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LAGRANGIAN SUBMANIFOLDS OF THE COMPLEX HYPERBOLIC SPACE

By

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Abstract. In previous papers [2, 3], B. Y. Chen introduced a Riemannian invariant δ_M for a Riemannian n-manifold M^n. He proved in [3] that every submanifold M^n in the complex hyperbolic m-space $CH^m(-4)$ satisfies the sharp inequality: $\delta_M \leq (n^2(n-2)/2(n-1))H^2 - 2(n+1)(n-2)$, where H^2 is the squared mean curvature. In this paper, we study Lagrangian submanifolds in $CH^n(-4)$ which satisfy the equality case of the inequality.

1. Introduction

Let $CH^m(-4)$ denote the complex hyperbolic space with constant holomorphic sectional curvature -4. An immersion $F: M^n \to CH^m(-4)$ is called totally real if the complex structure J of $CH^m(-4)$ maps at every point p of M the tangent space to M at p into the normal space to M at p (cf. [10]). If $n = m$, then a totally real immersion is called a Lagrangian immersion. In this case, J interchanges the tangent and normal spaces at every point p of M. Lagrangian geometry received renewed attention after its role in mirror symmetry was discovered in [22] (see also [1]).

It is known that every submanifold M^n of $CH^m(-4)$ satisfies

$$\delta_M(p) \leq \frac{n^2(n-2)}{2(n-1)} H^2 - 2(n+1)(n-2), \tag{1.1}$$

where H^2 denotes the squared mean curvature of the immersion and δ_M is the intrinsic invariant on M defined by

$$\delta_M(p) = \frac{n(n-1)}{2}\hat{\tau}(p) - (\inf K)(p).$$

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Here $\hat{\tau}$ denotes the normalized scalar curvature and

$$(\inf K)(p) = \inf\{K(\pi) \mid \pi \text{ a plane section in } T_pM\},$$

where $K(\pi)$ is the sectional curvature of π. (This inequality was first proved by the first author in [2] for submanifolds of real space forms, and later extended in [7] to totally real submanifolds of complex space forms, and in [3] to arbitrary submanifolds of the complex hyperbolic space. A similar inequality also appeared in affine differential geometry in [14, 15, 19, 20]. For various applications of the inequality (1.1) and of related inequalities to minimal immersions, symplectic geometry, spectral geometry and some other areas of mathematics, see [5, 6].)

For simplicity an n-dimensional submanifold of $CH^m(-4)$ is said to *satisfy the basic equality* if it satisfies equality case of (1.1) identically. Since the squared mean curvature H^2 simply measures the tension which the submanifold M receives from its ambient space, a submanifold M satisfying the basic equality implies that it receives the least possible tension among all isometric immersions of M in $CH^m(-4)$. Compact submanifolds satisfying the basic equality in $CH^m(-4)$ are stable critical points of the total mean curvature functional among the class of all isometric immersions of M in $CH^m(-4)$.

Proper CR-submanifolds in complex hyperbolic spaces satisfying the basic equality are classified in [11] (see also [21] for a generalization of [11]). On the other hand, Lagrangian submanifolds of a complex hyperbolic space are non-proper CR-submanifolds. The class of Lagrangian submanifolds satisfying the basic equality is much more complicated than the proper case. The purpose of this paper is thus to investigate Lagrangian submanifolds of a complex hyperbolic space which satisfy the basic equality.

For a Lagrangian submanifold satisfying the basic equality, it is known that the submanifold is always minimal and that

$$\mathscr{D}_p = \{v \in T_pM \mid h(v, w) = 0 \ \forall w \in T_pM\}$$

defines an integrable differentiable distribution on an open dense subset of M. Away from totally geodesic points this distribution is $(n-2)$-dimensional. We denote by $\mathscr{D}_p^{\perp}$ the 2-dimensional complementary distribution. Unfortunately, $\mathscr{D}^{\perp}$ need not be integrable. However, in case it is integrable, we succeed in Section 4 to determine all Lagrangian submanifolds of $CH^n(-4)$ which satisfy the basic equality. This is done by reducing the problem to the classification of minimal Lagrangian surfaces in $CP^2(4)$, $CH^2(-4)$ and $\boldsymbol{C}^2$.

In Section 5, we then restrict ourselves to the 3-dimensional case. We show how a 3-dimensional Lagrangian submanifold satisfying the basic equality with nonintegrable complementary distribution can be obtained from a solution of the following system of differential equations:

$$\Delta h = e^{-2k/3} \sin(2h),$$

$$\Delta k = -3e^{-2k/3}(2e^{2k} + \cos(2h))$$

defined on an open part D of $\boldsymbol{R}^2$.

2. The Complex Hyperbolic Space $CH^m(-4)$

Consider the complex $(m+1)$-dimensional space $\boldsymbol{C}_1^{m+1}$ endowed with the pseudo-Euclidean metric g_0 given by

$$g_0 = -dz_0 d\bar{z}_0 + \sum_{j=1}^{m} dz_j d\bar{z}_j, \tag{2.1}$$

where $\bar{z}_k$ denotes the complex conjugate of z_k.

On $\boldsymbol{C}_1^{m+1}$, we define

$$F(z,w) = -z_0 \overline{w}_0 + \sum_{k=1}^{m} z_k \overline{w}_k.$$

Put

$$H_1^{2m+1}(-1) = \{z = (z_0, z_1, \dots, z_m) \in \boldsymbol{C}_1^{m+1} \mid \langle z, z \rangle = -1\}, \tag{2.2}$$

where $\langle\ ,\ \rangle$ denotes the inner product on $\boldsymbol{C}_1^{n+1}$ induced from g_0. Then $H_1^{2m+1}(-1)$ is a real hypersurface of $\boldsymbol{C}^{m+1}$ whose tangent space at $z \in H_1^{2m+1}(-1)$ is given by

$$T_z H_1^{2m+1}(-1) = \{w \in \boldsymbol{C}^{m+1} \mid \operatorname{Re} F(z,w) = 0\}.$$

It is known that $H_1^{2m+1}(-1)$ together with the induced metric g is a pseudo-Riemannian manifold of constant sectional curvature -1, which is known as the anti-de Sitter space time.

We put

$$H_1^1 = \{\lambda \in \boldsymbol{C} \mid \lambda\bar{\lambda} = 1\}.$$

Then we have an H_1^1-action on $H_1^{2m+1}(-1)$ given by $z \mapsto \lambda z$. At each point z in $H_1^{2m+1}(-1)$, the vector iz is tangent to the flow of the action. Since g_0 is Hermitian, we have $\operatorname{Re} g_0(iz, iz) = -1$. Note that the orbit is given by $\tilde{z}(t) = e^{it}z$ and $d\tilde{z}(t)/dt = i\tilde{z}(t)$. Thus the orbit lies in the negative-definite plane spanned by z and iz. The quotient space $H_1^{2m+1}/\sim$, under the identification induced from the action, is the complex hyperbolic space $CH^m(-4)$ with constant holomorphic sectional curvature -4. The almost complex structure J on $CH^m(-4)$ is induced from the canonical almost complex structure J on $\boldsymbol{C}_1^{m+1}$ via the totally geodesic fibration

$$\pi : H_1^{2m+1}(-1) \to CH^m(-4). \tag{2.3}$$

More details about this construction can be found in [9] and [16].

Concerning Lagrangian submanifolds, the following lifting result by Reckziegel [18] is particularly useful.

THEOREM 2.1 [18]. *Let $f : M^n \to H_1^{2m+1}(-1)$ be a horizontal isometric immersion. Then $F = \pi \circ f : M^n \to CH^m(-4)$ is a Lagrangian immersion.*

Conversely, assume that M^n is simply connected and let $F : M^n \to CH^m(-4)$ be a Lagrangian immersion. Then there exists a horizontal isometric immersion $f : M^n \to H_1^{2m+1}(-1)$ such that $F = \pi \circ f$. Moreover any two such immersions f_1 and f_2 are related by $f_1 = e^{i\theta} f_2$, where θ is a constant.

Assume $f : M \to H_1^{2n+1}(-1)$ is an isometric immersion of M in $H_1^{2n+1}(-1)$. Denote by D and ∇ the Levi-Civita connections of $\boldsymbol{C}_1^{n+1}$ and of M, respectively. Let h denote the second fundamental form of M in $H_1^{2n+1}(-1)$. Then we have

$$D_X Y = \nabla_X Y + h(X, Y) + \langle X, Y \rangle f. \tag{2.4}$$

It is known that the second fundamental forms h^f and h^F of the immersions f and F given in Theorem 2.1 are related by

$$\pi_* h^f = h^F. \tag{2.5}$$

Moreover, the second fundamental form h^f of f is horizontal with respect to π.

3. Lagrangian Immersions

From this section on, we will assume that $F : M^n \to CH^n(-4)$ is a Lagrangian immersion. We denote by $\tilde{\nabla}$ the Levi-Civita connection on $CH^n(-4)$. Then the formula of Gauss decomposes $\tilde{\nabla}$ into tangential and normal parts which states that

$$\tilde{\nabla}_X Y = \nabla_X Y + h(X, Y), \tag{3.1}$$

for X, Y tangent vector fields. Similarly, we have

$$\tilde{\nabla}_X \xi = -A_\xi X + \nabla_X^\perp \xi. \tag{3.2}$$

Since M is a Lagrangian submanifold, it is well known that [10]

$$\nabla^\perp_X JY = J\nabla_X Y, \tag{3.3}$$

$$A_{JY}X = -Jh(X, Y), \tag{3.4}$$

which imply that $\langle h(X, Y), JZ\rangle$ is totally symmetric.

The equations (3.3) and (3.4) imply that the equation of Ricci reduces to the equation of Gauss. For a Lagrangian submanifold, the equations of Gauss and Codazzi reduce to:

$$R(X, Y)Z = \langle X, Z\rangle Y - \langle Y, Z\rangle X + A_{h(Y,Z)}X - A_{h(X,Z)}Y \tag{3.5}$$

$$(\nabla_X h)(Y, Z) = (\nabla_Y h)(X, Z), \tag{3.6}$$

where

$$(\nabla_X h)(Y, Z) = \nabla^\perp_X h(Y, Z) - h(\nabla_X Y, Z) - h(Y, \nabla_X Z). \tag{3.7}$$

We recall the following existence and uniqueness theorems from [4, 7].

Theorem 3.1. *Let $(M^n, \langle\,,\rangle)$ be a Riemannian manifold and let $F_1, F_2 : M \to CH^n(-4)$ be two isometric Lagrangian immersions. Suppose that*

$$\langle h^{F_1}(X, Y), JF_{1*}(Z)\rangle = \langle h^{F_2}(X, Y), JF_{2*}(Z)\rangle.$$

Then there exists an isometry Φ of $CH^n(-4)$ such that $\Phi \circ F_1 = F_2$.

Theorem 3.2. *Let $(M^n, \langle\,,\rangle)$ be a simply connected Riemannian manifold and let T be a symmetric $(1,2)$-tensor on M such that*

(1) *$\langle T(X, Y), Z\rangle$ is totally symmetric,*
(2) *$(\nabla_X T)(Y, Z) = \nabla_X(T(Y, Z)) - T(\nabla_X Y, Z) - T(Y, \nabla_X Z)$ is totally symmetric,*
(3) *$R(X, Y)Z = -(\langle Y, Z\rangle X - \langle X, Z\rangle Y) + T(X, T(Y, Z)) - T(Y, T(X, Z))$.*

Then there exists a Lagrangian isometric immersion $F : M^n \to CH^n(-4)$ whose second fundamental form h is given by $h(X, Y) = JT(X, Y)$.

Next, assume that $F : M^n \to CH^n(-4)$, $n \geq 3$, is a Lagrangian immersion which satisfies the equality. Then, it was proved in [8] that M^n has to be minimal. Moreover, assume that the immersion has no totally geodesic points, and so

$$\mathscr{D}_p = \{v \in T_pM \mid h(v, w) = 0 \ \ \forall w \in T_pM\}$$

defines an $(n-2)$-dimensional integrable distribution. This distribution is called the nullity distribution. It is easy to show that a Lagrangian submanifold satisfies the basic equality if and only if the nullity distribution is at least $(n-2)$-dimensional.

We recall the following lemmas from [8]:

Lemma 3.1. *Let $F : M^n \to CH^n(-4)$, $n \geq 3$, be a Lagrangian immersion satisfying the basic equality without totally geodesic points and $p \in M^n$. Then there exist a nonzero function λ and an orthonormal frame $\{E_1, E_2, \dots, E_n\}$ defined on a neighborhood of p such that*

$$h(E_1, E_1) = \lambda JE_1, \quad h(E_2, E_2) = -\lambda JE_1,$$

$$h(E_1, E_2) = -\lambda JE_2, \quad h(E_i, E_j) = 0, \quad i \geq 3 \ \text{or} \ j \geq 3.$$

If we introduce local functions γ_{ij}^k by

$$\gamma_{ij}^k = \langle \nabla_{E_i} E_j, E_k \rangle,$$

then it was shown in [8] that the Codazzi equation (3.6) implies the following relations for the functions γ_{ij}^k and λ:

Lemma 3.2. *Let M, p and $\{E_1, \dots, E_n\}$ be as in the previous lemma. Then*

(i) $\gamma_{11}^i - \gamma_{22}^i = 0$, $i > 2$,
(ii) $\gamma_{12}^i + \gamma_{21}^i = 0$, $i > 2$,
(iii) $\gamma_{ij}^1 = \gamma_{ij}^2 = 0$, $i, j > 2$,
(iv) $\gamma_{i1}^2 = -\frac{1}{3}\gamma_{12}^i$, $i > 2$.

Moreover, the function λ satisfies the following system of differential equation

(v) $E_1(\lambda) = -3\lambda\gamma_{21}^2$,
(vi) $E_2(\lambda) = 3\lambda\gamma_{11}^2$,
(vii) $E_i(\lambda) = -\lambda\gamma_{1i}^1$, $i > 2$.

4. Lagrangian Submanifolds with Integrable Complementary Nullity Distribution

Let $F : M^n \to CH^n(-4)$, $n \geq 3$, be a Lagrangian immersion without totally geodesic points which satisfies the basic equality. Let us now assume that the complementary nullity distribution $\mathscr{D}^\perp$ is also integrable. In this section, we will show the way to construct such immersions F starting from minimal Lagrangian immersions of surfaces. This construction will be divided into several lemmas.

Let $p \in M$ and let $\{E_1, \dots, E_n\}$ be the local frame constructed in Lemma 3.1. Since we assume that $\mathscr{D}^\perp$ is an integrable distribution, it follows from Lemma 3.2 and [8] that we can write:

$$
\begin{aligned}
&\nabla_{E_1} E_1 = -\gamma E_2 + \alpha E_3, \\
&\nabla_{E_2} E_2 = -\delta E_1 + \alpha E_3, \\
&\nabla_{E_1} E_2 = \gamma E_1, \quad \nabla_{E_2} E_1 = \delta E_2, \\
&\nabla_{E_1} E_3 = -\alpha E_1, \quad \nabla_{E_2} E_3 = -\alpha E_2, \quad \nabla_{E_3} E_3 = 0, \\
&\nabla_{E_i} E_1 = \nabla_{E_i} E_2 = 0, \quad i \geq 3, \\
&\nabla_{E_i} E_3 = -\frac{1}{\alpha} E_i, \quad i \geq 4, \\
&\nabla_{E_i} E_j = \frac{\delta_{ij}}{\alpha} E_3 + \sum_{k \geq 4} \gamma_{ij}^k E_k, \quad i, j \geq 4.
\end{aligned}
\tag{4.1}
$$

Remark that if $\alpha = 0$ on an open set, both the distribution $\mathscr{D}$ and $\mathscr{D}^\perp$ are parallel which implies that

$$\langle R(E_3, E_1)E_1, E_3 \rangle = 0$$

on this open set. On the other hand, it would follow from the Gauss equation and Lemma 3.1 that $\langle R(E_3, E_1)E_1, E_3 \rangle = -1$. Clearly, this is a contradiction. Therefore, by restricting ourselves to an open dense subset of M, we may assume that $\alpha \neq 0$.

Recall that a distribution T is called

(a) totally geodesic (or autoparallel) if and only if $\nabla_X Y \in T$ for all $X, Y \in T$
(b) spherical if and only if there exists a vector field $H \in T^\perp$ such that $\nabla_X Y - \langle X, Y \rangle H \in T$ and $\nabla_X H \in T$ for all vector fields $X, Y \in T$. The vector field H is called the mean curvature vector of the distribution.

We define distributions T_0, T_1 and T_2 by

$$
\begin{aligned}
T_0 &= \operatorname{span}\{E_3\}, \\
T_1 &= \operatorname{span}\{E_1, E_2\}, \\
T_2 &= \operatorname{span}\{E_4, \dots, E_n\}.
\end{aligned}
$$

As in [8] we may obtain the following lemmas.

Lemma 4.1. *The distribution T_0 is totally geodesic in M^n and in $CH^n(-4)$.*

Lemma 4.2. *The distribution T_1 is spherical with mean curvature vector parallel to αE_3 and the distribution $T_0 \oplus T_1$ is autoparallel in M^n.*

Lemma 4.3. *The distribution T_2 is spherical with mean curvature vector $(1/\alpha)E_3$ and the distribution $T_0 \oplus T_2$ is totally geodesic in M^n.*

In particular, the above lemmas imply that there exist local coordinates

$$(t, u = (u_1, u_2), v = (v_1, \dots, v_{n-3}))$$

on M such that
(i) $E_3 = \frac{\partial}{\partial t}$,
(ii) $\frac{\partial}{\partial u_1}, \frac{\partial}{\partial u_2}$ span T_1,
(iii) $\frac{\partial}{\partial v_1}, \dots, \frac{\partial}{\partial v_{n-3}}$ span T_2.

Moreover, $\langle \partial/\partial u_i, \partial/\partial u_j \rangle$ does not depend on v_k and $\langle \partial/\partial v_i, \partial/\partial v_j \rangle$ does not depend on u_k.

Lemma 4.4. *The function α satisfies the following system of differential equations*:

$$E_1\alpha = E_2\alpha = E_i\alpha = 0, \quad i \geq 4, \tag{4.2}$$

$$E_3\alpha = \alpha^2 - 1. \tag{4.3}$$

Proof. $E_1\alpha = 0$ (respectively, $E_2\alpha = 0$) follows from (4.1) and the identity

$$\langle R(E_2, E_1)E_2, E_3 \rangle = 0, \quad \text{(respectively, } \langle R(E_1, E_2)E_1, E_3 \rangle = 0.)$$

The remaining equations follow from (4.1), the Gauss equation and

$$\langle \tilde{R}(E_j, E_1)E_1, E_3 \rangle = \delta_{j3}, \quad j \geq 3.$$

It follows from the previous lemmas that we can choose coordinates $t, u_1, u_2, v_1, \dots, v_{n-3}$ such that $E_3 = \partial/\partial t$, $\{\partial/\partial u_1, \partial/\partial u_2\}$ span T_1 and $\{\partial/\partial v_1, \dots, \partial/\partial v_{n-3}\}$ span T_2. It then follows from Lemma 4.4 that depending on the initial conditions and after a translation of the t-coordinate, the function α is given either by

$$\alpha = -\tanh t \quad \text{or by,} \tag{α.1}$$

$$\alpha = -\coth t \quad \text{or by,} \tag{α.2}$$

$$\alpha = \pm 1. \tag{α.3}$$

In the last case, after replacing E_3 by $-E_3$ if necessary, we may assume that $\alpha = 1$.

Now, we can formulate the main result of this section as follows.

Theorem 4.1. *Let $F : M^n \to CH^n(-4)$ be a Lagrangian immersion satisfying the basic equality and without geodesic points. Assume that the orthogonal complement of the nullity distribution is integrable. Then, every point p of an open dense subset of M^n has a neighborhood U_p such that either*

(i) $F(t,u,v) = \pi(\cosh t(\psi(u), 0, \dots, 0) + \sinh t(0,0,0,\phi(v)))$, *where*

$$\phi : (v_1, \dots, v_{n-3}) \mapsto \phi(v)$$

describes the standard totally real $(n-3)$-sphere S^{n-3} in $E^{n-2} \subset \boldsymbol{C}^{n-2}$ and $\psi : (u_1, u_2) \mapsto \psi(u)$ describes a minimal horizontal immersion in $H_1^5(-1)$, or

(ii) $F(t,u,v) = \pi(\cosh t(\phi(v), 0,0,0)) - \sinh t(0, \dots, 0, \psi(u)))$, *where*

$$\phi : (v_1, \dots, v_{n-3}) \mapsto \phi(v)$$

describes the standard totally real hyperbolic space H_1^{n-3} in $E_1^{n-2} \subset \boldsymbol{C}_1^{n-2}$ and $\psi : (u_1, u_2) \mapsto \psi(u)$ describes a minimal horizontal immersion in $S^5(1)$, or

(iii) $F(t,u,v) = \pi((\cosh t, -\sinh t, 0, \dots, 0) + (1/2)e^{-t}z(u,v)(1,-1,0,\dots,0) + (1/2)e^{-t}(0,0,w_1(u), w_2(u), v_1, \dots, v_{n-3}))$, *where*

$$w : D \subset \boldsymbol{R}^2 \to \boldsymbol{C}^2 : (u_1, u_2) \mapsto (w_1(u_1,u_2), w_2(u_1,u_2))$$

is a minimal Lagrangian immersion and z is a complex-valued function determined by the condition that

$$2(z + \bar{z}) = w_1\overline{w}_1 + w_2\overline{w}_2 + \sum_{i=1}^{n-3} v_i^2,$$

and by the condition that its imaginary part depends only on u and satisfies the following system of differential equations:

$$(z - \bar{z})_{u_1} = \frac{1}{2}\{w_1(\overline{w}_1)_{u_1} + w_2(\overline{w}_2)_{u_1} - \overline{w}_1(w_1)_{u_1} - \overline{w}_2(w_2)_{u_1}\},$$

$$(z - \bar{z})_{u_2} = \frac{1}{2}\{w_1(\overline{w}_1)_{u_2} + w_2(\overline{w}_2)_{u_2} - \overline{w}_1(w_1)_{u_2} - \overline{w}_2(w_2)_{u_2}\},$$

where $\pi : H_1^{2m+1}(-1) \to CH^m(-4)$ is the projection defined in section 2.

Proof. First, we assume that (α.1) holds. Denote by $f : M^n \to H_1^{2n+1}(-1)$ a horizontal lift of the Lagrangian immersion $F : M^n \to CH^n(-4)$. We define a map ψ by

$$\psi(t,u,v) = (\cosh t) f(t,u,v) - (\sinh t) E_3(t,u,v). \tag{4.4}$$

Clearly, $\langle \psi, \psi \rangle = -1$. A straightforward computation shows that

$$\begin{aligned} &D_{E_3}\psi = D_{E_i}\psi = 0, \quad i > 3, \\ &D_{E_1}\psi = (\operatorname{sech} t) E_1, \\ &D_{E_2}\psi = (\operatorname{sech} t) E_2. \end{aligned} \tag{4.5}$$

The above implies that ψ depends only on $u = (u_1, u_2)$. Therefore, we write $\psi(u) = \psi(t_0, u, v_0)$. It follows that ψ determines an immersion of a surface into $H_1^{2n+1}(-1)$.

From (2.4), (2.5), (4.1), (4.4), (4.5), (α.1), and Lemma 3.1 we find

$$\begin{aligned} &D_{E_1}\psi_*(E_1) = (\operatorname{sech} t)(-\gamma E_2 + \lambda i E_1 + (\operatorname{sech} t)\psi), \\ &D_{E_1}\psi_*(E_2) = (\operatorname{sech} t)(\gamma E_1 - \lambda i E_2), \\ &D_{E_2}\psi_*(E_2) = (\operatorname{sech} t)(-\delta E_1 - \lambda i E_1 + (\operatorname{sech} t)\psi). \end{aligned} \tag{4.6}$$

Thus, ψ determines a minimal horizontal immersion in an $H_1^5(-1)$ which is totally geodesic in $H_1^{2n+1}(-1)$.

Next, we put

$$\phi(t,u,v) = (\operatorname{csch} t)\{f(t,u,v) - (\cosh t)\psi(u)\}. \tag{4.7}$$

It follows that

$$\begin{aligned} &\phi_*(E_1) = D_{E_1}\phi = 0, \\ &\phi_*(E_2) = D_{E_2}\phi = 0, \\ &\phi_*(E_3) = D_{E_3}\phi = 0, \\ &\phi_*(E_i) = D_{E_i}\phi = (\operatorname{csch} t)E_i, \quad i \geq 4. \end{aligned}$$

Hence ϕ depends only on the variable v. Also, we have that

$$\begin{aligned} \langle \phi, \phi \rangle &= (\operatorname{csch}^2 t)\{\langle f, f \rangle - 2\cosh t \langle f, \psi \rangle + \cosh^2 t \langle \psi, \psi \rangle\} \\ &= (\operatorname{csch}^2 t)(\cosh^2 t - 1) = 1. \end{aligned}$$

It now follows from the previous lemmas that

$$D_{E_j}\phi_*(E_i) = (\operatorname{csch} t)\left\{\nabla^{\phi}_{E_j}E_i + \frac{\delta_{ij}}{\alpha}E_3 + \delta_{ij}f\right\}$$
$$= (\operatorname{csch} t)\nabla^{\phi}_{E_j}E_i - (\operatorname{csch}^2 t)\delta_{ij}\phi,$$

where $\nabla^{\phi}_{E_j}E_i$ denotes the T_2-component of $D_{E_j}E_i$. This implies that the image of ϕ is a real hypersphere in an $(n-2)$-dimensional, totally real, positive-definite subspace of $\boldsymbol{C}_1^{n+1}$, which is also orthogonal to the 3-dimensional complex space containing the image of ψ. Combining (4.4) and (4.7) we get

$$f(t,u,v) = (\sinh t)\phi(v) + (\cosh t)\psi(u).$$

Applying now an isometry of $\boldsymbol{C}_1^{n+1}$ gives (i).

In order to prove (ii), we proceed similarly starting from (α.2). We define

$$\psi(t,u,v) = (\sinh t)f - (\cosh t)E_3$$

and

$$\phi(t,u,v) = (\operatorname{sech} t)\{f + (\sinh t)\psi)\}.$$

Following now exactly the same type of arguments as before, we obtain (ii).

Finally, we consider the case that $\alpha = 1$ on a neighborhood of the point p. Then, we have

$$D_{E_3}(e^{-t}(f+E_3)) = -e^{-t}(f+E_3) + e^{-t}(E_3+f) = 0,$$
$$D_{E_i}(e^{-t}(f+E_3)) = e^{-t}(E_i + D_{E_i}E_3) = e^{-t}\left(E_i - \frac{1}{\alpha}E_i\right) = 0, \quad i > 3,$$
$$D_{E_1}(e^{-t}(f+E_3)) = e^{-t}(E_1 - \alpha E_1) = 0,$$
$$D_{E_2}(e^{-t}(f+E_3)) = 0.$$

Since $\langle e^{-t}(f+E_3), e^{-t}(f+E_3)\rangle = 0$, we see that $e^{-t}(f+E_3)$ is a constant lightlike vector along M. Clearly, by applying an isometry of $\boldsymbol{C}_1^{n+1}$ we may assume

$$e^{-t}(f+E_3) = (1,-1,0,\ldots,0).$$

We now define a map η by

$$\eta(t,u,v) = e^t(f - E_3).$$

Then

$$D_{E_3}\eta = e^t(f - E_3) + e^t(E_3 - f) = 0,$$
$$D_{E_1}\eta = 2e^t E_1,$$
$$D_{E_2}\eta = 2e^t E_2,$$
$$D_{E_i}\eta = 2e^t E_i, \quad i > 3,$$

from which we deduce that η depends only on the u and v-variables. Therefore it defines a map from an open part of $\boldsymbol{R}^2 \times \boldsymbol{R}^{n-3}$ into $\boldsymbol{C}_1^{n+1}$. The above formulas also imply that

$$(4.8) \qquad \begin{aligned} &\left\langle \eta_*\left(\frac{\partial}{\partial u_i}\right), \eta_*\left(\frac{\partial}{\partial v_j}\right)\right\rangle = 0, \\ &\left\langle \eta_*\left(\frac{\partial}{\partial u_i}\right), \eta_*\left(\frac{\partial}{\partial u_j}\right)\right\rangle \quad \text{does not depend on } v_k, \\ &\left\langle \eta_*\left(\frac{\partial}{\partial v_i}\right), \eta_*\left(\frac{\partial}{\partial v_j}\right)\right\rangle \quad \text{does not depend on } u_k. \end{aligned}$$

This means that the metric induced by the map η defines a product metric on $\boldsymbol{R}^2 \times \boldsymbol{R}^{n-3}$. Computing now the second fundamental form for the immersion η of an open part of $\boldsymbol{R}^2 \times \boldsymbol{R}^{n-3}$ into $\boldsymbol{C}_1^{n+1}$, we get

$$D_{E_1}\eta_*(E_1) = 2e^t(-\gamma E_2 + \lambda i E_1) + 2e^t(E_3 + f),$$
$$D_{E_1}\eta_*(E_2) = 2e^t(\gamma E_1 - \lambda i E_2),$$
$$D_{E_2}\eta_*(E_2) = 2e^t(-\delta E_1 - \lambda i E_1) + 2e^t(E_3 + f),$$
$$D_{E_i}\eta_*(E_1) = 0,$$
$$D_{E_i}\eta_*(E_2) = 0,$$
$$D_{E_j}\eta_*(E_i) = 2e^t D_{E_j} E_i = 2e^t(\nabla_{E_j} E_i)_{T_2} + 2e^t \delta_{ij}(E_3 + f),$$

where $(\cdot)_{T_2}$ denotes the component of $(\cdot)$ in the direction of the distribution T_2. Recall that $e^{-t}(E_3 + f)$ is a constant vector and therefore the image of η is contained in a linear n-dimensional space parallel to the complex subspace spanned by $(1, -1, 0, \ldots, 0)$ and $E_1(p), E_2(p), E_4(p), \ldots, E_n(p)$. Choosing initial conditions at the point $p = (t_0, u_0, v_0)$, we may assume that

$$\begin{aligned}
&e^{-t_0}(f+E_3)(p) = (1,-1,0,\dots,0),\\
&\eta(p) = e^{t_0}(f-E_3)(p) = (1,1,0,\dots,0),\\
&E_1(p) = (0,0,1,0,0,\dots,0),\\
&E_2(p) = (0,0,0,1,0,\dots,0),\\
&E_4(p) = (0,0,0,0,1,0,\dots,0),\\
&\dots\dots\\
&E_n(p) = (0,0,0,0,\dots,0,1).
\end{aligned} \tag{4.9}$$

Since $\eta(p) = (1,1,0,\dots,0)$ and η is independent of the variable t, we can thus write

$$\begin{aligned}
\eta(u,v) &= \eta(t_0,u,v) = \eta(t,u,v)\\
&= (1,1,0,\dots,0) + z(u,v)(1,-1,0,\dots,0) + (0,0,\eta_1(u,v),\dots,\eta_{n-1}(u,v)).
\end{aligned}$$

Denote now by ξ the map from an open part of $\boldsymbol{R}^2 \times \boldsymbol{R}^{n-3}$ to $\boldsymbol{C}^{n-1}$ defined by

$$(u,v) \mapsto \xi(u,v) = (\eta_1(u,v),\dots,\eta_{n-1}(u,v)).$$

Clearly $\xi = \bar{p} \circ \eta$, where $\bar{p}$ denotes the projection on the last $(n-1)$ coordinates. It follows that

$$\begin{aligned}
&\xi_*(E_1) = \eta_*(E_1) - \alpha_1 e^{-t}(f+E_3) = 2e^t E_1 - \alpha_1 e^{-t}(f+E_3),\\
&\xi_*(E_2) = \eta_*(E_2) - \alpha_2 e^{-t}(f+E_3) = 2e^t E_2 - \alpha_2 e^{-t}(f+E_3),\\
&\xi_*(E_i) = \eta_*(E_i) - \alpha_i e^{-t}(f+E_3) = 2e^t E_i - \alpha_i e^{-t}(f+E_3), \quad i > 3,
\end{aligned} \tag{4.10}$$

where $\alpha_1, \alpha_2, \alpha_4, \dots, \alpha_n$ are local functions. The above formulas show that ξ defines a Lagrangian immersion from an open part of $\boldsymbol{R}^2 \times \boldsymbol{R}^{n-3}$ into $\boldsymbol{C}^{n-1}$. By using (4.8), (4.9) and (4.10), we see that the pull-back metric on $\boldsymbol{R}^2 \times \boldsymbol{R}^{n-3}$ is the product metric. Computing now the second fundamental form of this immersion, we get

$$\begin{aligned}
&D_{E_1}\xi_*(E_1) = \gamma\xi_*(E_2) + i\lambda\xi_*(E_1),\\
&D_{E_2}\xi_*(E_1) = \delta\xi_*(E_2) - i\lambda\xi_*(E_2),\\
&D_{E_2}\xi_*(E_2) = -\delta\xi_*(E_1) - i\lambda\xi_*(E_1),\\
&D_{E_i}\xi_*(E_j) = \xi_*((\nabla_{E_i}E_j)_{T_2}),\\
&D_{E_i}\xi_*(E_1) = D_{E_i}\xi_*(E_2) = 0.
\end{aligned}$$

Hence applying Moore's theorem, we conclude that the immersion ξ is a product immersion. Thus it decomposes into immersions $w : \boldsymbol{R}^2 \to \boldsymbol{C}^2$ and $\tau : \boldsymbol{R}^{n-3} \to \boldsymbol{C}^{n-3}$. Moreover, the second one is totally geodesic and Lagrangian, whereas the first one is Lagrangian and minimal. Therefore we can write

$$\begin{aligned}\eta &= (1,1,0,\dots,0) + z(u,v)(1,-1,0,\dots,0)\\ &\quad + (0,0,w_1(u_1,u_2),w_2(u_1,u_2),v_1,\dots,v_{n-3}),\end{aligned}$$

where $w : D \subset \boldsymbol{R}^2 \to \boldsymbol{C}^2 : (u_1,u_2) \mapsto (w_1(u_1,u_2), w_2(u_1,u_2))$ is a minimal Lagrangian immersion. Since $\langle \eta,\eta\rangle = 0$, we deduce that

$$2(z+\bar{z}) = w_1\bar{w}_1 + w_2\bar{w}_2 + \sum_{i=1}^{n-3} v_i^2. \tag{4.11}$$

Since

$$\begin{aligned}f + E_3 &= e^t(1,-1,0,\dots,0),\\ f - E_3 &= e^{-t}\eta = e^{-t}(1,1,0,\dots,0) + e^{-t}z(u,v)(1,-1,0,\dots,0)\\ &\quad + e^{-t}(0,0,w_1(u_1,u_2),w_2(u_1,u_2),v_1,\dots,v_{n-3}),\end{aligned}$$

we deduce that

$$\begin{aligned}f &= (\cosh t, -\sinh t, 0,\dots,0) + \frac{1}{2}e^{-t}z(u,v)(1,-1,0,\dots,0)\\ &\quad + \frac{1}{2}e^{-t}(0,0,w_1(u_1,u_2),w_2(u_1,u_2),v_1,\dots,v_{n-3}).\end{aligned}$$

Since f is horizontal, we have $\langle if, f_t\rangle = \langle if, f_{u_i}\rangle = \langle if, f_{v_j}\rangle = 0$ and

$$\begin{aligned}if &= (i\cosh t, -i\sinh t, 0,\dots,0) + \frac{1}{2}e^{-t}iz(u,v)(1,-1,0,\dots,0)\\ &\quad + \frac{i}{2}e^{-t}(0,0,w_1,w_2,v_1,\dots,v_{n-3}),\\ f_t &= (\sinh t, -\cosh t, 0,\dots,0) - \frac{1}{2}e^{-t}z(u,v)(1,-1,0,\dots,0)\\ &\quad - \frac{1}{2}e^{-t}(0,0,w_1,w_2,v_1,\dots,v_{n-3}),\end{aligned}$$

$$f_{u_1} = \frac{1}{2}e^{-t}z_{u_1}(1,-1,0,\dots,0) + \frac{1}{2}e^{-t}(0,0,(w_1)_{u_1},(w_2)_{u_1},0,\dots,0),$$

$$f_{u_2} = \frac{1}{2}e^{-t}z_{u_2}(1,-1,0,\dots,0) + \frac{1}{2}e^{-t}(0,0,(w_1)_{u_2},(w_2)_{u_2},0,\dots,0),$$

$$f_{v_1} = \frac{1}{2}e^{-t}z_{v_1}(1,-1,0,\dots,0) + \frac{1}{2}e^{-t}(0,0,0,0,1,0,\dots,0),\dots$$

$$f_{v_{n-3}} = \frac{1}{2}e^{-t}z_{v_{n-3}}(1,-1,0,\dots,0) + \frac{1}{2}e^{-t}(0,0,0,0,0,\dots,0,1).$$

we deduce that

$$\begin{aligned}
0 = \langle if, f_{u_1}\rangle &= \frac{ie^{-t}}{4}(-(\cosh t)z_{u_1} + (\sinh t)z_{u_1} + (\cosh t)\bar{z}_{u_1} - (\sinh t)\bar{z}_{u_1}) \\
&\quad + \frac{ie^{-2t}}{8}(w_1(\overline{w}_1)_{u_1} - \overline{w}_1(\xi_1^1)_{u_1} + w_2(\overline{w}_2)_{u_1} - \overline{w}_2(w_2)_{u_1}) \\
&= \frac{ie^{-2t}}{8}(2(\bar{z}-z)_{u_1} + w_1(\overline{w}_1)_{u_1} + w_2(\overline{w}_2)_{u_1} - \overline{w}_1(w_1)_{u_1} - \overline{w}_2(w_2)_{u_1}),
\end{aligned}$$

$$0 = \langle if, f_{u_2}\rangle = \frac{ie^{-2t}}{8}(2(\bar{z}-z)_{u_2} + w_1(\overline{w}_1)_{u_2} + w_2(\overline{w}_2)_{u_2} - \overline{w}_1(w_1)_{u_2} - \overline{w}_2(w_2)_{u_2}),$$

$$0 = \langle if, f_{v_i}\rangle = \frac{ie^{-2t}}{8}(2(\bar{z}-z)_{v_i}), \quad i > 3.$$

Hence the imaginary part of z depends only on u and is determined by

$$(z-\bar{z})_{u_1} = \frac{1}{2}\{w_1(\overline{w}_1)_{u_1} + w_2(\overline{w}_2)_{u_1} - \overline{w}_1(w_1)_{u_1} - \overline{w}_2(w_2)_{u_1}\},$$

$$(z-\bar{z})_{u_2} = \frac{1}{2}\{w_1(\overline{w}_1)_{u_2} + w_2(\overline{w}_2)_{u_2} - \overline{w}_1(w_1)_{u_2} - \overline{w}_2(w_2)_{u_2}\}.$$

This completes the proof of the theorem. □

Remark 4.1. Conversely, a straightforward computation shows that the immersion, as defined in Theorem 4.1, give rise to Lagrangian immersions of $CH^n(-4)$ satisfying the basic equality.

5. Lagrangian 3-Dimensional Immersions in $CH^3(-4)$

Let $F: M^3 \to CH^3(-4)$ be a Lagrangian immersion without totally geodesic points satisfying the basic equality. Let $p \in M$ and let $\{E_1, E_2, E_3\}$ be the local frame constructed in Lemma 3.1. So, we have from Lemma 3.2 that

$$
\begin{aligned}
&\nabla_{E_1}E_1 = -\gamma E_2 + \alpha E_3,\\
&\nabla_{E_2}E_2 = -\delta E_1 + \alpha E_3,\\
&\nabla_{E_3}E_3 = 0,\\
&\nabla_{E_1}E_2 = \gamma E_1 + \beta E_3,\\
&\nabla_{E_2}E_1 = \delta E_2 - \beta E_3,\\
&\nabla_{E_1}E_3 = -\alpha E_1 - \beta E_2,\\
&\nabla_{E_2}E_3 = \beta E_1 - \alpha E_2,\\
&\nabla_{E_3}E_1 = -\tfrac{1}{3}\beta E_2,\\
&\nabla_{E_3}E_2 = \tfrac{1}{3}\beta E_1.
\end{aligned}
\tag{5.1}
$$

We also assume that M^3 contains no points where the distribution $\mathscr{D}^\perp$ is integrable. In particular the function β is nowhere zero. Therefore, by changing the sign of E_2 if necessary we may assume that $\beta > 0$. Remark that from Lemma 3.2 we also have

$$
\begin{aligned}
&E_1(\lambda) = -3\lambda\delta,\\
&E_2(\lambda) = -3\lambda\gamma,\\
&E_3(\lambda) = \lambda\alpha.
\end{aligned}
\tag{5.2}
$$

Since we assumed that M^3 does not have any totally geodesic points, we may assume $\lambda > 0$, by replacing E_3 by $-E_3$ if necessary. So (5.2) yields

$$
\begin{aligned}
&E_1(\mu) = -3\delta,\\
&E_2(\mu) = -3\gamma,\\
&E_3(\mu) = \alpha, \quad \mu = \log\lambda.
\end{aligned}
\tag{5.3}
$$

Lemma 5.1. *The functions α, β, γ and δ satisfy the following system of differential equations*:

(i) $E_3(\alpha) = \alpha^2 - \beta^2 - 1$,
(ii) $E_3(\beta) = 2\alpha\beta$,
(iii) $E_1(\alpha) - E_2(\beta) = 0$,
(iv) $E_1(\beta) + E_2(\alpha) = 0$,
(v) $E_1(\beta) - 3E_3(\gamma) = 2\beta\delta - 3\alpha\gamma$,
(vi) $E_2(\beta) + 3E_3(\delta) = 2\beta\gamma + 3\alpha\delta$,

(vii) $E_1(\gamma) - E_2(\delta) = -\frac{2}{3}\alpha\beta$,
(viii) $E_1(\delta) + E_2(\gamma) + \gamma^2 + \delta^2 = 1 + 2\lambda^2 - \alpha^2 - \frac{5}{3}\beta^2$.

Proof. Statements (i)–(v) and (viii) follow from the Gauss equation and straight-forward computation. And statements (vi) and (vii) follow from (5.3). □

Remark that except for Lemma 5.1 (viii), these are exactly the same equations as those obtained in Lemma 2 of [15] determining an extremal class of 3-dimensional hyperbolic affine sphere. Therefore, following the ideas of [15], we get the following two lemmas:

Lemma 5.2. *For any function t with $E_3(t) = 1$ there are functions $h, k, \ell : M \to \boldsymbol{R}$ with $E_3(h) = E_3(k) = E_3(\ell) = 0$ such that*

$$\alpha = -\frac{\sinh(t-\ell)\cosh(t-\ell)}{\sinh^2(t-\ell) + \cos^2(h)}, \tag{ix}$$

$$\beta = \frac{\sin(h)\cos(h)}{\sinh^2(t-\ell) + \cos^2(h)}, \tag{x}$$

$$\lambda = \frac{e^k}{\sqrt{\sinh^2(t-\ell) + \cos^2(h)}}. \tag{xi}$$

Proof. Combining Lemma 5.1 (i) and Lemma 5.1 (ii) yields

$$E_3(\alpha + i\beta) = (\alpha + i\beta)^2 - 1.$$

Integrate this equation along the integral curves of E_3 gives

$$(\alpha + i\beta - 1) = (\alpha + i\beta + 1)e^{2(t-\ell+i(\pi/2-h))}$$

for some functions ℓ, h satisfying $E_3\ell = E_3 h = 0$, which implies equations (ix) and (x). Equation (xi) can now be obtained by solving the third differential equation of (5.3).

Lemma 5.3. *Let f be any function with $E_3(f) = -1$. Define functions a_1, a_2, b_1, b_2 by*

$$(a_1 + ia_2)^3 = \frac{1}{\lambda((\alpha + i\beta)^2 - 1)},$$

$$b_1 = E_1(f),$$

$$b_2 = E_2(f).$$

Then the vector fields

$$T = E_3,$$
$$(5.4) \qquad U = a_1E_1 + a_2E_2 + (a_1b_1 + a_2b_2)E_3,$$
$$V = -a_2E_1 + a_1E_2 + (a_1b_2 - a_2b_1)E_3,$$

satisfy $[T, U] = [T, V] = [U, V] = 0$.

This lemma can be verified by a direct long computation.

Lemma 5.3 implies that it is possible to find coordinates (t, u, v) such that

$$(5.5) \qquad \frac{\partial}{\partial t} = T, \quad \frac{\partial}{\partial u} = U, \quad \frac{\partial}{\partial v} = V.$$

It also follows from (5.2), Lemma 5.1 (i) and Lemma 5.1 (ii) that a possible choice of the function f is given by

$$(5.6) \qquad f = -\frac{1}{2}\sinh^{-1}\left(\frac{-2\alpha}{\sqrt{(\alpha^2 - \beta^2 - 1)^2 + 4\alpha^2\beta^2}}\right).$$

Using Lemma 5.2, we deduce that

$$(5.7) \qquad f = \ell - t.$$

Since $E_1(f) = b_1$, $E_2(f) = b_2$, $E_3(f) = -1$, it follows from (5.4), (5.5) and (5.7) that

$$\frac{\partial f}{\partial u} = \frac{\partial f}{\partial v} = 0.$$

Hence (5.7) and $E_3(f) = -1$ imply that ℓ is a constant. Thus, by a translation of the t-coordinate we may assume that $\ell = 0$. Thus, (ix), (x) and (xi) reduce to

$$(\text{ix}') \qquad \alpha = -\frac{\sinh(t)\cosh(t)}{\sinh^2(t) + \cos^2(h)},$$

$$(\text{x}') \qquad \beta = \frac{\sin(h)\cos(h)}{\sinh^2(t) + \cos^2(h)},$$

$$(\text{xi}') \qquad \lambda = \frac{e^k}{\sqrt{\sinh^2(t) + \cos^2(h)}}$$

where $h = h(u, v)$, $k = k(u, v)$.

We can formulate the main theorem of this section as follows.

THEOREM 5.1. *Let* $f : M^3 \to CH^3(-4)$ *be a Lagrangian immersion satisfying the basic equality and* $p \in M^3$. *If the immersion has no totally geodesic points and the distribution* $\mathscr{D}^\perp$ *is nowhere integrable, then there exist coordinates* (u, v, t) *defined in a neighborhood* $D \times I$ *of* p *and functions* $h : D \to \boldsymbol{R} : (u, v) \mapsto h(u, v)$ *and* $k : D \to \boldsymbol{R} : (u, v) \mapsto k(u, v)$ *satisfying*

$$\Delta h = e^{-2k/3} \sin(2h), \tag{5.8}$$

and

$$\Delta k = -3e^{-2k/3}(\cos(2h) + 2e^{2k}), \tag{5.9}$$

where $\Delta = \partial^2/\partial u^2 + \partial^2/\partial v^2$. *Moreover, the induced metric can be expressed by*

$$\begin{aligned}
&\left\langle \frac{\partial}{\partial t}, \frac{\partial}{\partial t} \right\rangle = 1, \quad \left\langle \frac{\partial}{\partial u}, \frac{\partial}{\partial v} \right\rangle = -h_u h_v, \\
&\left\langle \frac{\partial}{\partial t}, \frac{\partial}{\partial u} \right\rangle = h_v, \quad \left\langle \frac{\partial}{\partial u}, \frac{\partial}{\partial u} \right\rangle = e^{-2k/3}(\cos^2 h + \sinh^2 t) + h_v^2, \\
&\left\langle \frac{\partial}{\partial t}, \frac{\partial}{\partial v} \right\rangle = -h_u, \quad \left\langle \frac{\partial}{\partial v}, \frac{\partial}{\partial v} \right\rangle = e^{-2k/3}(\cos^2 h + \sinh^2 t) + h_u^2,
\end{aligned} \tag{5.10}$$

and the tensor $T = -Jh$ *induced from the second fundamental form satisfies*

$$\begin{aligned}
&T\left(\frac{\partial}{\partial u}, \frac{\partial}{\partial t}\right) = T\left(\frac{\partial}{\partial v}, \frac{\partial}{\partial t}\right) = T\left(\frac{\partial}{\partial t}, \frac{\partial}{\partial t}\right) = 0, \\
&T\left(\frac{\partial}{\partial u}, \frac{\partial}{\partial u}\right) = e^{2k/3}\left(1 - \frac{2\cos^2 h \cosh^2 t}{\sinh^2 t + \cos^2 h}\right)\left(\frac{\partial}{\partial u} - h_v \frac{\partial}{\partial t}\right), \\
&\qquad - \frac{1}{2} e^{2k/3} \frac{\sin 2h \sinh 2t}{\sinh^2 t + \cos^2 h}\left(\frac{\partial}{\partial v} + h_u \frac{\partial}{\partial t}\right), \\
&T\left(\frac{\partial}{\partial v}, \frac{\partial}{\partial v}\right) = -e^{2k/3}\left(1 - \frac{2\cos^2 h \cosh^2 t}{\sinh^2 t + \cos^2 h}\right)\left(\frac{\partial}{\partial u} - h_v \frac{\partial}{\partial t}\right), \\
&\qquad + \frac{1}{2} e^{2k/3} \frac{\sin 2h \sinh 2t}{\sinh^2 t + \cos^2 h}\left(\frac{\partial}{\partial v} + h_u \frac{\partial}{\partial t}\right), \\
&T\left(\frac{\partial}{\partial u}, \frac{\partial}{\partial v}\right) = -\frac{1}{2} e^{2k/3} \frac{\sin 2h \sinh 2t}{\sinh^2 t + \cos^2 h}\left(\frac{\partial}{\partial u} - h_v \frac{\partial}{\partial t}\right), \\
&\qquad - e^{2k/3}\left(1 - \frac{2\cos^2 h \cosh^2 t}{\sinh^2 t + \cos^2 h}\right)\left(\frac{\partial}{\partial v} + h_u \frac{\partial}{\partial t}\right).
\end{aligned} \tag{5.11}$$

Conversely, let h, k be any solutions of (5.8) *and* (5.9) *on an open set D of $\boldsymbol{R}^2$. We define on $M = D \times \boldsymbol{R}$ a metric by* (5.10) *and a tensor T by* (5.11). *Let*

$$M_0 = \left\{(x,t) \in M \mid h(x) \neq \frac{1}{2}(2k+1)\pi \in \boldsymbol{Z} \text{ or } t \neq 0\right\}$$

and let M_1 be a simply connected component of M_0. Then, up to rigid motions of $CH^3(-4)$, there exists a unique Lagrangian immersion $F : M_1 \to CH^3(-4)$ with nonintegrable distribution $\mathscr{D}$, the second fundamental form $h = JT$, and satisfying the basic equality.

Proof. Let $f : M^3 \to CH^3(-4)$ be a Lagrangian immersion satisfying the basic equality. Assume that the immersion has no totally geodesic points and the distribution $\mathscr{D}^{\perp}$ is nowhere integrable. We use the notations introduced in the beginning of this section.

First it follows from (5.4), (5.5), (ix′), (x′), and (iii) and (iv) of Lemma 5.1 and a straightforward computation that

$$b_1 = \frac{a_2 h_u + a_1 h_v}{a_1^2 + a_2^2}, \tag{5.12}$$

$$b_2 = -\frac{a_1 h_u - a_2 h_v}{a_1^2 + a_2^2}. \tag{5.13}$$

From (5.4) and (5.5) we get

$$E_1\mu = \frac{a_1\mu_u - a_2\mu_v}{a_1^2 + a_2^2} - b_1\mu_t, \tag{5.14}$$

$$E_2\mu = \frac{a_1\mu_v + a_2\mu_u}{a_1^2 + a_2^2} - b_2\mu_t. \tag{5.15}$$

Using (5.12)–(5.14), (5.3) becomes

$$\gamma = -\frac{a_2\mu_u + a_1\mu_v - (-a_1 h_u + a_2 h_v)\mu_t}{3(a_1^2 + a_2^2)},$$

$$\delta = -\frac{a_1\mu_u - a_2\mu_v - (a_2 h_u + a_1 h_v)\mu_t}{3(a_1^2 + a_2^2)}.$$

Thus, using Lemma 5.3 it follows after a long computation that we can express a_{1u}, a_{1v}, a_{2u} and a_{2v} in terms of $a_1, a_2, \mu, \mu_u, \alpha$ and β as follows:

$$a_{1u} = -\frac{a_1 k_u(\cos 2h + \cosh 2t) + 3a_1 h_u \sin 2h - 2a_2 h_u \sinh 2t}{3(\cos 2h + \cosh 2t)},$$

$$a_{1v} = -\frac{a_1 k_v(\cos 2h + \cosh 2t) + 3a_1 h_v \sin 2h - 2a_2 h_v \sinh 2t}{3(\cos 2h + \cosh 2t)},$$

$$a_{2u} = -\frac{a_2 k_u(\cos 2h + \cosh 2t) + 3a_2 h_u \sin 2h + 2a_1 h_u \sinh 2t}{3(\cos 2h + \cosh 2t)},$$

$$a_{2v} = -\frac{a_2 k_v(\cos 2h + \cosh 2t) + 3a_2 h_v \sin 2h + 2a_1 h_v \sinh 2t}{3(\cos 2h + \cosh 2t)}.$$

We now deduce after a lengthy computation that equations (v) and (vi) of Lemma 5.1 (v) and Lemma 5.1 are trivially satisfied and that equations (vii) and (viii) of Lemma 5.1 reduce to (5.8) and (5.9). In order to obtain (5.10) and (5.11), we use Lemma 5.3. First it follows that

$$\left\langle \frac{\partial}{\partial t}, \frac{\partial}{\partial t} \right\rangle = \langle E_3, E_3 \rangle = 1,$$

$$\left\langle \frac{\partial}{\partial t}, \frac{\partial}{\partial u} \right\rangle = \langle E_3, U \rangle = a_1 b_1 + a_2 b_2 = h_v,$$

$$\left\langle \frac{\partial}{\partial t}, \frac{\partial}{\partial v} \right\rangle = \langle E_3, V \rangle = a_1 b_2 - a_2 b_1 = -h_u,$$

$$\left\langle \frac{\partial}{\partial u}, \frac{\partial}{\partial u} \right\rangle = a_1^2 + a_2^2 + (a_1 b_1 + a_2 b_2)^2$$

$$= h_v^2 + e^{-(2/3)k}(\sinh^2 t + \cos^2 h),$$

$$\left\langle \frac{\partial}{\partial v}, \frac{\partial}{\partial v} \right\rangle = a_1^2 + a_2^2 + (a_1 b_2 - a_2 b_1)^2$$

$$= h_u^2 + e^{-(2/3)k}(\sinh^2 t + \cos^2 h),$$

$$\left\langle \frac{\partial}{\partial u}, \frac{\partial}{\partial v} \right\rangle = (a_1 b_1 + a_2 b_2)(a_1 b_2 - a_2 b_1)$$

$$= -h_v h_u.$$

In order to obtain (5.11) we proceed as follows. First, it follows from Lemma 5.3 that

$$E_1 = \frac{a_1}{a_1^2+a_2^2}\left(\frac{\partial}{\partial u} - h_v\frac{\partial}{\partial t}\right) - \frac{a_2}{a_1^2+a_2^2}\left(\frac{\partial}{\partial v} + h_u\frac{\partial}{\partial t}\right),$$

$$E_2 = \frac{a_2}{a_1^2+a_2^2}\left(\frac{\partial}{\partial u} - h_v\frac{\partial}{\partial t}\right) + \frac{a_1}{a_1^2+a_2^2}\left(\frac{\partial}{\partial v} + h_u\frac{\partial}{\partial t}\right),$$

$$E_3 = \frac{\partial}{\partial t}.$$

It also follows from Lemma 5.2 and 5.3 that

$$\begin{aligned}(\alpha^2-\beta^2-1)^2+4\alpha^2\beta^2 &= (\sinh^2 t+\cos^2 h)^{-2},\\ a_1^2+a_2^2 &= \lambda^{-2/3}((\alpha^2-\beta^2-1)^2+4\alpha^2\beta^2)^{-1/3},\\ &= e^{-2k/3}(\sinh^2 t+\cos^2 h),\end{aligned} \tag{5.16}$$

and

$$\begin{aligned}(a_1^3-3a_1a_2^2)-i(a_2^3-3a_1^2a_2) = (a_1+ia_2)^3 &= \frac{1}{\lambda((\alpha+i\beta)^2-1)}\\ &= \frac{1}{\lambda}\frac{(\alpha^2-\beta^2-1)-2i\alpha\beta}{((\alpha^2-\beta^2-1)^2)+4\alpha^2\beta^2)}.\end{aligned}$$

Applying now the previous formulas, we obtain

$$\begin{aligned}T\left(\frac{\partial}{\partial u},\frac{\partial}{\partial u}\right) &= (a_1^2-a_2^2)\lambda E_1 - 2a_1a_2\lambda E_2\\ &= \frac{\lambda}{(a_1^2+a_2^2)}\left\{a_1(a_1^2-3a_2^2)\left(\frac{\partial}{\partial u}-h_v\frac{\partial}{\partial t}\right)+a_2(a_2^2-3a_1^2)\left(\frac{\partial}{\partial v}+h_u\frac{\partial}{\partial t}\right)\right\}\\ &= e^{2k/3}(\sinh^2 t+\cos^2 h)\left((\alpha^2-\beta^2-1)\left(\frac{\partial}{\partial u}-h_v\frac{\partial}{\partial t}\right)\right.\\ &\qquad\left.+2\alpha\beta\left(\frac{\partial}{\partial v}+h_u\frac{\partial}{\partial t}\right)\right)\\ &= e^{2k/3}\left(1-\frac{2\cos^2 h\cosh^2 t}{\sinh^2 t+\cos^2 h}\right)\left(\frac{\partial}{\partial u}-h_v\frac{\partial}{\partial t}\right)\\ &\quad -\frac{1}{2}e^{2k/3}\frac{\sin 2h\sinh 2t}{\sinh^2 t+\cos^2 h}\left(\frac{\partial}{\partial v}+h_u\frac{\partial}{\partial t}\right).\end{aligned}$$

The other equations are obtained in a similar manner.

Since all integrability conditions have been checked, the proof of the converse follows straightforwardly from the fundamental existence and uniqueness theorems. □

Remark 5.1. There exist infinitely many solutions for the differential system (5.8) and (5.9), in particular, there exist infinitely many solutions of the differential system with $h = 0$ (see, for instance, [13]). Hence, it follows immediately from Theorem 5.1 that there exist infinitely many Lagrangian submanifolds in $CH^3(-4)$ with nonintegrable distribution $\mathscr{D}$ which satisfy the basic equality.

Remark 5.2. Theorem 4.1 and Theorem 5.2 together determine Lagrangian submanifolds of CH^3 satisfying the basic equality.

References

[1] R. L. Bryant, *Second order families of special Lagrangian* 3*-fold*, preprint.

[2] B. Y. Chen, *Some pinching and classification theorems for minimal submanifolds*, Arch. Math. **60** (1993), 568–578.

[3] B. Y. Chen, *A general inequality for submanifolds in complex space forms and its applications*, Arch. Math. **67** (1996), 519–528.

[4] B. Y. Chen, *Jacobi's elliptic functions and Lagrangian immersions*, Proc. Royal Soc. Edinburgh. Sect. A. Math. **126** (1996), 687–704.

[5] B. Y. Chen, *Some new obstruction to minimal and Lagrangian isometric immersions*, Japan. J. Math. **26** (2000), 105–127.

[6] B. Y. Chen, *Strings of Riemannian invariants, inequalities, ideal immersions and their applications*, Third Pacific Rim Geom. Conf. pp. 7–60, (1998) (Intern. Press, Cambridge, MA).

[7] B. Y. Chen, F. Dillen, L. Verstraelen and L. Vrancken, *An exotic totally real minimal immersion of S^3 in CP^3 and its characterization*, Proc. Royal Soc. Edinburgh. Sect. A. Math. **126** (1996), 153–165.

[8] B. Y. Chen, F. Dillen, L. Verstraelen and L. Vrancken, *Totally real submanifolds of CP^n satisfying a basic inequality*, Arch. Math. **63** (1994), 553–564.

[9] B. Y. Chen, G. D. Ludden and S. Montiel, *Real submanifolds of a Kaehler manifold*, Algebras, Groups and Geom. **1** (1984), 176–212.

[10] B.-Y. Chen and K. Ogiue, *On totally real submanifolds*, Trans. Amer. Math. Soc. **193** (1974), 257–266.

[11] B. Y. Chen and L. Vrancken, *CR-submanifolds of complex hyperbolic spaces satisfying a basic equality*, Israel J. Math. **110** (1999), 341–358.

[12] F. Dillen, M. Petrovic and L. Verstraelen, *Einstein, conformally flat and semi-symmetric submanifolds satisfying Chen's equality*, Israel J. Math. **100** (1997), 163–169.

[13] D. G. de Figueiredo, O. H. Miyagaki and B. Ruf, *Elliptic equations in $\boldsymbol{R}^2$ with nonlinearities in the critical growth range*, Calc. Var. Partial Diff. Equat. **3** (1995), 139–153; *Corrigendum*, ibid **4** (1996), 203.

[14] M. Kriele, C. Scharlach and L. Vrancken, *An extremal class of* 3*-dimensional elliptic affine hyperspheres*, Hokkaido Math. J. **30** (2001), 1–23.

[15] M. Kriele and L. Vrancken, *An extremal class of* 3*-dimensional hyperbolic affine hyperspheres*, Geom. Dedicata **77** (1999), 239–252.

[16] S. Montiel and A. Romero, *On some real hypersurfaces of a complex hyperbolic space*, Geom. Dedicata **20** (1986), 245–261.
[17] S. Nölker, *Isometric immersions of warped products*, Diff. Geom. and Appl. **6** (1996), 1–30.
[18] H. Reckziegel, *Horizontal lifts of isometric immersions into the bundle space of a pseudo-Riemannian submersion*, Global Differential Geometry and Global Analysis (1984), Lecture Notes in Math., vol. 1156, Springer Verlag, 1985, pp. 264–279.
[19] C. Scharlach, U. Simon, L. Verstraelen and L. Vrancken, *A new intrinsic curvature invariant for centroaffine hypersurfaces*, Beiträge Algebra Geom. **38** (1997), 437–458.
[20] C. Scharlach and L. Vrancken, *A curvature invariant for centroaffine hypersurfaces*, *Part II*, in Geometry and Topology of Submanifolds **VIII** (1996), 341–350.
[21] T. Sasahara, *CR-submanifolds in complex hyperbolic spaces satisfying an equality of Chen*, Tsukuba J. Math. **23** (1999), 565–583.
[22] A. Strominger, S.-T. Yau, and E. Zaslow, *Mirror symmetry is T-duality*, Nuclear Phys. B **479** (1996), 243–259.

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