{\alpha i}\bar{\varphi} = Da_{\alpha i} := da_{\alpha i} - \sum_\beta a_{\beta i}\omega_{\alpha\beta} + \sum_j a_{\alpha j}\omega_{ji} - \sqrt{-1}a_{\alpha i}\rho,$$

$$q_{\alpha i}\varphi + r_{\alpha i}\bar{\varphi} = Db_{\alpha i} := db_{\alpha i} - \sum_\beta b_{\beta i}\omega_{\alpha\beta} + \sum_j b_{\alpha j}\omega_{ji} + \sqrt{-1}b_{\alpha i}\rho. \quad (3.3)$$

In terms of matrix notation, we can rewrite (3.3) as

$$P\varphi + Q\bar{\varphi} = dA - \phi_{11}A + A\phi_{22} - \sqrt{-1}A\rho,$$

$$Q\varphi + R\bar{\varphi} = dB - \phi_{11}B + B\phi_{22} + \sqrt{-1}B\rho, \tag{3.4}$$

where $A = (a_{\alpha i})$, $B = (b_{\alpha i})$, $P = (p_{\alpha i})$, $Q = (q_{\alpha i})$, $R = (r_{\alpha i})$, $\phi_{11} = (\phi^*\omega_{\alpha\beta})$ and $\phi_{22} = (\phi^*\omega_{ij})$. It is known that ϕ is harmonic if and only if $Q = 0$ (see [2]). ϕ is an isometric immersion if and only if

$$\operatorname{tr}(A\bar{B}^t) = 0, \quad \operatorname{tr}(A\bar{A}^t + B\bar{B}^t) = 1. \tag{3.5}$$

If ϕ is an isometric immersion, then

$$\operatorname{tr}(A\bar{A}^t) = \cos^2\frac{\alpha}{2}, \quad \operatorname{tr}(B\bar{B}^t) = \sin^2\frac{\alpha}{2}, \tag{3.6}$$

where α is the Kaehler angle of ϕ (see [3, 4]). We say that ϕ is totally real if $\alpha = \pi/2$, and ϕ is strongly conformal if $A\bar{B}^t = 0$ (see [8]).

Let Δ be the Laplace-Beltrami operator on M. We consider $\phi : M \to G(m,n) \subset HM(n)$ to be the map to the Euclidean space. If there exist a constant $\lambda > 0$ and a constant matrix $T \in HM(n)$ such that

$$\Delta\phi = \lambda(\phi - T), \tag{3.7}$$

then ϕ is said to be of 1-type. In particular, when $T = (m/n)I$, ϕ is said to be mass-symmetric 1-type. We now want to compute $\Delta\phi$. From (2.6) and (3.2) we have

$$d\phi = \sum_{\alpha,i}(a_{\alpha i}Z_i\bar{Z}_\alpha^t + \bar{b}_{\alpha i}Z_\alpha\bar{Z}_i^t)\varphi + \sum_{\alpha,i}(b_{\alpha i}Z_i\bar{Z}_\alpha^t + \bar{a}_{\alpha i}Z_\alpha\bar{Z}_i^t)\bar{\varphi}. \tag{3.8}$$

Then, combining (2.1), (3.2), (3.3) and (3.8) we get

$$\frac{1}{4}\Delta\phi = -\operatorname{tr}_g D\, d\phi = \sum_{\alpha,\beta}(A\bar{A}^t + B\bar{B}^t)_{\beta\alpha}Z_\alpha\bar{Z}_\beta^t - \sum_{i,j}(\bar{A}^tA + \bar{B}^tB)_{ji}Z_i\bar{Z}_j^t - \sum_{\alpha,i}(q_{\alpha i}Z_i\bar{Z}_\alpha^t + \bar{q}_{\alpha i}Z_\alpha\bar{Z}_i^t). \tag{3.9}$$

If ϕ is mass-symmetric 1-type, then there exists a constant $\lambda > 0$ such that

$$\Delta\phi = \lambda\left(\phi - \frac{m}{n}I\right). \tag{3.10}$$

It is clear from Lemma 2.2 that $\phi, I \in T_\phi^\perp(G(m,n))$, which together with (3.10) yields that $\Delta\phi \in T_\phi^\perp(G(m,n))$. Therefore, from (3.9) we see that $q_{\alpha i} = 0$, i.e., ϕ is harmonic. The following theorem follows from (3.9), (3.10) combined with the fact that $\sum_A Z_A \bar{Z}_A^t = I$.

Theorem 3.1. *Let $\phi : M \to G(m,n)$ be a smooth map. ϕ is mass-symmetric 1-type if and only if ϕ is harmonic, and $A\bar{A}^t + B\bar{B}^t$ and $\bar{A}^t A + \bar{B}^t B$ are scalar matrices.*

For later use we write down the following lemma which can be shown similarly as Theorem 3.1.

Lemma 3.2. *Let $\phi : M \to G(m,n)$ be a smooth map. If there exist constants $\lambda > 0$ and a such that*

$$\Delta\phi = \lambda(\phi - aI),$$

then $a = m/n$, and consequently ϕ is mass-symetric 1-type.

From now on we assume that $\phi : M \to G(m,n)$ is harmonic. Then (3.9) is reduced to

$$\frac{1}{4}\Delta\phi = \sum_{\alpha,\beta}(A\bar{A}^t + B\bar{B}^t)_{\beta\alpha} Z_\alpha \bar{Z}_\beta^t - \sum_{i,j}(\bar{A}^t A + \bar{B}^t B)_{ji} Z_i \bar{Z}_j^t. \tag{3.11}$$

We are now in a position to prove the main result of this section.

Theorem 3.3. *Let $\phi : M \to G(m,n)$ be a harmonic map. ϕ is of 1-type if and only if $\phi = \phi_0 \oplus \phi_1 \oplus \cdots \oplus \phi_s$, where $\phi_p : M \to G(m_p, n_p) \subset HM(n_p)$ $(p = 1, \dots, s)$ are mass-symmetric 1-type with the same eigenvalue, ϕ_0 is a constant matrix in $HM(n_0)$ with $\phi_0^2 = \phi_0$, and that $\sum_{p=1}^s m_p + \operatorname{tr}\phi_0 = m$, $\sum_{p=0}^s n_p = n$. Here m_p, n_p $(p = 1, \dots, s)$ are positive integers, and n_0 is a non-negative integer.*

Proof. The sufficiency is obvious. So we need only to prove the necessity. If ϕ is of 1-type, then there are a constant $\lambda > 0$ and a constant matrix T such that (3.7) holds. Without loss of generality, we can assume that T is diagonal (otherwise we can use an isometry of $G(m,n)$ of the type $F \mapsto PF\bar{P}^t$, where P is in $U(n)$). Suppose that

$$T = \begin{pmatrix} a_1 I_{n_1} & & 0 \\ & \ddots & \\ 0 & & a_{s'} I_{n_{s'}} \end{pmatrix}, \tag{3.12}$$

where $a_1, \dots, a_{s'} \in R$ are different from each other and $n_1 + \cdots + n_{s'} = n$. From (3.11) we see that $\Delta\phi \in T_\phi^\perp(G(m,n))$, which together with (3.7) yields $T \in T_\phi^\perp(G(m,n))$. Therefore $\phi T = T\phi$ at any point in M, and hence we have the decomposition

$$\phi = \begin{pmatrix} \phi_1 & & 0 \\ & \ddots & \\ 0 & & \phi_{s'} \end{pmatrix}, \tag{3.13}$$

where $\phi_p : M \to HM(n_p)$, $\phi_p^2 = \phi_p$, and putting $\operatorname{tr} \phi_p = m_p$, then $m_p \in Z$, $m_1 + \cdots + m_{s'} = m$. So by (3.7) we get

$$\Delta\phi_p = \lambda(\phi_p - a_p I_{n_p}) \quad (p = 1, \dots, s'). \tag{3.14}$$

It is clear from (3.13) and (3.14) that $\phi_p = 0$ when $m_p = 0$, while $\phi_p = I_{m_p}$ when $m_p = n_p$. Now we consider the case $n_p > m_p > 0$. In this situation from (3.13) we see that ϕ_p defines a map $\phi_p : M \to G(m_p, n_p)$, and which together with (3.14) and Lemma 3.2 yields $a_p = m_p/n_p$. Thus $\phi_p : M \to G(m_p, n_p)$ is mass-symmetric 1-type with eigenvalue λ. Now the necessity follows easily.

Theorem 3.4. *Let M be a compact oriented Riemannian 2-manifold, and $\phi : M \to G(m,n)$ an isometric minimal immersion. Then the first eigenvalue λ_1 of the Laplace-Beltrami operator of M satisfies*

$$\lambda_1 \le 4 \int_M [\operatorname{tr}(A\bar{A}^t + B\bar{B}^t)^2 + \operatorname{tr}(\bar{A}^t A + \bar{B}^t B)^2]/\operatorname{Area}(M),$$

and the equality holds if and only if ϕ is of 1-type.

Proof. By (3.11) and $\phi = \sum_\alpha Z_\alpha \bar{Z}_\alpha^t$, a direct computation shows that

$$\langle \Delta\phi, \phi \rangle = \frac{1}{2} \operatorname{tr}(\Delta\phi \cdot \phi) = 2,$$

$$\langle \Delta\phi, \Delta\phi \rangle = 8 \operatorname{tr}(A\bar{A}^t + B\bar{B}^t)^2 + 8 \operatorname{tr}(\bar{A}^t A + \bar{B}^t B)^2. \tag{3.15}$$

Substituting (3.15) into the inequality (see [5])

$$\int_M \langle \Delta\phi, \Delta\phi \rangle - \lambda_1 \int_M \langle \Delta\phi, \phi \rangle \ge 0$$

we can obtain the desired inequality. From [5] we see that the equality holds if and only if ϕ is of 1-type.

4. The Gauss Map

Let $\phi : M \to G(m,n)$ be an isometric minimal immersion from a Riemann surface with the metric ds_M^2. Regarding ϕ as the map to Euclidean space $HM(n) \cong R^{n^2}$, we can define the Gauss map $g_\phi : M \to G_{2,n^2}$ of ϕ, where G_{2,n^2} is the real Grassmann manifold and the 2-plane $g_\phi(x)$ at each point $x \in M$ parallels the tangent plane of $\phi(M)$ in $HM(n)$ at the point $\phi(x)$. By (3.11), the mean curvature vector H of the isometric immersion $\phi : M \to HM(n)$ is given by

$$H = -\frac{1}{2}\Delta\phi = -2\sum_{\alpha,\beta}(A\bar{A}^t + B\bar{B}^t)_{\beta\alpha}Z_\alpha\bar{Z}_\beta^t + 2\sum_{i,j}(\bar{A}^tA + \bar{B}^tB)_{ji}Z_i\bar{Z}_j^t. \tag{4.1}$$

By (2.1), (3.2) and (3.4), the covariant derivative $DH = (D'H)\varphi + (D''H)\bar{\varphi}$ of H is given by

$$\begin{aligned} D'H = &-2\sum_{\alpha,\beta}(P\bar{A}^t + B\bar{R}^t)_{\beta\alpha}Z_\alpha\bar{Z}_\beta^t + 2\sum_{i,j}(\bar{A}^tP + \bar{R}^tB)_{ji}Z_i\bar{Z}_j^t \\ &- 2\sum_{\alpha,i}(2A\bar{A}^tA + A\bar{B}^tB + B\bar{B}^tA)_{\alpha i}Z_i\bar{Z}_\alpha^t \\ &- 2\sum_{\alpha,i}(2\bar{B}B^t\bar{B} + \bar{A}A^t\bar{B} + \bar{B}A^t\bar{A})_{\alpha i}Z_\alpha\bar{Z}_i^t. \end{aligned} \tag{4.2}$$

We denote by $T^\perp\phi(M)$ the normal space of $\phi(M)$ in $HM(n)$. If g_ϕ is harmonic, then by Ruh-Vilms' theorem [7], the projection of DH on $T^\perp\phi(M)$ vanishes. Therefore, from (4.2) we get

$$\begin{gathered} P\bar{A}^t + B\bar{R}^t = 0, \quad \bar{A}^tP + \bar{R}^tB = 0, \\ \sum_{\alpha,i}(2A\bar{A}^tA + A\bar{B}^tB + B\bar{B}^tA)_{\alpha i}(Z_i\bar{Z}_\alpha^t)^\perp \\ + \sum_{\alpha,i}(2\bar{B}B^t\bar{B} + \bar{A}A^t\bar{B} + \bar{B}A^t\bar{A})_{\alpha i}(Z_\alpha\bar{Z}_i^t)^\perp = 0, \end{gathered} \tag{4.3}$$

where $(Z_i\bar{Z}_\alpha^t)^\perp$ and $(Z_\alpha\bar{Z}_i^t)^\perp$ denote, respectively, the projection of $Z_i\bar{Z}_\alpha^t$ and $Z_\alpha\bar{Z}_i^t$ on the complexification of $T^\perp\phi(M)$. From (3.4) and (4.3) we have

$$\begin{aligned} &d(A\bar{A}^t + B\bar{B}^t) - \phi_{11}(A\bar{A}^t + B\bar{B}^t) + (A\bar{A}^t + B\bar{B}^t)\phi_{11} = 0, \\ &d(\bar{A}^tA + \bar{B}^tB) - \phi_{22}(\bar{A}^tA + \bar{B}^tB) + (\bar{A}^tA + \bar{B}^tB)\phi_{22} = 0. \end{aligned} \tag{4.4}$$

We can choose $Z_1, \ldots, Z_n$ suitably such that $A\bar{A}^t + B\bar{B}^t$ and $\bar{A}^t A + \bar{B}^t B$ are diagonal. Put

$$A\bar{A}^t + B\bar{B}^t = \begin{pmatrix} \mu_1 & & \\ & \ddots & \\ & & \mu_m \end{pmatrix}, \quad \bar{A}^t A + \bar{B}^t B = \begin{pmatrix} \mu_{m+1} & & \\ & \ddots & \\ & & \mu_n \end{pmatrix}. \tag{4.5}$$

Then from (4.4) it is clear that $\mu_1, \ldots, \mu_n$ are constant, and that $\phi^*\omega_{\alpha\beta} = 0$ when $\mu_\alpha \neq \mu_\beta$, while $\phi^*\omega_{ij} = 0$ when $\mu_i \neq \mu_j$. By virtue of (2.6) and (3.2) we get

$$d\phi = \sum_{\alpha, i}(a_{\alpha i} Z_i \bar{Z}_\alpha^t + \bar{b}_{\alpha i} Z_\alpha \bar{Z}_i^t)\varphi + \sum_{\alpha, i}(b_{\alpha i} Z_i \bar{Z}_\alpha^t + \bar{a}_{\alpha i} Z_\alpha \bar{Z}_i^t)\bar{\varphi}, \tag{4.6}$$

from which we can calculate out that

$$\begin{aligned} (Z_i\bar{Z}_\alpha^t)^\perp &= Z_i\bar{Z}_\alpha^t - \sum_{\beta, j}(\bar{a}_{\alpha i} a_{\beta j} + \bar{b}_{\alpha i} b_{\beta j}) Z_j \bar{Z}_\beta^t - \sum_{\beta, j}(\bar{a}_{\alpha i}\bar{b}_{\beta j} + \bar{b}_{\alpha i}\bar{a}_{\beta j}) Z_\beta \bar{Z}_j^t, \\ (Z_\alpha\bar{Z}_i^t)^\perp &= Z_\alpha\bar{Z}_i^t - \sum_{\beta, j}(a_{\alpha i} b_{\beta j} + b_{\alpha i} a_{\beta j}) Z_j \bar{Z}_\beta^t - \sum_{\beta, j}(a_{\alpha i}\bar{a}_{\beta j} + b_{\alpha i}\bar{b}_{\beta j}) Z_\beta \bar{Z}_j^t. \end{aligned} \tag{4.7}$$

Substituting (4.7) into the last equation in (4.3) we have

$$\begin{aligned} 0 &= A(\bar{A}^t A + \bar{B}^t B) + (A\bar{A}^t + B\bar{B}^t)A - 4\,\mathrm{tr}[(A\bar{A}^t + B\bar{B}^t)A\bar{B}^t]B \\ &\quad - [\mathrm{tr}(A\bar{A}^t + B\bar{B}^t)^2 + \mathrm{tr}(\bar{A}^t A + \bar{B}^t B)^2]A, \\ 0 &= B(\bar{A}^t A + \bar{B}^t B) + (A\bar{A}^t + B\bar{B}^t)B - 4\,\mathrm{tr}[(A\bar{A}^t + B\bar{B}^t)B\bar{A}^t]A \\ &\quad - [\mathrm{tr}(A\bar{A}^t + B\bar{B}^t)^2 + \mathrm{tr}(\bar{A}^t A + \bar{B}^t B)^2]B. \end{aligned} \tag{4.8}$$

(4.5) and (4.8) yields

$$\begin{aligned} a_{\alpha i}(\mu_\alpha + \mu_i) &= x a_{\alpha i} + y b_{\alpha i}, \\ b_{\alpha i}(\mu_\alpha + \mu_i) &= x b_{\alpha i} + \bar{y} a_{\alpha i}, \end{aligned} \tag{4.9}$$

where $x = \mu_1^2 + \cdots + \mu_n^2$, $y = 4\,\mathrm{tr}[(A\bar{A}^t + B\bar{B}^t)A\bar{B}^t]$. From (4.9) we get $y|b_{\alpha i}|^2 = y|a_{\alpha i}|^2$, so if $y \neq 0$, we must have $|a_{\alpha i}| = |b_{\alpha i}|$ for any α, i, and consequently, ϕ is totally real. We now assume that ϕ is not totally real, then $y = 0$, which together with (4.9) implies that either $a_{\alpha i} = b_{\alpha i} = 0$ or $\mu_\alpha + \mu_i = x$. We claim that for $\mu_{\alpha_0} \neq 0$, there exists i_0 such that $\mu_{\alpha_0} + \mu_{i_0} = x$. Otherwise, $a_{\alpha_0 i} = b_{\alpha_0 i} = 0$, so that $\mu_{\alpha_0} = \sum_i(|a_{\alpha_0 i}|^2 + |b_{\alpha_0 i}|^2) = 0$, which is a contradiction. Similarly, for $\mu_{i_0} \neq 0$, there exists α_0 such that $\mu_{\alpha_0} + \mu_{i_0} = x$. Hence we can assume that

$$A\bar{A}^t + B\bar{B}^t = \begin{pmatrix} a_1 I_{m_1} & & & 0 \\ & \ddots & & \\ & & a_s I_{m_s} & \\ 0 & & & 0_{m_0} \end{pmatrix},$$

$$\bar{A}^t A + \bar{B}^t B = \begin{pmatrix} b_1 I_{l_1} & & & 0 \\ & \ddots & & \\ & & b_s I_{l_s} & \\ 0 & & & 0_{l_0} \end{pmatrix}, \tag{4.10}$$

where $a_1, \dots, a_s$ (resp. $b_1, \dots, b_s$) are nonzero constants different from each other and satisfying $a_p + b_p = x \ (p = 1, \dots, s)$, and 0_{m_0} is the $m_0 \times m_0$-zero matrix. We remark that $\sum_{p=0}^{s} m_p = m$, $\sum_{p=0}^{s} l_p = n - m$, and m_0 and l_0 may equal to zero. In this situation, the matrices A, B can be written as

$$A = \begin{pmatrix} A_1 & & & \\ & \ddots & & \\ & & A_s & \\ & & & 0 \end{pmatrix}, \quad B = \begin{pmatrix} B_1 & & & \\ & \ddots & & \\ & & B_s & \\ & & & 0 \end{pmatrix},$$

where A_p, B_p are $(m_p \times l_p)$-matrices, and $A_p \bar{A}_p^t + B_p \bar{B}_p^t = a_p I_{m_p}$, $\bar{A}_p^t A_p + \bar{B}_p^t B_p = b_p I_{l_p}$, $p = 1, \dots, s$. Put $n_p = m_p + l_p$. First we consider the n_1-dimensional subspace π_1 defined by $Z_1 \wedge \cdots \wedge Z_{m_1} \wedge Z_{m+1} \wedge \cdots \wedge Z_{m+l_1}$ of C^n. Obviously it is a constant subspace of C^n and $Z_1 \wedge \cdots \wedge Z_{m_1}$ defines a m_1-dimensional subspace of π_1. We may consider π_1 as C^{n_1}. Then we can define a map $\phi_1 : M \to G(m_1, n_1) \subset HM(n_1)$, which A- and B-matrices coincide with A_1 and B_1 respectively. So a similar computation as in (3.11) shows that

$$\frac{1}{4}\Delta\phi_1 = (a_1 + b_1)\phi_1 - b_1 I_{n_1} = x\left(\phi_1 - \frac{m_1}{n_1} I_{n_1}\right),$$

which implies that ϕ_1 is mass-symmetric 1-type with eigenvalue $4x$. Similarly we can define $\phi_0, \phi_2, \dots, \phi_s$ so that $\phi = \phi_0 \oplus \phi_1 \oplus \cdots \oplus \phi_s$, where ϕ_0 is a constant matrix in $HM(n_0)$ with $\phi_0^2 = \phi_0$, and $\phi_p : M \to G(m_p, n_p)$ is mass-symmetric 1-type with eigenvalue $4x$, $p = 1, \dots, s$. Therefore, ϕ is of 1-type.

Conversely, if ϕ is of 1-type, from Theorems 3.1 and 3.3, we have the decomposition $\phi = \phi_0 \oplus \phi_1 \oplus \cdots \oplus \phi_s$ and (4.10). Here each $\phi_p \ (p = 1, \dots, s)$ is a map to $G(m_p, n_p) \subset HM(n_p)$ and satisfying that $\Delta\phi_p = \lambda(\phi_p - c_p I_p)$ for a constant

$\lambda = 4(a_p + b_p)$ and $c_p = m_p/n_p = b_p/\lambda$. We remark that λ is independent of p, and $a_p m_p = b_p l_p$ for every p. Then we obtain

$$x := \mathrm{tr}[(A\bar{A}^t + B\bar{B}^t)^2 + (\bar{A}^t A + \bar{B}^t B)^2] = \sum_{p=1}^{s} (m_p a_p^2 + l_p b_p^2)$$

$$= \frac{\lambda}{4} \sum_{p=1}^{s} m_p a_p = \frac{\lambda}{4} \mathrm{tr}(A\bar{A}^t + B\bar{B}^t) = \frac{\lambda}{4}. \tag{4.11}$$

Assume $y = 0$, then (4.11) implies that the equation (4.9) and hence the last equation in (4.3) holds. The first two equations in (4.3) follow from $d(A\bar{A}^t + B\bar{B}^t) = 0$ and $d(\bar{A}^t A + \bar{B}^t B) = 0$. Consequently g_ϕ is harmonic. So we have the following

Theorem 4.1. *Let $\phi : M \to G(m,n)$ be an isometric minimal immersion which is not totally real. Then the Gauss map g_ϕ of ϕ is harmonic if and only if ϕ is of* 1-*type and* $\mathrm{tr}[(A\bar{A}^t + B\bar{B}^t)A\bar{B}^t] = 0$.

Recall that ϕ is said to be strongly conformal if $A\bar{B}^t = 0$. The following theorem can be shown similarly.

Theorem 4.2. *Let $\phi : M \to G(m,n)$ be an isometric minimal immersion which is strongly conformal. Then the Gauss map g_ϕ of ϕ is harmonic if and only if ϕ is of* 1-*type*.

Acknowledgement

The author would like to thank the referee for helpful suggestions.

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