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ON THE BRAIDED STRUCTURES OF BICROSSPRODUCT HOPF ALGEBRAS*

By

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Abstract. In this paper we show that if $H \star A$ is a bicrossproduct Hopf algebra then $(H \star A, \sigma)$ is braided if and only if σ has a unique form: $\sigma(h\otimes a, g\otimes b) = \sum \beta(h_1, g_1)\omega(h_2, g_{2(-1)})\omega(h_3, b_1)\alpha(a_1, b_2)\tau(a_2, g_{20})$ such that β, ω, τ and α satisfy certain compatible conditions. The result is applied to a certain bicrossproduct of H and H^{cop}, where H is a Hopf algebra with bijective antipode.

An appropriately general setting in which to view the basic constructions is that of a braided Hopf algebra. Braided Hopf algebras are known as dual quasitriangular, coquasitriangular Hopf algebras. They play the role of the dual of a quasitriangular Hopf algebra and include all of the standard, multiparameter, and nonstandard quantizations of semisimple algebraic groups. Let A and H be two Hopf algebras such that H acts on A, A coacts on H, and the smash product multiplication together the smash coproduct comulitiplication on $H \otimes A$ make this a Hopf algebra, called a bicrossproduct Hopf algebra and denoted by $H \star A$ see ref. [Maj4]. It is natural that we ask when $H \star A$ admits a braided structure, and what forms the braided structure σ of $H \star A$ will take if $H \star A$ admits a braided structure.

In this paper we give a positive answer to the question above. We find necessary and sufficient conditions for A and H such that their bicrossproduct is a braided Hopf algebra. The result is applied to a certain bicrossproduct of H and H^{cop}, where H is a Hopf algebra with bijective antipode.

The paper is organized as follows. Section 1 contains a survey of known definitions and results for the bicrossproduct $H \star A$ obtained by S. Majid in ref. [Maj4], which will serve as a backgroud for our results. We also introduce some new notions (see Definition 1.3–1.4 below).

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In Section 2, we discuss the braided structures of $H \star A$. We show that $(H \star A, \sigma)$ is braided if and only if there exist some bilinear forms $\alpha : A \otimes A \to k$, $\omega : H \otimes A \to k$, $\tau : A \otimes H \to k$ and $\beta : H \otimes H \to k$, satisfying certain compatible conditions such that (A, α) is braided, (H, A, ω) is a dual ω-Hopf algebra pair, (A, H, τ) is an anti-skew compatible τ-Hopf algebra pair and (H, β) is a braided-like Hopf algebra associated to (ω, τ, δ_H), where δ_H is a comodule structure map of H, and σ has a unique form:

$$\sigma(h \otimes a, g \otimes b) = \sum \beta(h_1, g_1)\omega(h_2, g_{2(-1)})\omega(h_3, b_1)\alpha(a_1, b_2)\tau(a_2, g_{20})$$

What we do in Section 3 is to give a braided structure over $H \star H^{cop}$, where H is a Hopf algebra with bijective antipode.

1. Preliminaries

Throughout this paper, unless otherwise explicittly stated, k denotes an arbitrary field, $\otimes = \otimes_k$, and H is a Hopf algebra over the field k with a multiplication m_H, unit μ_H, comultiplication Δ_H, counit ε_H and antipode S. We follow the notation in [Mon] and [S], but we will write the comultiplication in H, $\Delta : H \to H \otimes H$, $\Delta(h) = \sum h_1 \otimes h_2$, for all $h \in H$. Denote by ${}^H Mod$ the category of left H-comodules and by Mod_H the category of right H-modules. For $(V, \delta_V) \in {}^H Mod$ write: for all $v \in V$

$$\delta_V(v) = \sum v_{(-1)} \otimes v_0 \in H \otimes V.$$

We say that A is an algebra in Mod_H (i.e. A is a right H-module algebra) if the following conditions hold:

(1.1) $(A, \leftharpoonup)$ is a right H-comodule,

(1.2) $ab \leftharpoonup h = \sum (a \leftharpoonup h_1)(b \leftharpoonup h_2)$ and $1 \leftharpoonup h = \varepsilon(h)1$,

for all $a, b \in A$, $h \in H$.

Similarly, a coalgebra C in ${}^H Mod$ (i.e. C is a left H-comodule coalgebra) means that the following conditions are satisfied:

(1.3) (C, δ_C) is a left H-comodule,

(1.4) $\sum c_{(-1)} \otimes c_{01} \otimes c_{02} = \sum c_{1(-1)}c_{2(-1)} \otimes c_{10} \otimes c_{20}$, $\sum \varepsilon(c_0)c_{(-1)} = \varepsilon(c)1$,

for all $c \in C$.

We recall now the definition of a bicrossproduct Hopf algebra. Let A, H be two Hopf algebras, H a coalgebra in ${}^A Mod$ and A an algebra in Mod_H. If the following conditions hold:

(i) $\Delta(a \leftharpoonup h) = \sum (a_1 \leftharpoonup h_1)h_{2(-1)} \otimes a_2 \leftharpoonup h_{20}$, $\varepsilon(a \leftharpoonup h) = \varepsilon(a)\varepsilon(h)$;

(ii) $\delta_H(hg) = \sum (h_{(-1)} \leftharpoonup g_1) g_{2(-1)} \otimes h_0 g_{20}$, $\delta_H(1) = 1 \otimes 1$;

(iii) $\sum h_{1(-1)}(a \leftharpoonup h_2) \otimes h_{10} = \sum (a \leftharpoonup h_1) h_{2(-1)} \otimes h_{20}$,

for all $h, g \in H$, $a \in A$. Then the tensor product $H \otimes A$ bears a Hopf algebra structure, called a bicrossproduct Hopf algebra and denoted $H \star A$, via the smash product and smash coproduct:

$$(h \star a)(g \star b) = \sum hg_1 \star (a \leftharpoonup g_2) b;$$

$$\Delta(h \star a) = \sum h_1 \star h_{2(-1)} a_1 \otimes h_{20} \star a_2.$$

It has an antipode given by

$$S(h \otimes a) = \sum (1 \star S(h_{(-1)}) a)(S(h_0) \star 1),$$

for all $h \in H$, $a \in A$.

We recall now the definition of a braided Hopf algebra:

Definition 1.1 ([D, LT]). A braided Hopf algebra is a pair (A, σ) with a bilinear form $\sigma : A \otimes A \to k$ satisfying

(BR1) $\sigma(ab, c) = \sum \sigma(a, c_1) \sigma(b, c_2)$;

(BR2) $\sigma(a, bc) = \sum \sigma(a_1, c) \sigma(a_2, b)$;

(BR3) $\sum \sigma(a_1, b_1) a_2 b_2 = \sum \sigma(a_2, b_2) b_1 a_1$;

(BR4) $\sigma(a, 1) = \varepsilon(a); \sigma(1, a) = \varepsilon(a)$.

In this case, σ is termed a braided structure over A. It is a consequence of the above that $\sigma^{-1}(a, b) = \sigma(S(a), b)$.

We next recall the definition of a dual Hopf algebra pair:

Definition 1.2 ([FS]). Let H, A be two Hopf algebras. Assume that there exists a bilinear form $\omega : H \otimes A \to k$. (H, A, ω) is called a dual skew ω-Hopf algebra pair if the following conditions hold:

(DP1) $\omega(hg, a) = \sum \omega(h, a_1) \omega(g, a_2)$;

(DP2) $\omega(h, ab) = \sum \omega(h_1, a) \omega(h_2, b)$;

(DP3) $\omega(1, a) = \varepsilon(a); \omega(h, 1) = \varepsilon(h)$,

and in this case, we also say that (H, A, ω) is a dual pairing.

In what follows we introduce two new conceptions as follows:

DEFINITION 1.3. Let H, A be two Hopf algebras. Assume that there exists a bilinear form $\tau : A \otimes H \to k$. (A, H, τ) is called an anti-skew τ-Hopf algebra pair if the following conditions hold:

(ASP1) $\tau(ab, h) = \sum \tau(a, h_2)\tau(b, h_1)$;

(ASP2) $\tau(a, hg) = \sum \tau(a_1, h)\tau(a_2, g)$;

(ASP3) $\tau(a, 1) = \varepsilon(a); \tau(1, h) = \varepsilon(h)$,

and in this case, we also call (H, A, τ) an anti-skew pairing.

DEFINITION 1.4. Let A, H be two Hopf algebras, and let $\omega : H \times A \to k$, $\tau : A \times H \to k$ be two bilinear maps. Assume that H is a left A-comodule with structure map δ_H. A braided-like Hopf algebra associated to (ω, τ, δ_H) is a pair (H, β) with a bilinear form $\beta : H \times H \to k$ satisfying

(BRL1) $\beta(hh', g) = \sum \beta(h_1, g_1)\beta(h', g_{20})\omega(h_2, g_{2(-1)})$;

(BRL2) $\beta(h, gg') = \sum \beta(h_1, g'_1)\beta(h_{30}, g)\omega(h_2, g'_{2(-1)})\tau(h_{3(-1)}, g'_{20})$;

(BRL3) $\sum \beta(h_2, g_2)g_1 h_1 = \sum h_{30}g_{30}\beta(h_1, g_1)\omega(h_2, g_{2(-1)})$
$\tau(h_{3(-1)}, g_{20})\omega(h_4, g_{3(-1)})$;

(BRL4) $\beta(1, h) = \beta(h, 1) = \varepsilon(h)$.

EXAMPLE 1.5. Let (H, σ) be a braided Hopf algebra and let A be arbitrary Hopf algebra. H is a left A-comodule with trivial comodule coaction $\delta_H(h) = 1 \otimes h$. Assume that $\omega : H \otimes A \to k$, $\tau : A \otimes H \to k$ be two trivial linear maps. Then it is easy to see that (H, A, ω) is a dual ω-Hopf algebra pair, (A, H, τ) is called an anti-skew τ-Hopf algebra pair, and (H, σ) is also a braided-like Hopf algebra associated to (ω, τ, δ_H).

REMARK. The example 1.5 means that Definition 1.4 is a generalization of the usual braided Hopf algebra.

EXAMPLE 1.6. Let (H, σ) be braided Hopf algebra with bijective antipode S. Then H^{cop} is also a Hopf algebra with the opposite comultiplication, i.e, $\Delta^{cop}(h) = \sum h_2 \otimes h_1$. Define

$$\delta_H : H \to H^{cop} \otimes H, \quad \delta_H(h) = \sum S(h_1)h_3 \otimes h_2.$$

Then, one has

1) (H,δ_H) is a coalgebra in ${}^{H^{cop}}Mod$;
2) If $\omega(h,a)=\sigma(a,h)$, then (H,H^{cop},ω) is a dual pairing;
3) If $\tau(a,h)=\sigma(h,a)$, (H^{cop},H,τ) is an anti-skew pairing;
4) (H,β) is a braided-like Hopf algebra associated to (ω,τ,δ_H) with

$$\beta(h,g)=\sum\sigma(h_1,g_1)\sigma(g_2,h_2),$$

where $\omega(h,a)=\sigma(a,h)$, $\tau(a,h)=\sigma(h,a)$, for all $a\in H^{cop}$, $h\in H$.

Proof. 1) is obvious. 2) and 3) follow that (H,σ) is a braided Hopf algebra.

4) It is easy to see that (BRL4) in Definition 1.4 holds. In order to show that (BRL1) is satisfied, one has:

$$\begin{aligned}
&\sum\beta(h_1,g_1)\omega(h_2,\underline{g_{2(-1)}})\beta(h',\underline{g_{20}})\\
&=\sum\underbrace{\beta(h_1,g_1)}\omega(h_2,S(g_2)g_4)\beta(h',g_3)\\
&=\sum\sigma(h_1,g_1)\underbrace{\sigma(g_2,h_2)\sigma(S(g_3)g_6,h_3)}\sigma(h'_1,g_4)\sigma(g_5,h'_2)\\
&\overset{(BR1)}{=}\sum\sigma(h_1,g_1)\sigma(\underbrace{g_2S(g_3)}g_6,h_2)\sigma(h'_1,g_4)\sigma(g_5,h'_2)\\
&=\sum\underline{\sigma(h_1,g_1)}\,\underbrace{\sigma(g_4,h_2)}\,\underline{\sigma(h'_1,g_2)}\,\underbrace{\sigma(g_3,h'_2)}\\
&\overset{(BR1)+(BR2)}{=}\sum\sigma(h_1h'_1,g_1)\sigma(g_2,h_2h'_2)=\beta(hh',g),
\end{aligned}$$

and the condition (BRL1) is proven.

Similarly, (BRL2) also holds.

We will show that the condition (BRL3) in Definition 1.4 holds as follows:

$$\begin{aligned}
&\sum\underbrace{h_{30}}\,g_{30}\beta(h_1,g_1)\omega(h_2,\underline{g_{2(-1)}})\tau(\underbrace{h_{3(-1)}},\underline{g_{20}})\omega(h_4,g_{3(-1)})\\
&=\sum h_4\,\underbrace{g_{50}}\,\beta(h_1,g_1)\omega(h_2,S(g_2)g_4)\tau(S(h_3)h_5,g_3)\omega(h_6,\underbrace{g_{5(-1)}})\\
&=\sum h_4g_6\underline{\beta(h_1,g_1)}\omega(h_2,S(g_2)g_4)\tau(S(h_3)h_5,g_3)\omega(h_6,S(g_5)g_7)\\
&=\sum h_4g_6\beta(h_1,g_1)\sigma(S(g_2)g_4,h_2)\sigma(g_3,S(h_3)h_5)\sigma(S(g_5)g_7,h_6)\\
&=\sum h_5g_7\sigma(h_1,g_1)\underbrace{\sigma(g_2,h_2)\sigma(S(g_3)g_5,h_3)}\sigma(g_4,S(h_4)h_6)\sigma(S(g_6)g_8,h_7)
\end{aligned}$$

$$\stackrel{(BR1)}{=} \sum h_4 g_7 \sigma(h_1, g_1)\sigma(g_2 S(g_3) g_5, h_2)\sigma(g_4, S(h_3)h_5)\sigma(S(g_6)g_8, h_6)$$

$$= \sum h_4 g_5 \sigma(h_1, g_1) \underbrace{\sigma(g_3, h_2)\sigma(g_2, S(h_3)h_5)} \sigma(S(g_4)g_6, h_6)$$

$$\stackrel{(BR2)}{=} \sum h_4 g_4 \sigma(h_1, g_1)\sigma(g_2, \underline{h_2 S(h_3)} h_5)\sigma(S(g_3)g_5, h_6)$$

$$= \sum h_2 g_4 \sigma(h_1, g_1) \underbrace{\sigma(g_2, h_3)\sigma(S(g_3)g_5, h_4)}$$

$$\stackrel{(BR1)}{=} \sum h_2 g_4 \sigma(h_1, g_1)\sigma(g_2 S(g_3) g_5, h_3) = \sum \underbrace{h_2 g_2 \sigma(h_1, g_1)} \sigma(g_3, h_3)$$

$$\stackrel{(BR3)}{=} \sum g_1 h_1 \underline{\sigma(h_2, g_2)\sigma(g_3, h_3)} = \sum \beta(h_2, g_2) g_1 h_1,$$

and (BRL3) is proved.

Thus (H, β) is a braided-like Hopf algebra associated to (ω, τ, δ_H).

2. The Braided Structures over $H \star A$

In this Section we will describe the braided structures over bicrossproduct Hopf algebra $H \star A$.

The following is obvious:

PROPOSITION 2.1. *Let $H \star A$ be a bicrossproduct bialgebra. Define maps as follows*:

$$p : H \star A \to H, p(h \otimes a) = \varepsilon(a)h, \quad \pi : H \star A \to A, \pi(h \otimes a) = \varepsilon(h)a,$$

$$i : A \to H \star A, i(a) = 1_H \otimes a, \quad j : H \to H \star A, j(h) = h \otimes 1_A.$$

Then 1) *p, i is a bialgebra map*,

2) *π is a coalgebra map and $\pi((h \star a)(g \star b)) = \varepsilon(h)(a \leftharpoonup g)b$*,

3) *j is a algebra map, and $\Delta(j(h)) = \sum h_1 \star h_{2(-1)} \otimes h_{20} \star 1_A$*.

Let $(H \star A, \sigma)$ be a bicrossproduct bialgebra, and $\sigma : H \star A \otimes H \star A \to k$ a bilinear form, Define:

$$\alpha : A \times A \to k, \alpha(a, b) = \sigma(i \otimes i)(a \otimes b) = \sigma(1 \otimes a, 1 \otimes b);$$

$$\beta : H \times H \to k, \beta(h, g) = \sigma(j \otimes j)(h \otimes g) = \sigma(h \otimes 1, g \otimes 1);$$

$$\omega : H \times A \to k, \omega(h, a) = \sigma(j \otimes i)(h \otimes a) = \sigma(h \otimes 1, 1 \otimes a);$$

$$\tau : A \times H \to k, \tau(a, h) = \sigma(i \otimes j)(a \otimes h) = \sigma(1 \otimes a, h \otimes 1).$$

The following Proposition 2.2 is obvious:

PROPOSITION 2.2. *With the notation above, let $H \star A$ be a bicrossproduct bialgebra. If σ satisfies condition* (BR4), *then*

1) $\alpha(a,1) = \varepsilon(a)1 = \alpha(1,a)$
2) $\beta(h,1) = \varepsilon(h)1 = \beta(1,h)$,
3) $\omega(h,1) = \varepsilon(h)1;\ \omega(1,a) = \varepsilon(a)1$,
4) $\tau(a,1) = \varepsilon(a)1;\ \tau(1,h) = \varepsilon(h)1$.

PROPOSITION 2.3. *Let $H \star A$ be a bicrossproduct Hopf algebra and $\sigma : H \star A \otimes H \star A \to k$ a bilinear form, if $(H \star A, \sigma)$ is a braided Hopf algebra, then we have*:

$$\sigma(h \otimes a, g \otimes b) = \sum \beta(h_1, g_1)\omega(h_2, g_{2(-1)})\omega(h_3, b_1)\alpha(a_1, b_2)\tau(a_2, g_{20}),$$

where $h, g \in H$, $a, b \in A$.

PROOF. For all $a, b, a', b' \in A$; $h, h', g, g' \in H$, we have:

$$\begin{aligned}
&\sigma((h \otimes a)(h' \otimes a'), (g \otimes b)(g' \otimes b')) \\
&\overset{(BR1)}{=} \sum \sigma(h \otimes a, ((g \otimes b)(g' \otimes b'))_1)\sigma(h' \otimes a', ((g \otimes b)(g' \otimes b'))_2) \\
&\overset{(BR2)}{=} \sum \sigma((h \otimes a)_1, (g' \otimes b')_1)\sigma((h \otimes a)_2, (g \otimes b)_1) \\
&\qquad \sigma((h' \otimes a')_1, (g' \otimes b')_2)\sigma((h' \otimes a')_2, (g \otimes b)_2).
\end{aligned} \tag{1}$$

Letting $a = 1$, $h' = 1$, $b = 1$, $g' = 1$ in both sides of the equation (1), we obtain

$$\sigma(h \otimes a', g \otimes b') = \sum \beta(h_1, g_1)\omega(h_2, g_{2(-1)})\omega(h_3, b'_1)\alpha(a'_1, b'_2)\tau(a'_2, g_{20}),$$

and so completing the proof of proposition 2.3.

The following give some useful identities concerning the forms σ, β, ω and τ:

PROPOSITION 2.4. *Let $B \star H$ be a bicrossproduct Hopf algebra. Assume that $\sigma(h \otimes a, g \otimes b) = \sum \beta(h_1, g_1)\omega(h_2, g_{2(-1)})\omega(h_3, b_1)\alpha(a_1, b_2)\tau(a_2, g_{20})$ is a braided structure over $H \star A$. Then we have the following identities*:

(BC1) $\sum \omega(h_2, b)h_1 = \sum h_{20}\omega(h_1, b_1)\alpha(h_{2(-1)}, b_2)$;

(BC2) $\sum \tau(a, g_2)g_1 = \sum g_{20}\tau(a_2, g_1)\alpha(a_1, g_{2(-1)})$;

(BC3) $\sum \omega(h, b_1)b_2 = \sum (b_1 \leftharpoondown h_1)h_{2(-1)}\omega(h_{20}, b_2))$;

(BC4) $\sum \tau(a_1, g_2)(a_2 \leftharpoonup g_1) = \sum g_{(-1)} a_1 \tau(a_2, g_0))$;

(BC5) $\sum \beta(h, g) = \sum (g_{(-1)} \leftharpoonup h_1) h_{2(-1)} \beta(h_{20}, g_0))$;

(BC6) $\sum \beta(h, g_1)\tau(a, g_2) = \sum \beta(h_1, g_1)\omega(h_2, g_{2(-1)})\tau(a \leftharpoonup h_3, g_{20})$;

(BC7) $\sum \alpha(a, b_1)\omega(h, b_2) = \sum \omega(h_1, b_1)\alpha(a \leftharpoonup h_2, b_2)$;

(BC8) $\sum \beta(h_1, g_1)\omega(h_2, b \leftharpoonup g_2) = \sum \beta(h_1, g_1)$
$\cdot\, \omega(h_2, g_{2(-1)})\tau(h_{3(-1)}, g_{20})\omega(h_{30}, b)$;

(BC9) $\sum \alpha(a_1, b \leftharpoonup g_2)\tau(a_2, g_1) = \sum \tau(a_1, g)\alpha(a_2, b)$.

PROOF. By (BR1), one gets:

$$\sigma((h \otimes a)(h' \otimes a'), g \otimes b) = \sum \sigma(h \otimes a, g_1 \otimes g_{2(-1)} b_1)\sigma(h' \otimes a', g_{20} \otimes b_2). \tag{A}$$

By (BR2) one has:

$$\sigma(h \otimes a, (g \otimes b)(g' \otimes b')) = \sum \sigma(h_1 \otimes h_{2(-1)} a_1, g' \otimes b')\sigma(h_{20} \otimes a_2, g \otimes b), \tag{B}$$

and by (BR3) one knows:

$$\sum \sigma((h \otimes a)_1, (g \otimes b)_1)(h \otimes a)_2(g \otimes b)_2 = \sum (g \otimes b)_1(h \otimes a)_1\sigma((h \otimes a)_2, (g \otimes b)_2). \tag{C}$$

Let $a = 1$ and $g = 1$ in the equation (C), then we get

$$\sum \sigma(h_1 \otimes h_{2(-1)}, 1 \otimes b_1)(h_{20} \otimes b_2) = \sum (h_1 \otimes (b_1 \leftharpoonup h_2) h_{3(-1)})\omega(h_{20}, b_2), \tag{D}$$

and by applying $(id \otimes \varepsilon)$ to both sides of the equation (D), and by proposition 2.3, we obtain (BC1); by applying $(\varepsilon \otimes id)$ to both sides of the equation (D), and by proposition 2.3, we get (BC3).

Letting $h = 1$, $b = 1$ in equation (C), then applying $(id \otimes \varepsilon)$ to both sides of the equation (C), and proposition 2.3, we can get (BC2); applying $(\varepsilon \otimes id)$ to both sides of the equation (C), and proposition 2.3, we can obtain (BC4).

By letting $a = b = 1$ in the equation (C), then applying $(\varepsilon \otimes id)$ to both sides of the equation (C), we can get (BC5).

Let $h = 1$, $a' = b = 1$ in the formula (A), then by proposition 2.3, and (BC2), we get (BC6).

By letting $h=1$, $a'=1$, $g=1$ in the equation (A), then by proposition 2.3, we get (BC7).

By letting $a=1$, $g=1$, $b'=1$ in the formula (B), then by proposition 2.3, we get (BC8).

Letting $h=g=1$, $b'=1$ in the equation (B), then by proposition 2.3, we get (BC9).

This completes the proof.

Proposition 2.5. *Let $B \star H$ be a bicrossproduct Hopf algebra. Assume that $\sigma(h\otimes a, g\otimes b)=\sum \beta(h_1,g_1)\omega(h_2,g_{2(-1)})\omega(h_3,b_1)\alpha(a_1,b_2)\tau(a_2,g_{20})$ is a braided structure over $H\star A$. Then we have*

1) *(A,α) is a braided Hopf algebra,*
2) *(H,A,ω) is a dual pairing,*
3) *(A,H,τ) is an anti-skew pairing;*
4) *(H,β) is a braided-like Hopf algebra associated to (ω,τ,δ_H).*

Proof. It follows from the proposition 2.1 that α, β, ω, and τ respectively satisfies (BR4), (BRL4), (DP3), and (ASP3).

1) Since $i: A\to H\otimes A$ is a bialgebra map, and $(B\star A,\sigma)$ a braided Hopf algebra, so is (A,α).

2) is obvious by letting $a=a'=1$, $g=1$ in the equation (A) and letting $a=1$, $g=g'=1$ in the equation (B).

3) is easy to be seen by letting $h=h'=1$, $b=1$ in the formula (A) and letting $h=1$, $b=b'=1$ in the equation (B).

4) Let $a=a'=b=1$ in the equation (A), one gets (BRL1); by letting $a=b'=b=1$ in the formula (B), one has (BRL2); by letting $a=b=1$ in the equation (C), we can complete the proof of that (A,β) is a braided-like Hopf algebra associated to (ω,τ,δ_H), including the proof.

Theorem 2.6. *Let $H\star A$ be a bicrossproduct Hopf algebra. If there exist forms $\alpha: A\times A\to k$, $\beta: H\times H\to k$, $\omega: H\times A\to k$, $\tau: A\times H\to k$, such that the following conditions hold:*

1) *(A,α) is a braided Hopf algebra;*
2) *(H,A,ω) is a dual pairing;*
3) *(A,H,τ) is an anti-skew pairing;*
4) *(H,β) is a braided-like Hopf algebra associated to (ω,τ,δ).*
5) *The conditions* (BC1)–(BC9) *in Proposition* 2.4 *hold.*

Then $(H\star A,\sigma)$ is a braided Hopf algebra with a braided structure given by:

$$\sigma(h\otimes a,g\otimes b)=\sum\beta(h_1,g_1)\omega(h_2,g_{2(-1)})\omega(h_3,b_1)\alpha(a_1,b_2)\tau(a_2,g_{20})$$

PROOF. It is obvious that σ satisfies (BR4). In what follows we show that (BR1) holds:

$$\sigma((h\otimes a)(h'\otimes a'),g\otimes b)=\sum\sigma(hh'_1\otimes(a\leftharpoondown h'_2)a'),g\otimes b)$$

$$=\sum\beta(h_1h'_1,g_1)\omega(h_2h'_2,g_{2(-1)})\omega(h_3h'_3,b_1)\alpha(\underbrace{(a\leftharpoondown h'_4)_1}a'_1,b_2)$$

$$\tau(\underbrace{(a\leftharpoondown h'_4)_2}a'_2,g_{20})$$

$$=\sum\underbrace{\beta(h_1h'_1,g_1)}\omega(h_2h'_2,g_{2(-1)})\omega(h_3h'_3,b_1)$$

$$\alpha((a_1\leftharpoondown h'_4)h'_{5(-1)}a'_1,b_2)\tau((a_2\leftharpoondown h'_{50})a'_2,g_{20})$$

$$\overset{\text{(BRL1)}}{=}\sum\beta(h_1,g_1)\omega(h_2,g_{2(-1)})\beta(h'_1,g_{20})\underbrace{\omega(h_3h'_2,g_{3(-1)})}\underline{\omega(h_4h'_3,b_1)}$$

$$\alpha((a_1\leftharpoondown h'_4)h'_{5(-1)}a'_1,b_2)\tau((a_2\leftharpoondown h'_{50})a'_2,g_{30})$$

$$\overset{\text{(DP1)}}{=}\sum\beta(h_1,g_1)\omega(h_2,g_{2(-1)})\beta(h'_1,g_{20})\omega(h_3,g_{3(-1)1})\omega(h'_2,g_{3(-1)2})$$

$$\omega(h_4,b_1)\omega(h'_3,b_2)\underbrace{\alpha((a_1\leftharpoondown h'_4)h'_{5(-1)}a'_1,b_3)}\underline{\tau((a_2\leftharpoondown h'_{50})a'_2,g_{30})}$$

$$\overset{\text{(BR1)+(ASP1)}}{=}\sum\beta(h_1,g_1)\omega(h_2,g_{2(-1)})\beta(h'_1,g_{20})\omega(h_3,g_{3(-1)1})\omega(h'_2,g_{3(-1)2})$$

$$\omega(h_4,b_1)\underbrace{\omega(h'_3,b_2)\alpha(a_1\leftharpoondown h'_4),b_3)}\alpha(h'_{5(-1)},b_4)\alpha(a'_1,b_5)$$

$$\tau(a'_2,g_{301})\tau(a_2\leftharpoondown h'_{50},g_{302})$$

$$\overset{\text{(BC7)}}{=}\sum\beta(h_1,g_1)\omega(h_2,g_{2(-1)})\beta(h'_1,g_{20})\omega(h_3,g_{3(-1)1})\omega(h'_2,g_{3(-1)2})$$

$$\omega(h_4,b_1)\alpha(a_1,b_2)\underbrace{\omega(h'_3,b_3)\alpha(h'_{4(-1)},b_4)}$$

$$\alpha(a'_1,b_5)\tau(a'_2,g_{301})\tau(a_2\leftharpoondown\underbrace{h'_{40}},g_{302})$$

$$\overset{\text{(BC1)}}{=}\sum\beta(h_1,g_1)\omega(h_2,g_{2(-1)})\beta(h'_1,g_{20})\omega(h_3,\underbrace{g_{3(-1)1}})$$

$$\omega(h'_2,\underbrace{g_{3(-1)2}})\omega(h_4,b_1)\alpha(a_1,b_2)\omega(h'_4,b_3)$$

$$\alpha(a'_1,b_4)\tau(a'_2,\underbrace{g_{301}})\tau(a_2\leftharpoondown h'_3,\underbrace{g_{302}})$$

$$= \sum \beta(h_1, g_1)\, \overbrace{\omega(h_2, g_{2(-1)})}\, \beta(h'_1, g_{20})\, \overbrace{\omega(h_3, \underbrace{g_{3(-1)1}}\, \underline{g_{4(-1)1}}}$$

$$\omega(h'_2, \underbrace{g_{3(-1)2}}\, \underline{g_{4(-1)2}})\omega(h_4, b_1)\alpha(a_1, b_2)\omega(h'_4, b_3)$$

$$\alpha(a'_1, b_4)\tau(a'_2, \underbrace{g_{30}})\tau(a_2 \leftharpoonup h'_3, \underline{g_{40}})$$

$$\overset{(DP1)}{=} \sum \beta(h_1, g_1)\omega(h_2, \underbrace{g_{2(-1)}g_{3(-1)}g_{4(-1)}})\beta(h'_1, \underbrace{g_{20}})$$

$$\omega(h'_2, \underline{\underbrace{g_{30(-1)}}\,\underbrace{g_{40(-1)}}})\omega(h_3, b_1)\alpha(a_1, b_2)\omega(h'_4, b_3)$$

$$\alpha(a'_1, b_4)\tau(a'_2, \underline{\underbrace{g_{300}}})\tau(a_2 \leftharpoonup h'_3, \underline{\underbrace{g_{400}}})$$

$$= \sum \beta(h_1, g_1)\, \underbrace{\omega(h_2, g_{2(-1)})}\beta(h'_1, g_{201})\omega(h'_2, g_{202(-1)})\, \underbrace{\omega(h_3, b_1)}$$

$$\alpha(a_1, b_2)\omega(h'_4, b_3)\alpha(a'_1, b_4)\tau(a'_2, g_{20201})\tau(a_2 \leftharpoonup h'_3, g_{20202})$$

$$\overset{(DP1)}{=} \sum \beta(h_1, g_1)\omega(h_2, g_{2(-1)}b_1)\beta(h'_1, g_{201})\omega(h'_2, g_{202(-1)})\alpha(a_1, b_2)$$

$$\omega(h'_4, b_3)\alpha(a'_1, b_4)\tau(a'_2, \underbrace{g_{20201}})\, \underbrace{\tau(a_2 \leftharpoonup h'_3, g_{20202})}$$

$$\overset{(BC2)}{=} \sum \beta(h_1, g_1)\omega(h_2, g_{2(-1)}b_1)\beta(h'_1, g_{201})\omega(h'_2, g_{202(-1)})$$

$$\alpha(a_1, b_2)\omega(h'_4, b_3)\alpha(a'_1, b_4)\underline{\alpha(\underbrace{(a_2 \leftharpoonup h'_3)_1}, g_{20202(-1)})}$$

$$\tau(\underbrace{(a_2 \leftharpoonup h'_3)_2}, g_{20201})\tau(a'_2, g_{202020})$$

$$\overset{(i)+(BR1)}{=} \sum \beta(h_1, g_1)\omega(h_2, g_{2(-1)}b_1)\beta(h'_1, g_{201})\, \underbrace{\omega(h'_2, \underline{g_{202(-1)}})}\, \alpha(a_1, b_2)$$

$$\alpha(a'_1, b_4)\omega(h'_5, b_3)\alpha((a_2 \leftharpoonup h'_3), \underline{g_{20202(-1)1}})\alpha(h'_{4(-1)}, \underline{g_{20202(-1)2}})$$

$$\tau(a_3 \leftharpoonup h'_{40}, \underline{g_{20201}})\tau(a'_2, \underline{g_{202020}})$$

$$\overset{(DP1)}{=} \sum \beta(h_1, g_1)\omega(h_2, g_{2(-1)}b_1)\beta(h'_1, g_{201})\omega(h'_2, g_{202(-1)})\, \underbrace{\omega(h'_3, \underline{g_{203(-1)}})}$$

$$\alpha(a_1, b_2)\alpha(a'_1, b_4)\omega(h'_6, b_3)\, \underbrace{\alpha((a_2 \leftharpoonup h'_4), \underline{g_{2030(-1)1}})}$$

$$\alpha(h'_{5(-1)}, \underline{g_{2030(-1)2}})\tau(a_3 \leftharpoonup h'_{50}, g_{2020})\tau(a'_2, \underline{g_{20300}})$$

$$\overset{\text{(BC7)}}{=} \sum \beta(h_1, g_1)\omega(h_2, g_{2(-1)}b_1)\beta(h'_1, g_{201})\omega(h'_2, g_{202(-1)})\underbrace{\omega(h'_3, g_{203(-1)2})}$$

$$\alpha(a_1, b_2)\alpha(a'_1, b_4)\omega(h'_5, b_3)\alpha(a_2, g_{203(-1)1})\underbrace{\alpha(h'_{4(-1)}, g_{203(-1)3})}$$

$$\tau(a_3 \leftharpoonup \underbrace{h'_{40}}, g_{2020})\tau(a'_2, g_{2030})$$

$$\overset{\text{(BC1)}}{=} \sum \beta(h_1, g_1)\omega(h_2, g_{2(-1)}b_1)\underbrace{\beta(h'_1, g_{201})\omega(h'_2, g_{202(-1)})}\alpha(a_2, \underline{g_{203(-1)1}})$$

$$\alpha(a_1, b_2)\alpha(a'_1, b_4)\omega(h'_5, b_3)\omega(h'_4, \underline{g_{203(-1)2}})$$

$$\underbrace{\tau(a_3 \leftharpoonup h'_3, g_{2020})}\tau(a'_2, \underline{g_{2030}})$$

$$\overset{\text{(BC6)}}{=} \sum \beta(h_1, g_1)\omega(h_2, g_{2(-1)}b_1)\beta(h'_1, g_{201})\underline{\alpha(a_2, g_{203(-1)})}$$

$$\alpha(a_1, b_2)\alpha(a'_1, b_4)\underbrace{\omega(h'_3, b_3)\omega(h'_2, g_{2030(-1)})}$$

$$\underline{\tau(a_3, g_{202})}\tau(a'_2, g_{20300})$$

$$\overset{\text{(DP1)+(BC2)}}{=} \sum \beta(h_1, g_1)\omega(h_2, g_{2(-1)}b_1)\beta(h'_1, g_{201})\alpha(a_2, g_{203(-1)})$$

$$\alpha(a_1, b_2)\alpha(a'_1, b_4)\omega(h'_2, g_{202(-1)}b_3)\tau(a_2, g_{203})\tau(a'_2, g_{2020}),$$

and

$$\sum \sigma(h \otimes a, g_1 \otimes g_{2(-1)}b_1)\sigma(h' \otimes a', g_{20} \otimes b_2)$$

$$= \sum \beta(h_1, g_1)\underbrace{\omega(h_2, g_{2(-1)})\omega(h_3, \underline{g_{3(-1)1}}b_1)}\alpha(a_1, \underline{g_{3(-1)2}}b_2)\tau(a_2, g_{20})$$

$$\beta(h'_1, \underline{g_{301}})\omega(h'_2, \underline{g_{302(-1)}})\omega(h'_3, b_3)\alpha(a'_1, b_4)\tau(a'_2, \underline{g_{3020}})$$

$$\overset{\text{(DP1)}}{=} \sum \beta(h_1, g_1)\omega(h_2, \underline{g_{2(-1)}g_{3(-1)}}b_1)\underbrace{\alpha(a_1, \underline{g_{30(-1)}}b_2)}\tau(a_2, \underline{g_{20}})$$

$$\beta(h'_1, \underline{g_{3001}})\omega(h'_2, \underline{g_{3002(-1)}})\omega(h'_3, b_3)\alpha(a'_1, b_4)\tau(a'_2, \underline{g_{30020}})$$

$$\overset{\text{(BR2)}}{=} \sum \beta(h_1, g_1)\omega(h_2, g_{2(-1)}b_1)\alpha(a_1, b_2)\underbrace{\alpha(a_2, g_{202(-1)})\tau(a_3, g_{201})}$$

$$\beta(h'_1, \underbrace{g_{20201}})\underline{\omega(h'_2, \underbrace{g_{20202(-1)}})\omega(h'_3, b_3)}\alpha(a'_1, b_4)\tau(a'_2, \underbrace{g_{202020}})$$

$$\overset{\text{(DP1)+(BC2)}}{=} \sum \beta(h_1, g_1)\omega(h_2, g_{2(-1)}b_1)\beta(h'_1, g_{201})\alpha(a_2, g_{203(-1)})$$

$$\alpha(a_1, b_2)\alpha(a'_1, b_4)\omega(h'_2, g_{202(-1)}b_3)\tau(a_2, g_{203})\tau(a'_2, g_{2020}),$$

and (BR1) is proved.

Similarly, we can check (with tedious calculation) that (BR2) and (BR3) hold.

This completes the proof of Theorem.

Thus it follows from Proposition 2.3, Proposition 2.4 and Theorem 2.6 that:

THEOREM 2.7. *The bicrossproduct Hopf algebra $H \star A$ is braided if and only if there exist forms $\alpha : A \times A \to k$, $\beta : H \times H \to k$, $\omega : H \times A \to k$, and $\tau : A \times H \to k$, such that (A, α) is a braided Hopf algebra, (H, A, ω) is a dual pairing, (A, H, τ) is an anti-skew pairing, (H, β) is a braided-like Hopf algebra associated to (ω, τ, δ_H) and the conditions* (BC1)–(BC9) *are satisfied. Moreover, σ has a unique decomposition*:

$$\sigma(h \otimes a, g \otimes b) = \sum \beta(h_1, g_1)\omega(h_2, g_{2(-1)})\omega(h_3, b_1)\alpha(a_1, b_2)\tau(a_2, g_{20}).$$

REMARK. Let H be arbitrary Hopf algebra. Theorem 2.7 shows that if A is not a braided Hopf algebra then the $H \star A$ is not a braided Hopf algebra either.

COROLLARY 2.8. *Let $H \star A$ be bicrossproduct Hopf algebra and (H, β), (A, α) braided Hopf algebras. Then $\sigma(h \otimes a, g \otimes b) = \sum \beta(h, g)\alpha(a, b)$ is a braided structure over $H \star A$ if and only if*

(BC1′) $\sum \varepsilon(b)h = \sum h_0 \alpha(h_{(-1)}, b)$;

(BC2′) $\sum \varepsilon(a)g = \sum g_0 \alpha(a, g_{(-1)})$;

(BC3′) $\sum \varepsilon(h)b = \sum b \leftharpoonup h$;

(BC4′) $\sum \beta(h, g) = \sum (g_{(-1)} \leftharpoonup h_1)h_{2(-1)}\beta(h_{20}, g_0))$;

PROOF. Letting $\beta : H \times H \to k$, $\omega : H \times A \to k$, be trivial in Theorem 2.7, we obtain this Corollary.

COROLLARY 2.9. *Let $H \star A$ be bicrossproduct Hopf algebra and (H, β) a braided Hopf algebras. Assume that A is cocommutative. Then $\sigma(h \otimes a, g \otimes b) = \sum \beta(h, g)\varepsilon(a)\varepsilon(b)$ is a braided structure over $H \star A$ if and only if*

(BC1″) $\sum \varepsilon(h)b = \sum b \leftharpoonup h$;

(BC2″) $\sum \beta(h, g) = \sum (g_{(-1)} \leftharpoonup h_1)h_{2(-1)}\beta(h_{20}, g_0))$.

COROLLARY 2.10. *Let $H \star A$ be bicrossproduct Hopf algebra and (A, α) a braided Hopf algebras. Assume that H is cocommutative. Then $\sigma(h \otimes a, g \otimes b) = \sum \alpha(a,b)\varepsilon(h)\varepsilon(g)$ is a braided structure over $H \star A$ if and only if*

(BC1″) $\sum \varepsilon(b)h = \sum h_0\alpha(h_{(-1)}, b)$;

(BC2″) $\sum \varepsilon(a)g = \sum g_0\alpha(a, g_{(-1)})$;

(BC3″) $\sum \varepsilon(h)b = \sum b \leftharpoonup h$.

COROLLARY 2.11. *Let $H \star A$ be bicrossproduct Hopf algebra, (H, A, ω) a dual pairing, and (A, H, τ) an anti-skew pairing. Assume that A, H are cocommutative. If the condition $gh = \sum h_{20}g_{20}\omega(h_1, g_{1(-1)})\tau(h_{2(-1)}, g_{10})\omega(h_3, g_{2(-1)})$ is satisfied, then $\sigma(h \otimes a, g \otimes b) = \sum \omega(h_1, g_{(-1)})\omega(h_2, b)\tau(a, g_0)$ is a braided structure over $H \star A$ if and only if*

(BC1‴) $\sum \omega(h, b_1)b_2 = \sum (b_1 \leftharpoonup h_1)h_{2(-1)}\omega(h_{20}, b_2)$;

(BC2‴) $\sum \tau(a_1, g_2)(a_2 \leftharpoonup g_1) = \sum g_{(-1)}a_1\tau(a_2, g_0)$;

(BC3‴) $\sum \varepsilon(h)\tau(a, g) = \sum \omega(h_1, g_{(-1)})\tau(a \leftharpoonup h_2, g_0)$;

(BC4‴) $\sum \omega(h, b \leftharpoonup g) = \sum \omega(h_1, g_{(-1)})\tau(h_{2(-1)}, g_0)\omega(h_{20}, b)$.

PROOF. Letting $\beta : H \times H \to k$, $\alpha : A \times A \to k$ be trivial in Theorem 2.7, we get this Corollary.

3. Application to $H \star H^{cop}$

Let H be an arbitrary Hopf algebra with a bijective antipode S. Then H^{cop} is also a Hopf algebra with an antipode S^{-1}. Define

$$\leftharpoonup : H^{cop} \otimes H \to H^{cop}, \quad a \leftharpoonup h = \sum S(h_1)ah_2,$$

for all $a \in H^{cop}$, $h \in H$,
and

$$\delta_H : H \to H^{cop} \otimes H, \quad \delta_H(h) = \sum S(h_1)h_3 \otimes h_2,$$

for all $h \in H$, then H is a coalgebra in ${}^{H^{cop}}Mod$ and H^{cop} is an algebra in Mod_H.

Then we can construct a bicrossproduct $M(H) = H \otimes H^{cop}$ with multiplication and comultiplication respectively as follows:

$$(h \otimes a)(g \otimes b) = \sum hg_1 \otimes S(g_2)ag_3b;$$

$$\Delta(h \otimes a) = \sum h_1 \otimes S(h_2)h_4a_1 \otimes h_3 \otimes a_2.$$

Let (A, α) be braided. Then (A^{cop}, α^t) is also braided with $\alpha^t(a,b) = \alpha(b,a)$. Thus, we have

THEOREM 3.1. *Let (H, σ) be a braided Hopf algebra. Then the bicrossproduct Hopf algebra $H \star H^{cop}$ is a braided Hopf algebra with a braided structure given by:*

$$\tilde{\sigma}(h \otimes a, g \otimes b) = \sum \sigma(h_1, g_1)\sigma(g_2 b, h_2 a),$$

for all $a, b \in H^{cop}$, $h, g \in H$

PROOF. Let $\beta(h,g) = \sum \sigma(h_1, g_1)\sigma(g_2, h_2)$, $\omega(h,a) = \sigma(a,h)$, $\tau(a,h) = \sigma(h,a)$ for all $a \in H^{cop}$, $h, g \in H$. By Example 1.6 (H, H^{cop}, ω) is a dual pairing, (H^{cop}, H, τ) is an anti-skew pairing, and (H, β) is a braided-like Hopf algebra associated to (ω, τ, δ_H). Thus we will only check that the conditions (BC1)–(BC9) are satisfied.

we first have:

$$\begin{aligned}\sum \omega(h_1, b_2)\alpha(h_{2(-1)}, b_1)h_{20} &= \sum \sigma(b_2, h_1)\sigma(b_1, \underline{h_{2(-1)}})h_{20} \\ &= \sum \underbrace{\sigma(b_2, h_1)\sigma(b_1, S(h_2)h_4)}\, h_3 \overset{(\mathrm{BR2})}{=} \sum \sigma(b, h_1 \underbrace{S(h_2)}\, h_4)h_3 \\ &= \sum \sigma(b, h_2)h_1 = \sum \omega(h_2, b)h_1,\end{aligned}$$

and (BC1) is proved.

Similarly, we can show that the condition (BC2) is also true.

Secondly, we have:

$$\begin{aligned}\sum (b_2 \leftharpoonup h_1)\underline{h_{2(-1)}}\omega(\underline{h_{20}}, b_1) &= \sum (\underbrace{b_2 \leftharpoonup h_1})S(h_2)h_4\omega(h_3, b_1) \\ &= \sum S(h_1)b_2\underline{h_2 S(h_3)}h_5\omega(h_4, b_1) = \sum S(h_1)b_2h_3\omega(h_2, b_1) \\ &= \sum S(h_1)\,\underbrace{b_2h_3\sigma(b_1, h_2)} \overset{(\mathrm{BR3})}{=} \sum \underline{S(h_1)h_2}b_1\sigma(b_2, h_3) \\ &= \sum b_1\sigma(b_2, h) = \sum b_1\omega(h, b_2),\end{aligned}$$

and (BC3) is proved.

Similarly, it is not hard to verify that the condition (BC4) also holds.

Third, we check (BC5) as follows:

$$
\begin{aligned}
&\sum(\underbrace{g_{(-1)}}_{} \leftharpoonup h_1)\underline{h_{2(-1)}}\beta(\underline{h_{20}}, \underbrace{g_0}_{})\\
&\quad = \sum(\underbrace{S(g_1)g_3 \leftharpoonup h_1}_{})S(h_2)h_4\beta(h_3, g_2)\\
&\quad = \sum S(h_1)S(g_1)g_3\underline{h_2S(h_3)}h_5\beta(h_3, g_2)\\
&\quad = \sum S(h_1)S(g_1)g_3h_3\underline{\beta(h_2, g_2)}\\
&\quad = \sum S(h_1)S(g_1)\underbrace{g_4h_4}_{}\sigma(h_2, g_2)\underbrace{\sigma(g_3, h_3)}_{}\\
&\quad \overset{(BR3)}{=} \sum S(h_1)S(g_1)\underbrace{h_3g_3\sigma(h_2, g_2)}_{}\sigma(g_4, h_4)\\
&\quad \overset{(BR3)}{=} \sum \underline{S(g_1h_1)g_2h_2}\sigma(h_3, g_3)\sigma(g_4, h_4)\\
&\quad = \sum \sigma(h_1, g_1)\sigma(g_2, h_2) = \beta(h, g),
\end{aligned}
$$

and (BC5) is proven.

$$
\begin{aligned}
&\sum \beta(h_1, g_1)\omega(h_2, g_{2(-1)})\tau(a \leftharpoonup h_3, g_{20})\\
&\quad = \sum \sigma(h_1, g_1)\sigma(g_2, h_2)\sigma(\underline{g_{3(-1)}}, h_3)\sigma(\underline{g_{30}}, \underbrace{a \leftharpoonup h_4}_{})\\
&\quad = \sum \sigma(h_1, g_1)\underbrace{\sigma(g_2, h_2)\sigma(S(g_3)g_5, h_3)}_{}\sigma(g_4, S(h_4)ah_5)\\
&\quad \overset{(BR1)}{=} \sum \sigma(h_1, g_1)\sigma(\underline{g_2S(g_3)}g_5, h_2)\sigma(g_4, S(h_3)ah_4)\\
&\quad = \sum \sigma(h_1, g_1)\underbrace{\sigma(g_3, h_2)\sigma(g_2, S(h_3)ah_4)}_{}\\
&\quad \overset{(BR2)}{=} \sum \sigma(h_1, g_1)\sigma(g_2, \underline{h_2S(h_3)}ah_4) = \sum \sigma(h_1, g_1)\underbrace{\sigma(g_2, ah_2)}_{}\\
&\quad \overset{(BR2)}{=} \sum \sigma(h_1, g_1)\sigma(g_2, h_2)\sigma(g_3, a) = \sum \beta(h, g_1)\tau(a, g_2),
\end{aligned}
$$

and this proves (BC6).

It is easy to check that the condition (BC7).

Finally, we have:

$$
\begin{aligned}
&\sum \beta(h_1, g_1)\omega(h_2, g_{2(-1)})\tau(h_{2(-1)}, g_{20})\omega(h_{20}, b)\\
&\quad = \sum \sigma(h_1, g_1)\sigma(g_2, h_2)\sigma(\underline{g_{3(-1)}}, h_3)\sigma(\underline{g_{30}}, h_{4(-1)})\sigma(b, h_{40})\\
&\quad = \sum \sigma(h_1, g_1)\underbrace{\sigma(g_2, h_2)\sigma(S(g_3)g_5, h_3)}_{}\sigma(g_4, h_{4(-1)})\sigma(b, h_{40})
\end{aligned}
$$

$$\overset{\text{(BR1)}}{=} \sum \sigma(h_1, g_1)\sigma(\underline{g_2(S(g_3)}g_5, h_2)\sigma(g_4, h_{3(-1)})\sigma(b, h_{30})$$

$$= \sum \sigma(h_1, g_1)\sigma(g_3, h_2)\sigma(g_2, \underline{h_{3(-1)}})\sigma(b, \underline{h_{30}})$$

$$= \sum \sigma(h_1, g_1)\underbrace{\sigma(g_3, h_2)\sigma(g_2, S(h_3)h_5)}\sigma(b, h_4)$$

$$\overset{\text{(BR2)}}{=} \sum \sigma(h_1, g_1)\sigma(g_2, \underline{h_2S(h_3)}h_5)\sigma(b, h_4)$$

$$= \sum \sigma(h_1, g_1)\underline{\sigma(g_2, h_3)\sigma(b, h_2)}$$

$$\overset{\text{(BR1)}}{=} \sum \underbrace{\sigma(h_1, g_1)\sigma(g_2, h_2)}\sigma(\underline{S(g_3)bg_4}, h_3)$$

$$= \sum \beta(h_1, g_1)\sigma(b \leftharpoonup g_2, h_2) = \sum \beta(h_1, g_1)\omega(h_2, b \leftharpoonup g_2),$$

and (BC8) is proven.

A similar proof shows that (BC9) is also true.

Thus, by Theorem 2.6 we have

$$\tilde{\sigma}(h \otimes a, g \otimes b)$$

$$= \sum \underbrace{\beta(h_1, g_1)}\omega(h_2, g_{2(-1)})\omega(h_3, b_2)\alpha(a_2, b_1)\tau(a_1, g_{20})$$

$$= \sum \sigma(h_1, g_1)\sigma(g_2, h_2)\sigma(\underline{g_{3(-1)}}, h_3)\sigma(b_2, h_4)\sigma(b_1, a_2)\sigma(\underline{g_{30}}, a_1)$$

$$= \sum \sigma(h_1, g_1)\underbrace{\sigma(g_2, h_2)\sigma(S(g_3)g_5, h_3)}\sigma(b_2, h_4)\sigma(b_1, a_2)\sigma(g_4, a_1)$$

$$\overset{\text{(BR1)}}{=} \sum \sigma(h_1, g_1)\underbrace{\sigma(g_3, h_2)\sigma(b_2, h_3)}\underline{\sigma(b_1, a_2)\sigma(g_2, a_1)}$$

$$\overset{\text{(BR1)}}{=} \sum \sigma(h_1, g_1)\underbrace{\sigma(g_3b_2, h_2)\sigma(g_2b_1, a)} \overset{\text{(BR2)}}{=} \sum \sigma(h_1, g_1)\sigma(g_2b, h_2a).$$

This concludes the proof.

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