TSUKUBA J. MATH.
Vol. 22 No. 2 (1998), 379-392

ON D-PARACOMPACT p- AND Σ-SPACES

By

Norihito SHIMANE and Takemi MIZOKAMI

1. Introduction

All spaces are assumed to be T_1 topological spaces and all mappings to be continuous and onto. The letter N always denotes all positive integers and τ_X the topology of a space X.

As well known as Dowker's Theorem, a T_2-space X is paracompact if and only if for each open cover $\mathscr{U}$ of X there exists a $\mathscr{U}$-mapping f of X onto a metric space M, where a mapping f is called a *$\mathscr{U}$-mapping* if there exists an open cover $\mathscr{V}$ of M such that $f^{-1}(\mathscr{V}) < \mathscr{U}$. Taking into account that developable spaces is one of the nicest generalizations of metric spaces, it is quite natural to substitute a metric space M in the above with a developable space D in order to get a generalization of both paracompact spaces and developable spaces.

DEFINITION 1.1 [12]. A space X is called a *D-paracompact* if for each open cover $\mathscr{U}$ of X there exists a $\mathscr{U}$-mapping of X onto a developable spaces.

Pareek originally gave its inner characterization to D-paracompact spaces [12]. Besides many inner characterizations are given by Brandenburg [1], Chaber [6] and Mizokami [9]. As for the overview of D-paracompact spaces, refer to [2]. In this paper, we consider the mapping properties of D-paracompact spaces on the classes of D-paracompact p-spaces and D-paracompact Σ-spaces.

2. D-paracompact p-spaces

With respect to the mapping property of D-paracompact spaces, the following problem remains unsolved.

PROBLEM [1], [6]. Let $f: X \to Y$ be a perfect mapping of a D-paracompact space onto a space Y. Then is Y D-paracompact?

Received January 13, 1997

Let us note that D-paracompactness is preserved by neither of perfect preimages and closed images. The former is due to [6, Example 3.3] and the latter due to [9, Example 3]. But we have the following positive partial answers given by Chaber [6] and by Mizokami [9]: Let $\mathscr{C}$ be a class of spaces such that $\mathscr{C} \subset \{D\text{-paracompact spaces}\}$. Then $\mathscr{C}$ is closed under perfect images when $\mathscr{C}$ is either of the class of D-paracompact p-spaces [6] of D-paracompact σ-spaces [9]. According to his definition there [6], a space X is a D-paracompact p-space if and only if for any open cover $\mathscr{U}$ of X there exists a perfect $\mathscr{U}$-mapping of X onto a Moore space, that is a regular developable space. Originally, p-spaces are defined for completely regular spaces by Arhangelskii as follows: A completely regular space X is a p-space if X has a sequence $\{\mathscr{U}_n \mid n \in N\}$ of open covers of X in βX such that $\bigcap\{S(x, \mathscr{U}_n) \mid n \in N\} \subset X$ for each $x \in X$. A few inner characterizations are given by Burke [4], Burke and Stoltenberg [5] and Pareek [13]. But, as observed in Remark and the part preceding to Theorem 3.16 in [8, p. 442], since the Stone-Čech compactification βX can be changed by any compactification of X, their discussions are applicable to regular spaces. In this sense, we consider here p-spaces, strict p-spaces, Pareek's p-spaces for regular spaces. Pareek gave the definition of p-spaces in his paper and showed the equivalence of (iv) and (v) below [12, Theorem 4.4]. But this was criticized to be based on a dubious lemma by Mack [1974, Math. Reviews 47 (#1034)]. Here, we can show the equivalence by a different way.

THEOREM 2.1. *For a regular space X, the following are equivalent:*

(i) *X is a D-paracompact $w\Delta$-space.*

(ii) *X is a D-paracompact p-space in the sense of Burke* [4]. (*Refer to* [8, *Theorem* 3.21]).

(iii) *X is a D-paracompact strict p-space in the sense of Burke and Stoltenberg* [5]. (*Refer to* [8, *Theorem* 3.17]).

(iv) *X is a D-paracompact p-space in the sense of Pareek* [12, *Definition* 4.6].

(v) *For any open cover $\mathscr{U}$ of X, there exists a perfect $\mathscr{U}$-mapping of X onto a Moore space.*

(vi) *X is a D-paracompact space and has a perfect mapping of X onto a Moore space.*

PROOF. Since D-paracompact spaces are submetacompact, the arguments of [8, Theorem 3.19 and 3.21] can apply to get the equivalence of (i), (ii) and (iii). If we again note the remark in [8, p. 442], the discussion of [13] holds true for regular spaces, so that we have the equivalence of (iv) and (iii). (iii) $\rightarrow$ (v): Let $\mathscr{U}$

be an open cover of X and let $\{\mathscr{G}_n : n \in N\}$ be a strict p-sequence for X satisfying the following:

(1) $C_x = \bigcap\{S(x, \mathscr{G}_n) : n \in N\}$ is compact.

(2) $\{S(x, \mathscr{G}_n) : n \in N\}$ is an open neighborhood base of C_x in X.

Since X is regular and D-paracompact, for some open cover $\mathscr{V}_1$ of X such that $\overline{\mathscr{V}_1} < \mathscr{G}_1 \wedge \mathscr{U}$, there exists a $\mathscr{V}_1$-mapping f_1 of X onto a developable space D_1. Without loss of generality, we can assume that D_1 has a decreasing development $\{\mathscr{A}_{1n} : n \in N\}$ such that $f_1^{-1}(\mathscr{A}_{11}) < \mathscr{V}_1$. By regularity of X, there exists an open cover $\mathscr{V}_2$ of X such that

$$\overline{\mathscr{V}_2} < \mathscr{G}_2 \wedge f_1^{-1}(\mathscr{A}_{12}) \wedge \mathscr{U}.$$

Using D-paracompactness of X again, there exists a $\mathscr{V}_2$-mapping f_2 of X onto a developable space D_2 which has a decreasing development $\{\mathscr{A}_{2n} : n \in N\}$ such that $f_2^{-1}(\mathscr{A}_{21}) < \mathscr{V}_2$. Repeating this process, we can get sequences $\{\mathscr{V}_n : n \in N\}$, $\{\mathscr{A}_{ni} : i \in N\}$, $\{f_n : n \in N\}$ and $\{D_n : n \in N\}$ satisfying the following:

(3) D_n has a decreasing development $\{\mathscr{A}_{nk} : k \in N\}$ such that $f_n^{-1}(\mathscr{A}_{n1}) < \mathscr{V}_n$.

(4) For each n, f_n is a $\mathscr{V}_n$-mapping of X onto D_n.

(5) $\mathscr{V}_n$ is an open cover of X such that

$$\overline{\mathscr{V}_n} < \mathscr{G}_n \wedge \left(\bigwedge_{i=1}^{n-1} f_i^{-1}(\mathscr{A}_{in}) \right) \wedge \mathscr{U} \text{ for } n \geq 2.$$

Let $f = \prod f_i : X \to \prod D_i$ be defined by $f(x) = (f_i(x))_i$, $x \in X$. Then it is easily seen from (4) and (5) that f is a $\mathscr{U}$-mapping of X onto a developable space $D = f(X) \subset \prod D_n$. We show that f is a perfect mapping, and consequently D is a Moore space. For each $p \in D$, by virtue of (3) and (5) we have

$$f^{-1}(p) \subset \bigcap_n S(x, \mathscr{G}_n),$$

where $x \in f^{-1}(p)$. So, because of (1), $f^{-1}(p)$ is compact. To see the closedness of f, it suffices to show that for each point $p = (p_i)_i \in D$ and each open subset U of X such that $f^{-1}(p) \subset U$, there exists a neighborhood V of p in D such that $f^{-1}(V) \subset U$. Let

$$C_x = \bigcap_n S(x, \mathscr{G}_n), \quad x \in f^{-1}(p).$$

We can easily observe by virtue of (1) that $f(C_x \backslash U)$ is a compact subset of D and $p \notin f(C_x \backslash U)$. Take a neighborhood G of p in D such that

$$G = \left(\prod_{i=1}^{k} S(p_{n(i)}, \mathscr{A}_{n(i)m(i)}) \times \prod\{D_t : t \neq n(i)\} \right) \cap D$$

$$\bar{G} \cap f(C_x \setminus U) = \emptyset.$$

By virtue of (3), (4) and (5), we can find some $n(0) \in N$ such that

(6) $\overline{f_{n(0)}^{-1}(S(p_{n(0)}, \mathscr{A}_{n(0)1}))} \cap (C_x \setminus U) = \emptyset.$

Set

$$O = X \setminus (\overline{f_{n(0)}^{-1}(S(p_{n(0)}, \mathscr{A}_{n(0)1}))} \setminus U).$$

Then O is an open neighborhood of C_x. By virtue of (2), there exists $s \in N$ such that

(7) $C_x \subset S(x, \mathscr{G}_s) \subset O.$

Using all of (3) through (7), we can find some $t \in N$ such that

$$V = \left(S(p_t, \mathscr{A}_{t1}) \times \prod\{D_n : n \neq t\} \right) \cap D$$

is an open neighborhood of p in D such that $f^{-1}(V) \subset U$. Hence f is a perfect mapping. Since (vi) $\to$ (i) is trivial, we have completed the proof. □

Let us note that in most cases, D-paracompact p-spaces go parallel to paracompact p-spaces. For example, the following theorem on making the space Moore corresponds to the metrization theorem of paracompact p-spaces.

THEOREM 2.2. *A regular D-paracompact p-space X is a Moore space if and only if X has a G_δ-diagonal.*

PROOF. Only if part is trivial. If part: Let $\{\mathscr{U}_n : n \in N\}$ be a sequence of open covers of X such that $\bigcap_n S(p, \mathscr{U}_n) = \{p\}$ for each point $p \in X$. By the above theorem, for each n there exists a perfect $\mathscr{U}_n$-mapping f_n of X onto a Moore space D_n. Let $f : X \to \prod_n D_n$ be defined by

$$f(x) = (f_n(x))_n, \quad x \in X.$$

Then easily we can observe that f is a homeomorphism of X onto $f(X) \subset \prod_n D_n$. Since Moore spaces have countably productive and hereditary properties, $f(X)$ is a Moore space. This completes the proof. □

Nagata characterized a paracompact p-space as a space which is embedded in the closed subspace of the product of a metrizable space and a compact space

[11]. But this type of characterization does not work for D-paracompact p-spaces stated below:

THEOREM 2.3. *A regular D-paracompact p-space is embedded in a closed subspace of the product of a Moore space and a compact space. But the converse is not true.*

PROOF. The former is straightforward from [8, Lemma 3.13] and Theorem 2.1. For the latter, it suffices to consider the product space of a Moore space $S = N \cup \mathscr{A}$ and a compact space $Z = A(\aleph_1)$ for which $S \times Z$ is not D-paracompact [6, Example 3.3].

3. D-paracompact Σ-spaces

As stated above, D-paracompact p-spaces and D-paracompact σ-spaces are preserved by perfect mappings. Both are Σ-spaces in the sense of Nagami. So it is quite natural to ask whether D-paracompact Σ-spaces are preserved by perfect mappings. In this section, we give the positive answer to it. Here, we use the definition of Σ-spaces due to Michael, which is equivalent to the original one due to Nagami.

DEFINITION 3.1 [8, Definition 4.13]. A regular space X is called a (*strong*) Σ-*space* if X has a cover $\mathscr{C}$ by (resp. compact) countably compact subsets and has a σ-locally finite family $\mathscr{F}$ of closed subsets of X such that for $C \in \mathscr{C}$ and $U \in \tau_X$, if $C \subset U$, then $C \subset F \subset U$ for some $F \in \mathscr{F}$.

Since D-paracompact space is subparacompact, a D-paracompact Σ-space is a strong Σ-space. We state the terminology used in the proof. We call $\mathscr{P}$ a *pair-collection* of a space X if $\mathscr{P}$ is a collection of ordered pairs $P = (P_1, P_2)$ of subsets of X such that $P_1 \subset P_2$ and P_1, P_2 are closed, open in X, respectively. We call $\mathscr{P}$ *discrete*, *locally finite*, *σ-discrete* or *σ-locally finite in X* if the family $\{P_1 : P \in \mathscr{P}\}$ is so in X, that is, each point p of X has a neighborhood in X intersecting P_1 for at most one $P \in \mathscr{P}$, and so forth. Let $\mathscr{U}$ be a family of open subsets of X. Then we call that $\mathscr{P}$ is a *pair-network for $\mathscr{U}$ in X* if for each point $p \in X$ and each $U \in \mathscr{U}$, if $p \in U$, then $p \in P_1 \subset P_2 \subset U$ for some $P = (P_1, P_2) \in \mathscr{P}$. As known already [7], a space X is developable if and only if there exists a σ-discrete pair-network for the topology τ_X of X. We prepare two lemmas for the main theorem.

LEMMA 3.2. *Let X be a subparacompact space and let $\mathscr{F}$ be a locally finite family of closed subsets of X and $\{U(F) : F \in \mathscr{F}\}$ its open expansion in X. Then there exists a σ-discrete pair-collection $\mathscr{P}$ of X such that for each point $p \in X$ and each $F \in \mathscr{F}$, if $p \in F$, then $p \in P_1 \subset P_2 \subset U(F)$ for some $P = (P_1, P_2) \in \mathscr{P}$.*

PROOF. For each point $p \in X$, take an open neighborhood $V(p)$ of p in X such that

$$V(p) \subset X \setminus \bigcup \{F \in \mathscr{F} : p \notin F\}$$

and such that if $p \in \bigcup \mathscr{F}$, then

$$V(p) \subset \bigcap \{U(F) : p \in F \in \mathscr{F}\}.$$

By subparacompactness of X, there exists a σ-discrete closed refinement $\mathscr{H}$ of $\{V(p) : p \in X\}$. For each $H \in \mathscr{H}$ with $H \cap \left(\bigcup \mathscr{F}\right) \neq \emptyset$, choose an open subset $W(H)$ of X such that

$$H \subset W(H) \subset \bigcap \{U(F) : F \cap H \neq \emptyset\}.$$

Then

$$\mathscr{P} = \left\{(H, W(H)) : H \in \mathscr{H} \text{ with } H \cap \left(\bigcup \mathscr{F}\right) \neq \emptyset\right\}$$

is the required pair-collection of X. □

For brevity, in the next lemma we call that a space X satisfies the *condition* $(*)$ if for each discrete pair-collection $\{(F, U(F)) : F \in \mathscr{F}\}$ of X there exists a pair $\langle \mathscr{V}, \mathscr{P} \rangle$ of a family $\mathscr{V}$ of subsets of X and a σ-discrete pair-collection $\mathscr{P}$ of X satisfying the following (1) and (2):

(1) $\mathscr{V} = \{V(F) : F \in \mathscr{F}\}$ is an open expansion of $\mathscr{F}$ in X such that $F \subset V(F) \subset U(F)$ for each $F \in \mathscr{F}$.

(2) For each point $p \in X$ and each $F \in \mathscr{F}$ if $p \in V(F)$ then $p \in P_1 \subset P_2 \subset U(F)$ for some $P = (P_1, P_2) \in \mathscr{P}$.

(We call the pair $\langle \mathscr{V}, \mathscr{P} \rangle$ the *$(*)$-pair for* $\{(F, U(F)) : F \in \mathscr{F}\}$.)

LEMMA 3.3. *Let X be a subparacompact space satisfying the condition $(*)$. Then X is D-paracompact.*

PROOF. By [1, Theorem 1, (iii)], it suffices to show that X is *D-expandable*, that is, for each discrete pair-collection $\{(F, U(F)) : F \in \mathscr{F}\}$ of X with $F \cap U(F') = \emptyset$ if $F \neq F'$ and F, $F' \in \mathscr{F}$, there exists a "dissectable" family $\mathscr{V} =$

$\{V(F) : F \in \mathscr{F}\}$ of open subsets of X such that $F \subset V(F) \subset U(F)$ for each $F \in \mathscr{F}$. To show the existence of such $\mathscr{V}$, by argument of the proof of [1, Theorem 1, (ii) $\to$ (iii)], it suffices to find a σ-discrete pair-network $\mathscr{P}$ for $\mathscr{V}$ in X. Thus we will construct such $\mathscr{V}$ and $\mathscr{P}$ for a given discrete pair-collection $\{(F, U(F)) : F \in \mathscr{F}\}$ of X. First, by (*) there exists a (*)-pair $\langle \mathscr{V}_1, \mathscr{P}_1 \rangle$ for $\{(F, U(F)) : F \in \mathscr{F}\}$ satisfying (1) and (2):

(1) $\mathscr{V}_1 = \{V_1(F) : F \in \mathscr{F}\}$ is an open expansion of $\mathscr{F}$ such that $F \subset V_1(F) \subset U(F)$ for each $F \in \mathscr{F}$.

(2) $\mathscr{P}_1$ is a σ-discrete pair-collection of X such that for each $p \in X$ and each $F \in \mathscr{F}$, if $p \in V_1(F)$, then $p \in P_1 \subset P_2 \subset U(F)$ for some $P = (P_1, P_2) \in \mathscr{P}_1$. Write $\mathscr{P}_1 = \bigcup\{\mathscr{P}_{1n} : n \in N\}$, where each $\mathscr{P}_{1n} = \{P_\alpha : \alpha \in A_{1n}\}$ is a discrete pair-collection of X. By (*), for each n there exists a (*)-pair

$$\langle \{P'_{\alpha 2} : \alpha \in A_{1n}\}, \mathscr{P}_{2n} \rangle$$

for $\mathscr{P}_{1n}$ satisfying the following (3) and (4):

(3) $P_{\alpha 1} \subset P'_{\alpha 2} \subset P_{\alpha 2}$ for each $\alpha \in A_{1n}$.

(4) $\mathscr{P}_{2n}$ is a σ-discrete pair-collection of X such that for each $\alpha \in A_{1n}$ and each $p \in X$, if $p \in P'_{\alpha 2}$, then $p \in P_1 \subset P_2 \subset P_{\alpha 2}$ for some $P = (P_1, P_2) \in \mathscr{P}_{2n}$. For each $F \in \mathscr{F}$ set

$$V_2(F) = \bigcup \left\{ P'_{\alpha 2} : \alpha \in \bigcup_n A_{1n}, P_{\alpha 1} \cap V_1(F) \neq \emptyset \text{ and } P_{\alpha 2} \subset U(F) \right\}$$

and set

$$\mathscr{P}'_1 = \left\{ (P_{\alpha 1}, P'_{\alpha 2}) : \alpha \in \bigcup_n A_{1n} \right\}.$$

Then $\{V_2(F) : F \in \mathscr{F}\}$ is an open expansion of $\mathscr{F}$ and $\mathscr{P}'_1$ is a σ-discrete pair-collection of X such that for each $p \in X$ and each $F \in \mathscr{F}$, if $p \in V_1(F)$, then $p \in P_1 \subset P_2 \subset V_2(F)$ for some $P = (P_1, P_2) \in \mathscr{P}'_1$. Write each σ-discrete pair-collection $\mathscr{P}_{2n}$ as

$$\mathscr{P}_{2n} = \bigcup\{\mathscr{P}_{2nm} : m \in N\},$$

where each $\mathscr{P}_{2nm} = \{(P_{\alpha 1}, P_{\alpha 2}) : \alpha \in A_{2nm}\}$ is a discrete pair-collection of X. For each n, $m \in N$, by (*) there exists a (*)-pair

$$\langle \{P'_{\alpha 2} : \alpha \in A_{2nm}\}, \mathscr{P}_{3nm} \rangle$$

for $\mathscr{P}_{2nm}$ satisfying the following (5) and (6):

(5) $P_{\alpha 1} \subset P'_{\alpha 2} \subset P_{\alpha 2}$ for each $\alpha \in A_{2nm}$.

(6) $\mathscr{P}_{3nm}$ is a σ-discrete pair-collection of X such that for each $\alpha \in A_{2nm}$ and each $P \in X$, if $p \in P'_{\alpha 2}$, then $p \in P_1 \subset P_2 \subset P_{\alpha 2}$ for some $P = (P_1, P_2) \in \mathscr{P}_{3nm}$. Set

$$V_3(F) = \bigcup \{P'_{\alpha 2} : \alpha \in \bigcup \{A_{2nm} : n, m \in N\}, P_{\alpha 1} \cap V_2(F) \neq \emptyset \text{ and } P_{\alpha 2} \subset U(F)\}$$

for each $F \in \mathscr{F}$ and set

$$\mathscr{P}'_2 = \{(P_{\alpha 1}, P'_{\alpha 2}) : \alpha \in \bigcup \{A_{2nm} : n, m \in N\}\}.$$

Then $\{V_3(F) : F \in \mathscr{F}\}$ is an open expansion of $\mathscr{F}$ satisfying the following (7) and (8):

(7) $F \subset V_1(F) \subset V_2(F) \subset V_3(F) \subset U(F)$ for each $F \in \mathscr{F}$.

(8) $\mathscr{P}'_2$ is a σ-discrete pair-collection of X such that for each $p \in X$ and each $F \in \mathscr{F}$, if $p \in V_2(F)$, then $p \in P_1 \subset P_2 \subset V_3(F)$ for some $P = (P_1, P_2) \in \mathscr{P}'_2$.

By repeating this process, we can construct a sequence $\{V_n(F) : F \in \mathscr{F}\}$ of open expansion of $\mathscr{F}$ and a sequence $\{\mathscr{P}'_n : n \in N\}$ of σ-discrete pair-collections of X satisfying the following (9) and (10):

(9) $F \subset V_1(F) \subset V_2(F) \subset \cdots \subset V_n(F) \subset V_{n+1}(F) \subset \cdots \subset U(F)$ for each $F \in \mathscr{F}$.

(10) For each $p \in X$ and $F \in \mathscr{F}$, if $p \in V_n(F)$, then $p \in P_1 \subset P_2 \subset V_{n+1}(F)$ for some $P = (P_1, P_2) \in \mathscr{P}'_n$.

Set

$$V(F) = \bigcup \{V_n(F) : n \in N\} \text{ for each } F \in \mathscr{F}$$

and

$$\mathscr{P}' = \bigcup \{\mathscr{P}'_n : n \in N\}.$$

Then each $V(F)$ is an open subset of X such that $F \subset V(F) \subset U(F)$ and obviously $\mathscr{P}'$ is a σ-discrete pair-network for $\{V(F) : F \in \mathscr{F}\}$ in X. This completes the proof. □

For a closed mapping $f : X \to Y$, we use the following notation: For each open subset U of X, we write

$$f^*(U) = Y \backslash f(X \backslash U),$$

which is open in Y.

THEOREM 3.4. *Let f be a perfect mapping of a space X onto a space Y. If X is a D-paracompact Σ-space, then so is Y.*

PROOF. By [10, Theorem 1.8], Y is a Σ-space. Since subparacompactness is preserved by perfect mappings, Y is subparacompact. Thus by Lemma 3.3, it suffices to show that Y satisfies the condition $(*)$. Let $\{(F, U(F)) : F \in \mathscr{F}\}$ be a discrete pair-collection of Y. We may assume that $F \cap U(F') = \emptyset$ for F, $F' \in \mathscr{F}$ with $F \neq F'$. Since X is D-paracompact, there exists a $\mathscr{U}_1$-mapping g_1 of X onto a developable space D_1, where

$$\mathscr{U}_1 = \{f^{-1}(U(F)) : F \in \mathscr{F}\} \cup \{X \setminus \bigcup f^{-1}(\mathscr{F})\}.$$

Obviously there exists an open expansion $\{V_1(F) : F \in \mathscr{F}\}$ of $f^{-1}(\mathscr{F})$ in X such that for each $F \in \mathscr{F}$

$$f^{-1}(F) \subset V_1(F) \subset f^{-1}(U(F))$$

and $V_1(F) = g_1^{-1}(O)$ with O open in D_1. For each $F \in \mathscr{F}$,

$$V_1(F)^* = f^{-1}(f^*(V_1(F)))$$

is an open subset of X such that

$$f^{-1}(F) \subset V_1(F)^* \subset V_1(F) \subset f^{-1}(U(F)).$$

Using the D-paracompactness of X, there exists a $\mathscr{U}_2$-mapping g_2 of X onto a developable space D_2, where

$$\mathscr{U}_2 = \{V_1(F)^* : F \in \mathscr{F}\} \cup \{X \setminus \bigcup f^{-1}(\mathscr{F})\}.$$

Then there exists an open expansion $\{V_2(F) : F \in \mathscr{F}\}$ of $f^{-1}(\mathscr{F})$ in X such that for each $F \in \mathscr{F}$

$$f^{-1}(F) \subset V_2(F) \subset V_1(F)^*$$

and $V_2(F) = g_2^{-1}(O)$ with O open in D_2. Let $g : X \to g(X) \subset D_1 \times D_2$ be a mapping defined by

$$g(x) = (g_1(x), g_2(x)) \text{ for each } x \in X.$$

Obviously both $V_1(F)$ and $V_2(F)$ are the inverse images of open subsets of $X' = g(X)$ for each $F \in \mathscr{F}$. Since X' is a developable space, there exists a σ-discrete pair-network $\mathscr{P}'$ for the topology of X'. Set

$$\mathscr{P} = \{(g^{-1}(P_1), g^{-1}(P_2)) : P = (P_1, P_2) \in \mathscr{P}'\}.$$

and write newly

$$\mathscr{P} = \{(F_\alpha, V_\alpha) : \alpha \in A_n' \text{ and } n \in N\}.$$

where for each n, $\{F_\alpha : \alpha \in A'_n\}$ is a discrete family of closed subsets of X. Obviously $\mathscr{P}$ satisfies the following (1):

(1) $\mathscr{P}$ is a pair-network for $\{V_1(F), V_2(F) : F \in \mathscr{F}\}$ in X.

By the definition of a strong Σ-space, Y has a cover $\mathscr{C}$ by compact subsets and has a σ-locally finite family $\mathscr{H} = \{H_\lambda : \lambda \in \Lambda\}$ of closed subsets of Y such that:

(2) For each $O \in \tau_Y$ and each $C \in \mathscr{C}$, if $C \subset O$, then $C \subset H_\lambda \subset O$ for some $\lambda \in \Lambda$.

Without loss of generality, we can assume that $\mathscr{H}$ is closed under any finite intersections. For each n, let $A_n = \bigcup\{A'_i : i \leq n\}$. Then $\{F_\alpha : \alpha \in A_n\}$ is locally finite in X and $A_n \subset A_{n+1}$. For each n, let Δ_n be the totality of finite subsets of A_n and for each $(\delta, \lambda) \in \Delta_n \times \Lambda$, $(\delta, \delta') \in \Delta_n \times \Delta_m$, n, $m \in N$, set

$$F(\delta) = \bigcap\{f(F_\alpha) : \alpha \in \delta\},$$

$$f(\delta, \lambda) = F(\delta) \cap H_\lambda,$$

$$W(\delta) = f^*(\bigcup\{V_\alpha : \alpha \in \delta\}),$$

$$W(\delta, \delta') = W(\delta) \cup W(\delta').$$

For each n, $m \in N$ let $T(m, n)$ be the set of all combinations $(\delta_1, \lambda, n) \in \Delta_m \times \Lambda \times \{n\}$ such that

$$A_n(\delta_1, \lambda) = \{\alpha \in A_n : f(F_\alpha) \cap (F(\delta_1, \lambda) \backslash W(\delta_1)) \neq \emptyset\}$$

is finite. ($T(m, n)$ may be empty for some m, n.) For each combination $(\delta_1, \lambda, n) \in T(m, n)$, let

$$\Delta(\delta_1, \lambda, n) = \{\delta_2 \in \Delta_n : \delta_2 \subset A_n(\delta_1, \lambda) \text{ and } F(\delta_1, \lambda) \subset W(\delta_1) \cup W(\delta_2)\}.$$

From the definition of $T(m, n)$, $\Delta(\delta_1, \lambda, n)$ is finite. For each $\delta_2 \in \Delta(\delta_1, \lambda, n)$ with $(\delta_1, \lambda, n) \in T(m, n)$, m, $n \in N$, construct an order pair of subsets of Y

$$P(\delta_1, \lambda, \delta_2) = (P_1(\delta_1, \lambda, \delta_2), P_2(\delta_1, \lambda, \delta_2))$$

where

$$P_1(\delta_1, \lambda, \delta_2) = F(\delta_1, \lambda)$$

and

$$P_2(\delta_1, \lambda, \delta_2) = W(\delta_1, \delta_2).$$

Set

$$\mathscr{P}(\delta_1, \lambda, n) = \{P(\delta_1, \lambda, \delta_2) : \delta_2 \in \Delta(\delta_1, \lambda, n)\}$$

and

$$\mathcal{Q} = \bigcup\{\mathcal{P}(\delta_1, \lambda, n) : (\delta_1, \lambda, n) \in T(m, n) \text{ and } m, n \in N\}.$$

Then obviously $\mathcal{Q}$ is a σ-locally finite pair-collection of Y. We establish the following claim:

Claim: For each $p \in Y$ and each $F \in \mathcal{F}$, if $p \in f^*(V_2(F))$, then $p \in Q_1 \subset Q_2 \subset f^*(V_1(F))$ for some $Q = (Q_1, Q_2) \in \mathcal{Q}$.

Suppose $p \in f^*(V_2(F))$. Then $f^{-1}(p) \subset V_2(F)$. By the compactness of $f^{-1}(p)$ and by (1), there exists $n_0 \in N$ such that for each $n \geq n_0$ there exists $\delta_n \in \Delta_n$ such that

$$f^{-1}(p) \cap F_\alpha \neq \emptyset \text{ for each } \alpha \in \delta_n,$$
$$f^{-1}(p) \subset \bigcap\{V_\alpha : \alpha \in \delta_n\} \subset V_2(F)$$

and $\delta_n \subset \delta_{n+1}$, which imply

$$p \in F(\delta_n) \cap W(\delta_n), \quad W(\delta_n) \subset f^*(V_2(F)).$$

Take $C \in \mathcal{C}$ with $p \in C$ and let $\{H_{\lambda(i)} : i \in N\}$ be a decreasing sequence of members of $\mathcal{H}$ containing C satisfying the following (3):

(3) For each $O \in \tau_Y$, if $C \in O$, then $C \subset H_{\lambda(i)} \subset O$ for some i.

In fact, such a sequence $\{H_{\lambda(i)}\}$ exists because of (2) and of the assumption on $\mathcal{H}$. We show the following (4):

(4) For each $t \in N$, there exists $i_0 \in N$ such that

$$(\delta_{n_0}, \lambda(i_0), t) \in T(n_0, t).$$

To show (4), assume the contrary, i.e., for some $s \in N$, $A_s(\delta_{n_0}, \lambda(i))$ is infinite for each i. Then, since $\{f(F_\alpha) : \alpha \in A_s\}$ is locally finite in Y, we can choose a sequence $\{\alpha_i : i \in N\} \subset A_s$ and a sequence $\{p_i : i \in N\}$ of points of Y such that $p_i \in Y \backslash \{p_1, \dots, p_{i-1}\}$ and

$$p_i \in f(F_{\alpha_i}) \cup (F(\delta_{n_0}, \lambda(i)) \backslash W(\delta_{n_0}))$$

and $F_{\alpha_i} \neq F_{\alpha_j}$ whenever $i \neq j$. By (3) $\{p_i : i \in N\}$ has a cluster point in Y. But this is a contradiction, because $p_i \in f(F_{\alpha_i})$ for each i. This establishes (4). Since

$$C \cap (F(\delta_{n_0}) \backslash W(\delta_{n_0}))$$

is a compact subset and is contained in $f^*(V_1(F))$, there exists $n_1 \geq n_0$ and $\delta_1 \in \Delta_{n_1}$ such that

$$C \cap (F(\delta_{n_0}) \backslash W(\delta_{n_0})) \subset W(\delta_1) \subset f^*(V_1(F)).$$

Using (4), there exists $i_1 \in N$ such that $(\delta_{n_0}, \lambda(i_1), n_1) \in T(n_0, n_1)$. By (3), we can easily find $i_2 \geq i_1$ such that

$$F(\delta_{n_0}, \lambda(i_2)) \subset W(\delta_{n_0}, \delta_1).$$

Since $\{H_{\lambda(i)}\}$ is decreasing, it is obvious that $(\delta_{n_0}, \lambda(i_2), n_1) \in T(n_0, n_1)$. Recalling the definition of $\mathscr{P}(\delta_{n_0}, \lambda(i_2), \delta_1)$, we have

$$p \in P_1(\delta_{n_0}, \lambda(i_2), \delta_1) \subset P_2(\delta_{n_0}, \lambda(i_2), \delta_1) \subset f^*(V_1(F))$$

and $P(\delta_{n_0}, \lambda(i_2), \delta_1) \in \mathscr{Q}$. This establishes the validity of the claim. Using Lemma 3.3, we can conclude that Y is D-paracompact. This completes the proof. □

Finally, we give a positive result to the mapping property of D-paracompact spaces. To state it, we need the definition of β-spaces. Σ-spaces and Moore spaces are β-spaces [8, Theorem 7.8(i)].

DEFINITION 3.5 [8, Definition 7.7]. A space X is called a *β-space* if there exists a β-function $g : N \times X \to \tau_X$ such that

(i) $x \in g(n, x)$ for each $n \in N$, $x \in X$.

(ii) If $x \in g(n, x_n)$ for each $n \in N$, then $\{x_n : n \in N\}$ has a cluster point in X.

THEOREM 3.6. *Let $f : X \to Y$ be a perfect mapping. If X is a D-paracompact β-space with a G_δ-diagonal, then Y is a D-paracompact β-space.*

PROOF. Since as easily checked β-spaces are preserved by perfect mappings, Y has a β-function $g : N \times Y \to \tau_Y$. To see that Y satisfies the condition (*) in Lemma 3.3, let $\{(F, U(F)) : F \in \mathscr{F}\}$ be a discrete pair-collection. Without loss of generality, we can assume that $U(F) \cap F' = \emptyset$ whenever $F \neq F'$. Since X is subdevelopable [12, Proposition 5.1], in the sense of [3], there exists a one-to-one $\mathscr{U}$-mapping h of X onto a developable space D, where

$$\mathscr{U} = \{f^{-1}(U(F)) : F \in \mathscr{F}\} \cup \{X \setminus \bigcup f^{-1}(\mathscr{F})\}.$$

Then there exists a family $\mathscr{V} = \{V(F) : F \in \mathscr{F}\}$ of open subsets of X and a σ-locally finite pair-network

$$\mathscr{P} = \{(F_\alpha, V_\alpha) : \alpha \in A_n, n \in N\}$$

for $\mathscr{V} \cup h^{-1}(\tau_D)$ in X satisfying the following:

(1) $f^{-1}(F) \subset V(F) \subset f^{-1}(U(F))$, $F \in \mathscr{F}$.

(2) For each n, $\{F_\alpha : \alpha \in A_n\}$ is locally finite in X and $A_n \subset A_{n+1}$.

(3) For each $p \in X$ and $F \in \mathscr{F}$, if $p \in V(F)$, then there exists $\alpha \in A_n$, $n \in N$, such that $p \in F_\alpha \subset V_\alpha \subset V(F)$.

Let Δ_n be the totality of finite subsets of A_n and for each $\delta \in \Delta_n$, $k \in N$, let

$$H(\delta, k) = \bigcap\{f(F_\alpha) : \alpha \in \delta\} \setminus \bigcup\{g(k, y) : y \in K(\delta)\},$$
$$K(\delta) = \bigcap\{f(F_\alpha) : \alpha \in \delta\} \setminus f^*(\bigcup\{V_\alpha : \alpha \in \delta\})$$

and

$$W(\delta, k) = f^*(\bigcup\{V_\alpha : \alpha \in \delta\}).$$

Then obviously $H(\delta, k) \subset W(\delta, k)$ for each δ and k, and by virtue of (2), $\{H(\delta, k) : \delta \in \Delta_n\}$ is locally finite in Y. Construct the pair-collection of Y

$$\mathscr{Q} = \{(H(\delta, k), W(\delta, k)) : \delta \in \Delta_n, k, n \in N\}.$$

Then we show that $\mathscr{Q}$ is a σ-locally finite pair-network for $\mathscr{W} = \{W(F) : F \in \mathscr{F}\}$ in Y, where $W(F) = f^*(V(F))$, $F \in \mathscr{F}$. It is trivial that $\mathscr{Q}$ is σ-locally finite in Y. To see that $\mathscr{Q}$ is a pair-network for $\mathscr{W}$ in Y, let $p \in W(F)$, $F \in \mathscr{F}$. Then there exists a sequence $\{\delta_n : n \geq n_0\}$ with $\delta_n \in \Delta_n$ for each $n \geq n_0$, satisfying for each $n \geq n_0$

$$p \in W(\delta_n, k), \quad \delta_n \subset \delta_{n+1} \text{ and}$$
$$\delta_n = \{\alpha \in A_n : F_\alpha \cap f^{-1}(p) \neq \emptyset \text{ and } V_\alpha \subset V(F)\}.$$

In this case we have $\bigcap\{K(\delta_n) : n \geq n_0\} = \emptyset$. For, if $q \in \bigcap_n K(\delta_n)$, then $q \in \bigcap\{f(F_\alpha) : \alpha \in \delta_n\}$ for each n, which implies

$$h(f^{-1}(p)) \cap h(f^{-1}(q)) \neq \emptyset,$$

but this is a contradiction to $f^{-1}(p) \cap f^{-1}(q) = \emptyset$. Assume $p \notin H(\delta_n, n)$ for each n. Then $p \in g(n, p_n)$ for some point $p_n \in K(\delta_n)$. Since g is a β-function, $\{p_n\}$ has a cluster point p_0, which must belong to $\bigcap_n K(\delta_n)$. But this is a contradiction to the above. Hence we have

$$p \in Q_1 \subset Q_2 \subset W(F)$$

for some $Q = (Q_1, Q_2) \in \mathscr{Q}$. This completes the proof. □

Remark. (i) Y need not have a G_δ-diagonal. In fact, there exists a perfect mapping of a disjoint topological sum of two Michael lines onto a space which has no G_δ-diagonal [14].

(ii) This theorem is not a corollary to the result in [9] that if X is a perfect image of a perfect D-paracompact space, then so is X because there exists a compact subdevelopable space X but not perfect.

References

[1] Brandenburg, H., On D-paracompact spaces, Top. Appl. **20** (1985), 17–25.
[2] Brandenburg, H., Separation axioms, covering properties, and inverse limits generated by developable topological spaces, Dissertations Math. CCLXXXIV (WARSZAWA, 1989).
[3] Brandenburg, H., Husek, M., On mapping from products into developable spaces, Top. Appl. **26** (1987), 229–238.
[4] Burke, D., On p-spaces and $w\Delta$-spaces, Pacific J. Math. **11** (1970), 105–126.
[5] Burke, D., Stoltenberg, R., A note on p-spaces and Moore spaces, Pacific J. Math. **30** (1969), 601–608.
[6] Chaber, J., On d-paracompactness and related properties, Fund. Math. **122** (1985), 175–186.
[7] Green, J. W., Completion and semicompletion of Moore spaces, Pacific J. Math. **57** (1975), 153–165.
[8] Gruenhage, G., Generalized Metric Spaces in Handbook of Set-theoretic Topology. North-Holland, 1984.
[9] Mizokami, T., On D-paracompact σ-spaces, Tsukuba J. Math. **15** (1991), 425–449.
[10] Nagami, K., Σ-spaces, Fund. Math. **65** (1969), 169–192.
[11] Nagata, J., A note on M-spaces and topologically complete spaces, Proc. Japan Acad. **45** (1969), 541–543.
[12] Pareek, C. M., Moore spaces, semi-metric spaces and continuous mappings connected with them. Canad. J. Math. **24** (1972), 1033–1042.
[13] Pareek, C. M., Characterization of p-spaces, Canad. Math. Bull. **14** (1971), 459–460.
[14] Popov, V., A perfect map need not preserve G_δ-diagonal, General Top. Appl. **7** (1977), 31–33.

The joint Graduate School (Ph. D. Program)
in Science of School Education,
Department of Natural Science Education
Yashiro, Katogun, Hyogo 673-1494
Japan

Department of Mathematics
Joetsu University of Education
Joetsu, Niigata 943-8512
Japan