TSUKUBA J. MATH.
Vol. 22 No. 1 (1998), 1–29

A FAMILY OF BRAIDED COSEMISIMPLE HOPF ALGEBRAS OF FINITE DIMENSION

By

Satoshi SUZUKI

0. Introduction

Recently some finite dimensional cosemisimple Hopf algebras were constructed [Mas2] [F] [G]. We aim to give a plain and systematic description of cosemisimple Hopf algebras of low dimension. For this purpose we construct them as quotient bialgebras of a sufficiently large bialgebra. This way has the advantage of defining homomorphisms and determining braidings.

In this paper we define and study a family of finite dimensional cosemisimple Hopf algebras

$$\mathscr{F} = \{A_{NL}^{(++)}, A_{NL}^{(+-)}, A_{NL}^{(-+)}, A_{NL}^{(--)} \mid N \geqq 1, L \geqq 2\},$$

which consists of quotients of a bialgebra B over an algebraically closed field k with $chk \neq 2$.

This family contains the "non-trivial" cosemisimple Hopf algebras of dimension 8, 12 if $chk \neq 3$.

In Section 1 we review basic definitions and results.

In Section 2 quadratic bialgebras B, $B^{(+)}$ and $B^{(-)}$ are constructed. We use B to construct the family $\mathscr{F}$, and $B^{(\pm)}$ to obtain braidings on the members of a subfamily of $\mathscr{F}$. These bialgebras B, $B^{(\pm)}$ are cosemisimple, and we determine all braidings on them.

In Section 3 we define the family $\mathscr{F}$ as a set of quotient bialgebras of the bialgebra B. We write $A_{NL}^{(+1,-1)} = A_{NL}^{(+-)}$, etc. Let $\nu, \lambda = \pm 1$. Our main results are as follows.

i) $A_{NL}^{(\nu\lambda)}$ is a non-cocommutative involutory cosemisimple Hopf algebra of dimension $4NL$, which is non-commutative unless $(L, \lambda) = (2, +1)$. $A_{NL}^{(\nu\lambda)}$ is furthermore semisimple if $(dim\, A_{NL}^{(\nu\lambda)}) \cdot 1 \neq 0$.

Received April 8, 1996.
Revised October 7, 1996.

ii) Any non-commutative subHopf algebra of $A_{NL}^{(\nu\lambda)}$ generated by a simple subcoalgebra is a member of the family.

iii) All braidings on $A_{NL}^{(\nu\lambda)}$ are determined.

iv) We determine when $A_{N_1L_1}^{(\nu_1\lambda_1)}$ and $A_{N_2L_2}^{(\nu_2\lambda_2)}$ are isomorphic.

1. Preliminaries [D]

We follow Sweedler's book [S] and Montgomery's book [M] for terminology of Hopf algebras.

In this section we review basic definitions and results. They are due to Doi [D].

Let B be a bialgebra over a field k, $\tau : B \otimes B \to k$ a k-linear map which is invertible with respect to the convolution product. (B, τ) is called a *braided* bialgebra if the following three conditions hold:

$$\Sigma\tau(x_1, y_1)x_2y_2 = \Sigma y_1x_1\tau(x_2, y_2) \tag{1}$$

$$\tau(xy, z) = \Sigma\tau(x, z_1)\tau(y, z_2) \tag{2}$$

$$\tau(x, yz) = \Sigma\tau(x_1, z)\tau(x_2, y) \tag{3}$$

for $x, y, z \in B$.

Then the following conditions are automatically satisfied:

$$\tau(x, 1) = \varepsilon(x) = \tau(1, x),$$

$$\Sigma\tau(x_1, y_1)\tau(x_2, z_1)\tau(y_2, z_2) = \Sigma\tau(y_1, z_1)\tau(x_1, z_2)\tau(x_2, y_2) \qquad \text{for } x, y, z \in B.$$

We call this τ a *braiding* on B.

PROPOSITION 1.1 ([H, Proposition 1.2]). *Let (B, τ) be a braided bialgebra generated by a subcoalgebra C, (I) the bi-ideal generated by a coideal I of B. Then τ induces a braiding on the bialgebra $B/(I)$ iff $\tau = 0$ on $C \otimes I + I \otimes C$.*

If (B, τ) is a braided bialgebra, ${}^t\tau^{-1}$ is another braiding on B, where ${}^t\tau^{-1}(x, y) = \tau^{-1}(y, x)$, and the braiding τ is said to be *symmetric* if ${}^t\tau^{-1} = \tau$.

Let C be a coalgebra over k, $\sigma : C \otimes C \to k$ an invertible k-linear map. For any bialgebra B, a linear map $f : C \to B$ is called a *σ-map* if

$$\Sigma\sigma(x_1, y_1)f(x_2)f(y_2) = \Sigma f(y_1)f(x_1)\sigma(x_2, y_2), \quad x, y \in C.$$

Let $T(C)$ be the tensor (bi-)algebra and I_σ is the (bi-)ideal generated by

$$\Sigma\sigma(x_1, y_1)x_2y_2 - \Sigma y_1x_1\sigma(x_2, y_2), \quad x, y, z \in C. \tag{4}$$

We can form the bialgebra $M(C,\sigma) = T(C)/I_\sigma$, which is called is the *quadratic bialgebra associated with* (C,σ).

REMARK 1.2. i) The map $i : C \hookrightarrow T(C) \to M(C,\sigma)$ is an injective coalgebra σ-map.

ii) If B is a bialgebra and $f : C \to B$ is a σ-(coalgebra) map, then there is a unique (*bi-*) algebra map $\hat{f} : M(C,\sigma) \to B$ such that $\hat{f} \circ i = f$.

iii) $M(C,\sigma)$ has a natural algebra-gradation $\{C^n\}_{n \geqq 0}$.

iv) $M(C,\sigma)^{op} = M(C,\sigma^{-1}) = M(C,{}^t\sigma)$, $M(C,\sigma) = M(C,{}^t\sigma^{-1})$.

Let (C,σ) be as above. The map σ is called a *Yang-Baxter form* (or YB*-form*) if for all $x,y,z \in C$,

$$\Sigma\sigma(x_1,y_1)\sigma(x_2,z_1)\sigma(y_2,z_2) = \Sigma\sigma(y_1,z_1)\sigma(x_1,z_2)\sigma(x_2,y_2). \tag{5}$$

We call (C,σ) a YB*-coalgebra* if σ is a YB-form.

REMARK 1.3. If σ is a YB-form on C, so is ${}^t\sigma^{-1}$.

A YB-form σ is said to be *symmetric* if ${}^t\sigma^{-1} = \sigma$.

PROPOSITION 1.4 ([D, Theorem 2.6]). *If* (C,σ) *is a* YB*-coalgebra,* σ *uniquely extends to a braiding* $\tilde{\sigma}$ *on* $M(C,\sigma)$.

We note that if (C,σ) is a YB-coalgebra then $M(C,\sigma)$ has another braiding ${}^t\tilde{\sigma}^{-1}$.

COROLLARY 1.5. $\tilde{\sigma}$ *is symmetric iff* σ *is symmetric.*

For a bialgebra B, a Hopf algebra H and a bialgebra map $\imath : B \to H$, we call $(H,\imath)$ (or simply H) a *Hopf closure* of B if the following universality holds: for any Hopf algebra A and any bialgebra map $f : B \to A$, there is a unique Hopf algebra map $\tilde{f} : H \to A$ such that $\tilde{f} \circ \imath = f$. See [Man] [H] [D].

PROPOSITION 1.6 ([T2] [D, Theorem 3.6] [H]). *Let* $M(C,\sigma)$ *be the quadratic bialgebra associated with* (C,σ), $d(\neq 0)$ *a grouplike element of* $M(C,\sigma)$*. If there is a map* $j : C \to M(C,\sigma)$ *such that*

$$\Sigma i(x_1)j(x_2) = \varepsilon(x)d = \Sigma j(x_1)i(x_2) \qquad \textit{for all } x \in C,$$

then d is central and the (well-defined) localization $M(C,\sigma)[d^{-1}]$ becomes a Hopf algebra. Moreover it is a Hopf closure of $M(C,\sigma)$, and it follows that $M(C,\sigma)[d^{-1}] = M(C,\sigma)[G^{-1}]$, where G is the set of grouplike elements of $M(C,\sigma)$. If (C,σ) is a YB-coalgebra, $M(C,\sigma)[d^{-1}]$ has a braiding.

2. YB-coalgebras and quadratic bialgebras

From now on we work over an algebraically closed field k whose characteristic, chk, is not 2. Indices of Kronecker's δ_{ij}, X_{ij}, etc. are considered modulo 2.

In this section we define some YB-coalgebras and examine quadratic bialgebras associated with them.

Set $C = M_2(k)^*$, the dual coalgebra of the 2×2-matrix algebra $M_2(k)$, and let $\{X_{ij}\}_{1 \leqq i,j \leqq 2}$ be the comatrix basis of C, namely it spans C and satisfies

$$\Delta(X_{ij}) = \Sigma_{k=1}^{2} X_{ik} \otimes X_{kj}, \quad \varepsilon(X_{ij}) = \delta_{ij}.$$

For any coalgebra D and $Y_{ij} \in D$, $1 \leqq i,\ j \leqq 2$, if the linear map $C \to D$, $X_{ij} \mapsto Y_{ij}$, is an injective coalgebra map, we denote the image by

$$span_k(Y_{ij}) = span_k\begin{pmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{pmatrix}.$$

Let $\lambda = \pm 1$. Now for any $\alpha \in k^{\times} = k - \{0\}$, we define linear maps $\sigma_{(\alpha)}$, $\tau_{(\alpha)}^{(\pm 1)} = \tau_{(\alpha)}^{(\pm)} : C \otimes C \to k$ as follows (see [D, Example 2.8] for $\tau^{(\lambda)}$):

$\sigma_{(\alpha)}$	X_{11}	X_{12}	X_{21}	X_{22}
X_{11}	0	0	0	0
X_{12}	0	α	1	0
X_{21}	0	1	α	0
X_{22}	0	0	0	0 ,

$\tau_{(\alpha)}^{(\lambda)}$	X_{11}	X_{12}	X_{21}	X_{22}
X_{11}	α	0	0	1
X_{12}	0	0	0	0
X_{21}	0	0	0	0
X_{22}	λ	0	0	α .

PROPOSITION 2.1. *$\sigma_{(\alpha)}$, $\tau_{(\alpha)}^{(\lambda)}$ $(\alpha \in k^{\times})$ are YB-forms on C.*

PROOF. We show that $\sigma_{(\alpha)} = \sigma$ is a YB-form.
We can write $\sigma(X_{i,j+1}, X_{l,m+1}) = \delta_{ij}\delta_{lm}\alpha^{\delta_{il}}$.
For X_{ij}, X_{lm} and X_{uv}, observe that

$$\Sigma_{a,b,c}\sigma(X_{ia}, X_{lb})\sigma(X_{aj}, X_{uc})\sigma(X_{bm}, X_{cv})$$
$$= \sigma(X_{i,i+1}, X_{l,l+1})\sigma(X_{i+1,j}, X_{u,u+1})\sigma(X_{l+1,m}, X_{u+1,v})$$
$$= \delta_{ij}\delta_{lm}\delta_{uv}\alpha^{\delta_{il}}\alpha^{\delta_{i+1,u}}\alpha^{\delta_{lu}},$$

and

$$\Sigma_{a,b,c}\sigma(X_{lb}, X_{uc})\sigma(X_{ia}, X_{cv})\sigma(X_{aj}, X_{bm})$$
$$= \sigma(X_{l,l+1}, X_{u,u+1})\sigma(X_{i,i+1}, X_{u+1,v})\sigma(X_{i+1,j}, X_{l+1,m})$$
$$= \delta_{uv}\delta_{ij}\delta_{lm}\alpha^{\delta_{lu}}\alpha^{\delta_{i,u+1}}\alpha^{\delta_{il}}.$$

Thus Condition (5) is satisfied.

The inverse is given by

$$\sigma_{(\alpha)}^{-1} = \sigma_{(\alpha^{-1})}.$$

Therefore $\sigma_{(\alpha)}$ is a YB-form for $\alpha \in k^\times$.

It is easy to check that $\tau_{(\alpha)}^{(\lambda)}$ is also a YB-form on C. □

Therefore $(C, \sigma_{(\alpha)})$ and $(C, \tau_{(\alpha)}^{(\lambda)})$ are YB-coalgebras for all $\alpha \in k^\times$.

REMARK 2.2. $\{\sigma_{(\alpha)}, \tau_{(\beta)}^{(+)} \mid \alpha, \beta \in k^\times\}$, $\{\tau_{(\alpha)}^{(+)}, \tau_{(\beta)}^{(-)} \mid \alpha, \beta \in k^\times\}$ form subgroups of the unit group of $M_2(k)^{\otimes 2}$.

Next we examine the defining relations of the quadratic bialgebras associated with them.

PROPOSITION 2.3.

i) *The ideal* I_σ, *where* $\sigma = \sigma_{(\alpha)}$, *is generated by the following*:

$$\{X_{11}^2 - X_{22}^2, X_{12}^2 - X_{21}^2, X_{j,j+1}X_{ii} - \alpha X_{i+1,i+1}X_{j+1,j}\} \quad \text{if } \alpha^2 = 1,$$
$$\{X_{11}^2 - X_{22}^2, X_{12}^2 - X_{21}^2, X_{ij}X_{lm}(i+j+l+m \equiv 1)\} \quad \text{if } \alpha^2 \neq 1.$$

ii) *The ideal* $I_{\tau^{(\lambda)}}$, *where* $\tau^{(\lambda)} = \tau_{(\alpha)}^{(\lambda)}$, *is generated by the following*:

$$\{X_{11}X_{22} - X_{22}X_{11}, X_{12}X_{21} - \lambda X_{21}X_{12}, X_{i2}X_{i1} - \alpha X_{i1}X_{i2}, X_{2j}X_{1j} - \lambda\alpha X_{1j}X_{2j}\} \quad \text{if } \alpha^2 = \lambda,$$

$$\{X_{11}X_{22} - X_{22}X_{11}, X_{12}X_{21} - \lambda X_{21}X_{12}, X_{ij}X_{lm}(i+j+l+m \equiv 1)\} \quad \text{if } \alpha^2 \neq \lambda.$$

PROOF. i) For X_{ij}, X_{lm}, observe that

$$\Sigma\sigma(X_{ia}, X_{lb})X_{aj}X_{bm} = \sigma(X_{i,i+1}, X_{l,l+1})X_{i+1,j}X_{l+1,m}$$
$$= \alpha^{\delta_{il}}X_{i+1,j}X_{l+1,m},$$

$$\Sigma X_{lb}X_{ia}\sigma(X_{aj},X_{bm}) = X_{l,m+1}X_{i,j+1}\sigma(X_{j+1,j},X_{m+1,m})$$
$$= X_{l,m+1}X_{i,j+1}\alpha^{\delta_{jm}}.$$

Thus the subset

$$\{\alpha^{\delta_{il}}X_{ij}X_{lm} - X_{l+1,m+1}X_{i+1,j+1}\alpha^{\delta_{jm}} \mid 1 \leqq i,j,l,m \leqq 2\}$$

generates the ideal I_σ. The above polynomials are written as follows:

$$\begin{cases} \alpha X_{ij}^2 - X_{i+1,j+1}^2\alpha & \text{if } i=l, j=m, \\ X_{ij}X_{lj} - X_{l+1,j+1}X_{i+1,j+1}\alpha & \text{if } i\neq l, j=m, \\ \alpha X_{ij}X_{im} - X_{i+1,m+1}X_{i+1,j+1} & \text{if } i=l, j\neq m, \\ X_{ij}X_{lm} - X_{l+1,m+1}X_{i+1,j+1} & \text{if } i\neq l, j\neq m \text{ (i.e., } l\equiv i+1, m\equiv j+1). \end{cases}$$

ii) This is similarly shown as i). □

REMARK 2.4. i) For the bialgebra $M(C,\sigma_{(-1)})$, see the quantum conformal group in [Man].

ii) $M(C,\tau^{(+)}_{(\pm 1)})$ are the quantum matrix bialgebras $M_{\pm 1}(2)$.

iii) $M(C,\tau^{(-)}_{(\sqrt{-1})})$ is Takeuchi's two-parameter bialgebra $M_{\alpha,\beta}(2)$ for $\alpha=\sqrt{-1}$, $\beta=-\sqrt{-1}$ ([T1], [D]).

Define $B=M(C,\sigma_{(\alpha)})$ for $\alpha^2\neq 1$ and $B^{(\lambda)}=M(C,\tau^{(\lambda)}_{(\alpha)})$ for $\alpha^2\neq\lambda$. We write $B^{(\pm 1)}=B^{(\pm)}$. These definitions, ignoring choice of α, are reasonable by Proposition 2.3.

On the other hand, we see by Proposition 1.1 that braidings $\tilde{\sigma}_{(\pm 1)}$, $\tilde{\tau}^{(\lambda)}_{(\pm\sqrt{\lambda})}$ are induced on B, $B^{(\lambda)}$, respectively, via the canonical surjections

$$M(C,\sigma_{(\pm 1)}) \to B, \quad M(C,\tau^{(\lambda)}_{(\pm\sqrt{\lambda})}) \to B^{(\lambda)}.$$

Note that $\{X_{ij}X_{lm} | i+j+l+m\equiv 1\}$ spans a coideal of $T(C)$.
Therefore we have the following claim:

CLAIM 2.5.
i) $\sigma_{(\alpha)}: C\otimes C\to k$ *extends to a braiding* $\tilde{\sigma}_{(\alpha)}$ *on* B *for every* $\alpha\in k^\times$.
ii) $\tau^{(\lambda)}_{(\alpha)}: C\otimes C\to k$ *extends to a braiding* $\tilde{\tau}^{(\lambda)}_{(\alpha)}$ *on* $B^{(\lambda)}$ *for every* $\alpha\in k^\times$.

We examine the coalgebra structure of B.

PROPOSITION 2.6.

i) *B has the following set as a basis*

$$\{X_{11}^{n-r}\overbrace{X_{22}X_{11}X_{22}\dots}^{r}, X_{12}^{n-r}\overbrace{X_{21}X_{12}X_{21}\cdots}^{r} \mid n \geqq 0, 0 \leqq r \leqq n\}.$$

ii) *The grouplike elements but* 1 *in B are given by*

$$X_{11}^{2s} \pm X_{12}^{2s} \quad (s \geqq 1).$$

Then are central non-zero divisors.

iii) *The simple subcoalgebras of B which are not spanned by grouplike elements are of dimension* 4. *They are given by*

$$C_{st} = span_k \begin{pmatrix} X_{11}^{2s}\overbrace{X_{11}X_{22}X_{11}\cdots}^{t} & X_{12}^{2s}\overbrace{X_{12}X_{21}X_{12}\cdots}^{t} \\ X_{12}^{2s}\overbrace{X_{21}X_{12}X_{21}\cdots}^{t} & X_{11}^{2s}\overbrace{X_{22}X_{11}X_{22}\cdots}^{t} \end{pmatrix} \quad (s \geqq 0, t \geqq 1).$$

iv) *B is cosemisimple. The nth component C^n $(n \geqq 1)$ of B is decomposed as the sum of simple subcoalgebras as follows*:

$$C^n = \begin{cases} \Sigma_{n=2s+t} C_{st}, & \text{if } n \text{ is odd;} \\ \Sigma_{n=2s+t} C_{st} + k(X_{11}^n \pm X_{12}^n), & \text{if } n \text{ is even.} \end{cases}$$

PROOF. i) It is verified in the same manner as Theorem 3.1.i) below.

ii), iii), iv) It is easy to see that $X_{11}^{2s} \pm X_{12}^{2s}$ is grouplike for $s \geqq 1$. By i) and the defining relations of B, it is a central non-zero divisor. C is isomorphic to C_{st} as coalgebras by

$$X_{11} \mapsto X_{11}^{2s}\overbrace{X_{11}X_{22}X_{11}\dots}^{t},$$

$$X_{12} \mapsto X_{12}^{2s}X_{12}X_{21}X_{12}\dots,$$

$$X_{21} \mapsto X_{12}^{2s}X_{21}X_{12}X_{21}\dots,$$

$$X_{22} \mapsto X_{11}^{2s}X_{22}X_{11}X_{22}\dots.$$

By i) we have that

$$\begin{aligned} B &= k \cdot 1 + \Sigma k(X_{11}^{2s} \pm X_{12}^{2s}) + \Sigma C_{st} \\ &= k \cdot 1 \oplus \{\oplus_{s \geqq 1} k(X_{11}^{2s} \pm X_{12}^{2s}\} \oplus \{\oplus_{s \geqq 0, t \geqq 1} C_{st}\}. \end{aligned}$$

Thus ii), iii), iv) are done. □

PROPOSITION 2.7.

i) *$B^{(\lambda)}$ has the following set as a basis*

$$\{X_{11}^u X_{22}^v, X_{12}^u X_{21}^v \mid u+v \geqq 0\}.$$

ii) *The grouplike elements but* 1 *in $B^{(\lambda)}$ are given by*

$$X_{11}^u X_{22}^u \pm \sqrt{\lambda^u} X_{12}^u X_{21}^u \quad (u \geqq 1).$$

They are non-zero divisors.

iii) *The simple subcoalgebras of $B^{(\lambda)}$ which are not spanned by grouplike elements are all of dimension* 4. *They are given by*

$$D_{uv} = span_k \begin{pmatrix} X_{11}^u X_{22}^v & X_{12}^u X_{21}^v \\ X_{21}^u X_{12}^v & X_{22}^u X_{11}^v \end{pmatrix}, \quad (u \lneqq v).$$

iv) *$B^{(\lambda)}$ is cosemisimple. The nth component C^n $(n \geqq 1)$ of $B^{(\lambda)}$ is decomposed as the sum of simple subcoalgebras as follows:*

$$C^n = \begin{cases} \Sigma_{n=u+v, u \lneqq v} D_{uv}, & \text{if } n \text{ is odd;} \\ \Sigma_{n=u+v, u \lneqq v} D_{uv} + k(X_{11}^{n/2} X_{22}^{n/2} \pm \sqrt{\lambda^{n/2}} X_{12}^{n/2} X_{21}^{n/2}), & \text{if } n \text{ is even.} \end{cases}$$

We omit the proof.

COROLLARY 2.8. *Let $\langle C_{st} \rangle$ denote the sub-bialgebra generated by the simple subcoalgebra $C_{st} \subset B$. Then as bialgebras,*

$$B \supsetneqq \langle C_{st} \rangle \simeq \begin{cases} B, & \text{if } t \text{ is odd;} \\ B^{(+)}, & \text{if } t \text{ is even.} \end{cases}$$

We omit the proof. See the proof of Theorem 3.5 below.

Define linear maps $\sigma_{\alpha\beta} = \beta\sigma_{(\alpha\beta^{-1})}$, $\tau_{\alpha\beta}^{(\lambda)} = \beta\tau_{(\alpha\beta^{-1})}^{(\lambda)}$ for α, $\beta \in k^\times$, $\lambda = \pm 1$. They are also YB-forms on C. The YB-form $\sigma_{\alpha\beta}$ extends to a braiding $\tilde{\sigma}_{\alpha\beta}$ on B, and $\tau_{\alpha\beta}^{(\lambda)}$ extends to a braiding $\tilde{\tau}_{\alpha\beta}^{(\lambda)}$ on $B^{(\lambda)}$.

PROPOSITION 2.9. i) *$\sigma_{\alpha\beta}$ is symmetric iff $\alpha^2 = 1 = \beta^2$. $\tau_{\alpha\beta}^{(\lambda)}$ is symmetric iff $\alpha^2 = 1$, $\beta^2 = \lambda$.*

ii) *The set of braidings on B is $\{\tilde{\sigma}_{\alpha\beta} \mid \alpha, \beta \in k^\times\}$, and that on $B^{(\lambda)}$ is $\{\tilde{\tau}_{\alpha\beta}^{(\lambda)} \mid \alpha, \beta \in k^\times\}$.*

PROOF. i) We note that ${}^t\sigma_{\alpha\beta} = \sigma_{\alpha\beta}$, ${}^t\tau_{\alpha\beta}^{(\lambda)} = \tau_{\alpha,\lambda\beta}^{(\lambda)}$. The statement follows from these.

ii) We show the statement with B. The statement with $B^{(\lambda)}$ is similarly verified.

We have obtained braidings $\tilde{\sigma}_{\alpha\beta}(\alpha, \beta \in k^{\times})$ on B.

Let σ be a braiding. Note that the second component C^2 of B has a basis

$$\{X_{11}^2, X_{12}^2, X_{11}X_{22}, X_{22}X_{11}, X_{12}X_{21}, X_{21}X_{12}\}.$$

So for X_{ij}, X_{lm}, it follows that

$$\Sigma\sigma(X_{ia}, X_{lb})X_{aj}X_{bm} = \sigma(X_{ij}, X_{lm})X_{jj}X_{mm} + \sigma(X_{i,j+1}, X_{l,m+1})X_{j+1,j}X_{m+1,m},$$

$$\Sigma X_{lb}X_{ia}\sigma(X_{aj}, X_{bm}) = X_{ll}X_{ii}\sigma(X_{ij}, X_{lm}) + X_{l,l+1}X_{i,i+1}\sigma(X_{i+1,j}, X_{l+1,m}).$$

These must be equal, so we obtain the following by Proposition 2.6.i):

$$\sigma(X_{ij}, X_{lm})X_{jj}X_{mm} = X_{ll}X_{ii}\sigma(X_{ij}, X_{lm}),$$

$$\sigma(X_{i,j+1}, X_{l,m+1})X_{j+1,j}X_{m+1,m} = X_{l,l+1}X_{i,i+1}\sigma(X_{i+1,j}, X_{l+1,m}).$$

The above equations imply that $\sigma|_{C\otimes C}$ is given as follows with some α, β, $\gamma \in k$:

σ	X_{11}	X_{12}	X_{21}	X_{22}
X_{11}	γ	0	0	0
X_{12}	0	α	β	0
X_{21}	0	β	α	0
X_{22}	0	0	0	γ .

Moreover it follows by Condition (2) that

$$\begin{aligned}0 &= \sigma(0, X_{12}) = \sigma(X_{11}X_{12}, X_{12}) \\ &= \sigma(X_{11}, X_{11})\sigma(X_{12}, X_{12}) + \sigma(X_{11}, X_{12})\sigma(X_{12}, X_{22}) = \gamma\alpha,\end{aligned}$$

and

$$\begin{aligned}0 &= \sigma(0, X_{12}) = \sigma(X_{11}X_{21}, X_{12}) \\ &= \sigma(X_{11}, X_{11})\sigma(X_{21}, X_{12}) + \sigma(X_{11}, X_{12})\sigma(X_{21}, X_{22}) = \gamma\beta.\end{aligned}$$

We have that $\gamma = 0$, α, $\beta \in k^{\times}$ since σ is invertible.

Therefore $\sigma|_{C\otimes C} = \sigma_{\alpha\beta}$, so $\sigma = \tilde{\sigma}_{\alpha\beta}$. □

We describe a Hopf closure of the bialgebra B.

Set $d_{\pm} = X_{11}^2 \pm X_{12}^2$. These are central grouplike elements. For example, observe that

$$\begin{pmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{pmatrix}\begin{pmatrix} X_{11} & X_{21} \\ X_{12} & X_{22} \end{pmatrix} = d_+\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} X_{11} & X_{21} \\ X_{12} & X_{22} \end{pmatrix}\begin{pmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{pmatrix},$$

and

$$\begin{pmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{pmatrix}\begin{pmatrix} X_{11} & -X_{21} \\ -X_{12} & X_{22} \end{pmatrix} = d_-\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$
$$= \begin{pmatrix} X_{11} & -X_{21} \\ -X_{12} & X_{22} \end{pmatrix}\begin{pmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{pmatrix}.$$

Using Proposition 1.6 and Proposition 2.6, we have the following.

PROPOSITION 2.10. *The Hopf closure H of B is given by*

$$H = B[d_+^{-1}] = B[d_-^{-1}] = B[G(B)^{-1}],$$

where $G(B)$ is the set of grouplike elements in B. This Hopf algebra is braided and cosemisimple, and includes B as a sub-bialgebra. Furthermore, H is involutory. In fact, the antipode S is determined by

$$S(X_{ij}) = X_{ji}d_+^{-1} = (-1)^{i+j}X_{ji}d_-^{-1}.$$

3. Quotients of the bialgebra B

In this section we define and study a family of finite dimensional cosemisimple bi(Hopf) algebras which are quotients of the bialgebra B over an algebraically closed field k with $chk \neq 2$.

It will be shown that the family contains the "non-trivial" cosemisimple Hopf algebras of dimension 8 ([Mas2]) and of dimension 12 ([F]) if $chk \neq 3$. See also Gelaki's Hopf algebras of dimension $4p$, where $p(\geqq 3)$ is prime ([G]).

We construct the family. It is easy to see by Proposition 2.6 that for $L \geqq 2$, $N \geqq 1$ and λ, $\nu = \pm 1$, the following subsets

$$\{\overbrace{X_{22}X_{11}X_{22}\cdots}^{L} - \overbrace{X_{11}X_{22}X_{11}\ldots}^{L},\ \overbrace{X_{21}X_{12}X_{21}\cdots}^{L} - \lambda\overbrace{X_{12}X_{21}X_{12}\cdots}^{L}\},$$
$$\{1 - (X_{11}^{2N} + \nu X_{12}^{2N})\}$$

span coideals of B. Let J_L^λ and I_N^ν be the ideals generated by these coideals respectively, which are bi-ideals.

We can form the bialgebra

$$A_{NL}^{(\nu\lambda)} = B/J_L^{\lambda} + I_N^{\nu}.$$

We write $A_{NL}^{(+-)} = A_{NL}^{(+1,-1)}$, etc. Let π be the following surjective bialgebra map:

$$\pi : B \to A_{NL}^{(\nu\lambda)}, \quad X_{ij} \mapsto \bar{X}_{ij} = x_{ij}.$$

THEOREM 3.1.
i) $A_{NL}^{(\nu\lambda)}$ *has the following set as a basis*

$$\{x_{11}^s \overbrace{x_{22}x_{11}x_{22}\cdots}^{t}, x_{12}^s \overbrace{x_{21}x_{12}x_{21}\cdots}^{t} \mid 1 \leqq s \leqq 2N,\ 0 \leqq t \leqq L-1\}.$$

Thus $\dim A_{NL}^{(\nu\lambda)} = 4NL$.
ii) *Let* $G(A_{NL}^{(\nu\lambda)}) = G$ *be the set of grouplike elements of* $A_{NL}^{(\nu\lambda)}$. *Then*

$$G = \{x_{11}^{2s} \pm x_{12}^{2s}, x_{11}^{2s} \overbrace{x_{11}x_{22}x_{11}\cdots}^{L} \pm \sqrt{\lambda} x_{12}^{2s} \overbrace{x_{12}x_{21}x_{12}\cdots}^{L} \mid 1 \leqq s \leqq N\}.$$

iii) *The simple subcoalgebras of* $A_{NL}^{(\nu\lambda)}$ *which are not spanned by grouplike elements are given by*

$$C_{st} = span_k \begin{pmatrix} x_{11}^{2s} \overbrace{x_{11}x_{22}x_{11}\cdots}^{t} & x_{12}^{2s} \overbrace{x_{12}x_{21}x_{12}\cdots}^{t} \\ x_{12}^{2s} \overbrace{x_{21}x_{12}x_{21}\cdots}^{t} & x_{11}^{2s} \overbrace{x_{22}x_{11}x_{22}\cdots}^{t} \end{pmatrix}$$

for $0 \leqq s \leqq N-1,\ 1 \leqq t \leqq L-1$.

iv) $|G(A_{NL}^{(\nu\lambda)})| = 4N$, *and there are exactly* $N(L-1)$ *simple subcoalgebras of dimension* 4.
v) $A_{NL}^{(\nu\lambda)}$ *is non-cocommutative and cosemisimple. It is non-commutative unless* $(L, \lambda) = (2, +1)$.
vi) $A_{NL}^{(\nu\lambda)}$ *is an involutory Hopf algebra.*
vii) *Let* $\Lambda = \Sigma x_{11}^s \overbrace{x_{22}x_{11}x_{22}\cdots}^{t} (1 \leqq s \leqq 2N, 0 \leqq t \leqq L-1)$. *Then* Λ *is a non-zero two-sided integral.*
viii) $A_{NL}^{(\nu\lambda)}$ *is semisimple if* $\operatorname{ch} k \nmid NL$.

PROOF. i) Let B' be the algebra $k\langle X, Y\rangle/\{X^2 - Y^2\}$ and λ, $\nu = \pm 1$. Let V be the k-vector space with a basis $\{\langle s, t\rangle \in V \mid s \geqq 1, 0 \leqq t \leqq L-1\}$.

We define the following ideals of B':

$$J_L^{\lambda\prime} = (\overbrace{YXYX\cdots}^{L} - \lambda\overbrace{XYXY\cdots}^{L}),$$
$$I_N^{\nu\prime} = (1 - \nu X^{2N}).$$

We prove i) step-by-step.

(Step 1) We define a right B'-module structure on V.

Define the actions of X and Y as follows:

$$X : \langle s,t\rangle \mapsto \begin{cases} \langle s,t+1\rangle, & \text{if } t \text{ is odd}, \quad t \leqq L-2, \\ \lambda\langle s+1,L-1\rangle, & \qquad\qquad\quad t = L-1, \\ \langle s+1,0\rangle, & \text{if } t \text{ is even}, \quad t = 0, \\ \langle s+2,t-1\rangle, & \qquad\qquad\quad t \geqq 2, \end{cases}$$

$$Y : \langle s,t\rangle \mapsto \begin{cases} \langle s+2,t-1\rangle, & \text{if } t \text{ is odd}, \\ \langle s,t+1\rangle, & \text{if } t \text{ is even}, \quad t \leqq L-2, \\ \lambda\langle s+1,L-1\rangle, & \qquad\qquad\quad t = L-1. \end{cases}$$

It is easy to see $X^2 \equiv Y^2$ in $End_k(V)$.

Thus we have a right B'-module structure on V.

(Step 2) We claim the subspace W spanned by

$$\{\langle q(2N)+s,t\rangle - \nu^q\langle s,t\rangle \mid 1 \leqq s \leqq 2N, q \geqq 1, 0 \leqq t \leqq L-1\}$$

is a submodule of V.

For example, when $t = L-1$ is odd and $s = 2N$, observe the following:

$$\begin{aligned} X : \langle q(2N)+2N, L-1\rangle \mapsto\ & \lambda\langle q(2N)+2N+1, L-1\rangle \\ &= \lambda\langle (q+1)(2N)+1, L-1\rangle \\ &\equiv \lambda\nu^{q+1}\langle 1, L-1\rangle \pmod W, \end{aligned}$$

and

$$\begin{aligned} X : \nu^q\langle 2N, L-1\rangle \mapsto\ & \nu^q\lambda\langle 2N+1, L-1\rangle \\ &= \nu^q\lambda\langle 1\cdot(2N)+1, L-1\rangle \\ &\equiv \nu^q\lambda\nu\langle 1, L-1\rangle \pmod W. \end{aligned}$$

(Step 3) The action of B' induces the $B'/J_L^{\lambda\prime}$-module structure on V. We check it case-by-case.

When L is even, for each $0 \leqq 2u \leqq L-2$, observe the following:

$$\overbrace{YX\cdots X}^{L} : \cdot\ \langle s, 2u\rangle \xmapsto{(YX)^{L/2-u-1}} \langle s, L-2\rangle$$

$$\xmapsto{YX} \lambda\langle s+1, L-1\rangle$$

$$\xmapsto{(YX)^{u}} \lambda\langle s+1+4u, L-1-2u\rangle,$$

$$\cdot\ \langle s, 2u+1\rangle \xmapsto{(YX)^{u}} \langle s+4u, 1\rangle \xmapsto{YX} \langle s+4u+3, 0\rangle$$

$$\xmapsto{(YX)^{L/2-u-1}} \langle s+4u+3, L-2u-2\rangle.$$

$$\overbrace{XY\cdots Y}^{L} : \cdot\ \langle s, 2u\rangle \xmapsto{(XY)^{u}} \langle s+4u, 0\rangle \xmapsto{XY} \langle s+4u+1, 1\rangle$$

$$\xmapsto{(XY)^{L/2-u-1}} \langle s+4u+1, L-2u-1\rangle,$$

$$\cdot\ \langle s, 2u+1\rangle \xmapsto{(XY)^{L/2-u-1}} \langle s, L-1\rangle$$

$$\xmapsto{XY} \lambda\langle s+3, L-2\rangle$$

$$\xmapsto{(XY)^{u}} \lambda\langle s+3+4u, L-2-2u\rangle.$$

Thus it follows that $\overbrace{YX\cdots X}^{L} \equiv \lambda\overbrace{XY\cdots Y}^{L}$ in $End_k(V)$.

When L is odd (so $L \geqq 3$), for each $2 \leqq 2u \leqq L-1$, observe the following:

$$\overbrace{YX\cdots Y}^{L} : \cdot\ \langle s, 0\rangle \xmapsto{(YX)^{(L-1)/2}} \langle s, L-1\rangle \overset{Y}{\mapsto} \lambda\langle s+1, L-1\rangle,$$

$$\cdot\ \langle s, 2u\rangle \xmapsto{(YX)^{(L-1)/2-u}} \langle s, L-1\rangle \overset{Y}{\mapsto} \lambda\langle s+1, L-1\rangle$$

$$\xmapsto{(XY)^{u}} \lambda\langle s+1+4u, L-1-2u\rangle.$$

$$\cdot\ \langle s, 2u-1\rangle \xmapsto{(YX)^{u-1}} \langle s+4u-4, 1\rangle \xmapsto{YX} \langle s+4u-1, 0\rangle$$

$$\xmapsto{(YX)^{(L-1)/2-u}} \langle s+4u-1, L-1-2u\rangle \overset{Y}{\mapsto} \langle s+4u-1, L-2u\rangle.$$

$$\overbrace{XY\cdots X}^{L} : \cdot \langle s,0\rangle \mapsto \langle s+1, L-1\rangle,$$

$$\cdot \langle s,2u\rangle \overset{(XY)^u}{\longmapsto} \langle s+4u,0\rangle \overset{X}{\mapsto} \langle s+4u+1,0\rangle$$

$$\overset{(YX)^{(L-1)/2-u}}{\longmapsto} \langle s+4u+1, L-2u-1\rangle,$$

$$\cdot \langle s,2u-1\rangle \overset{(XY)^{(L-1)/2-u}}{\longmapsto} \langle s,L-2\rangle \overset{XY}{\longmapsto} \lambda\langle s+1,L-1\rangle$$

$$\overset{(XY)^{u-1}}{\longmapsto} \lambda\langle s+4u-3, L-2u+1\rangle \overset{X}{\mapsto} \lambda\langle s+4u-1, L-2u\rangle.$$

Thus we have that $\overbrace{YX\cdots Y}^{L} \equiv \lambda \overbrace{XY\cdots X}^{L}$ in $End_k(V)$.

In either case V becomes a right $B'/J_L^{\lambda'}$-module by the action.

(Step 4) V/W is a $B'/J_L^{\lambda'} + I_N^{\nu'}$-module of dimension $2NL$.

Since V/W has the set $\{\langle s,t\rangle \mid 1 \leqq s \leqq 2N, 0 \leqq t \leqq L-1\}$ as a basis, V/W has dimension $2NL$.

The action of X^2 is given by $X^2 : \langle s,t\rangle \mapsto \langle s+2,t\rangle$.

Thus for $1 \leqq s \leqq 2N$, $0 \leqq t \leqq L-1$, it follows that

$$X^{2N} : \langle s,t\rangle \mapsto \langle s+2N,t\rangle = \langle 1\cdot(2N)+s,t\rangle \equiv \nu\langle s,t\rangle \bmod W.$$

So we have that $1 \equiv \nu X^{2N}$ in $End_k(V/W)$.

Thus it is done.

(Step 5) We construct a right $A_{NL}^{\nu\lambda}$-module $M = (V/W)\oplus(V/W)$.

There are two algebra maps

$$\pi_0' : B \to B'/J_L^{+'} + I_N^{+'},$$
$$X_{11} \mapsto \overline{X} = x, \quad X_{22} \mapsto \overline{Y} = y,$$
$$X_{i,i+1} \mapsto 0,$$

and

$$\pi_1' : B \to B'/J_L^{\lambda'} + I_N^{\nu'},$$
$$X_{12} \mapsto \overline{X} = x, \quad X_{21} \mapsto \overline{Y} = y,$$
$$X_{ii} \mapsto 0.$$

They induce algebra maps

$$\pi_0 : A_{NL}^{(\nu\lambda)} \to B'/J_L^{+'} + I_N^{+'},$$
$$x_{11} \mapsto x, \quad x_{22} \mapsto y, \quad x_{i,i+1} \mapsto 0,$$
$$\pi_1 : A_{NL}^{(\nu\lambda)} \to B'/J_L^{\lambda'} + I_N^{\nu'},$$
$$x_{12} \mapsto x, \quad x_{21} \mapsto y, \quad x_{ii} \mapsto 0.$$

Using these, we obtain the right $A_{NL}^{(\nu\lambda)}$-module $V/W = V_0$ through π_0 with a basis

$$\{\langle s,t\rangle_0 = \langle s,t\rangle \mid 1 \leqq s \leqq 2N, 0 \leqq t \leqq L-1\},$$

and the right $A_{NL}^{(\nu\lambda)}$-module $V/W = V_1$ through π_1 with a basis

$$\{\langle s,t\rangle_1 = \langle s,t\rangle \mid 1 \leqq s \leqq 2N, 0 \leqq t \leqq L-1\}.$$

Let M be the right $A_{NL}^{(\nu\lambda)}$-module $V_0 \oplus V_1$. We note that M has dimension $4NL$.

(Step 6) It follows that $M \simeq A_{NL}^{(\nu\lambda)}$ as right $A_{NL}^{(\nu\lambda)}$-modules.

Define an $A_{NL}^{(\nu\lambda)}$-module map $\phi : A_{NL}^{(\nu\lambda)} \to M$ and a k-linear map $\psi : M \to A_{NL}^{(\nu\lambda)}$ as follows:

$$\phi : A_{NL}^{(\nu\lambda)} \to M, \quad a \mapsto \{\langle 2N,0\rangle_0 + \nu\langle 2N,0\rangle_1\} \cdot a,$$

$$\psi : M \to A_{NL}^{(\nu\lambda)}, \quad \langle s,t\rangle_0 \mapsto x_{11}^s \overbrace{x_{22}x_{11}x_{22}\cdots}^{t},$$

$$\langle s,t\rangle_1 \mapsto x_{12}^s \overbrace{x_{21}x_{12}x_{21}\cdots}^{t}.$$

It is easy to see that ψ is surjective and that $\phi \circ \psi$ is the identity map on M. Therefore we have that $M \simeq A_{NL}^{(\nu\lambda)}$ as $A_{NL}^{(\nu\lambda)}$-modules, in particular $dim\, A_{NL}^{(\nu\lambda)} = dim\, M = 4NL$.

This completes the proof of i).

ii) $\sim$ v) These are easily verified by i). Since $A_{NL}^{(\nu\lambda)}$ is generated by $\{x_{ij}\}$, it is commutative iff $(L,\lambda) = (2,+1)$.

vi) There is an algebra map $B \to B^{op}$, $X_{ij} \mapsto X_{ji} \cdot (X_{11}^{2(2N-1)} + X_{12}^{2(2N-1)})$, and this induces an algebra map S,

$$\begin{array}{ccc} B & \longrightarrow & B^{op} \\ \downarrow{\scriptstyle \pi} & & \downarrow \\ A_{NL}^{(\nu\lambda)} & \xrightarrow{S} & (A_{NL}^{(\nu\lambda)})^{op}. \end{array}$$

The anti-algebra map S is an antipode of $A_{NL}^{(\nu\lambda)}$, which is given by

$$S : x_{ij} \mapsto x_{ji}(x_{11}^{2(2N-1)} + x_{12}^{2(2N-1)})$$
$$= x_{ji}(x_{11}^2 + x_{12}^2)^{-1}.$$

So $A_{NL}^{(\nu\lambda)}$ is an involutory Hopf algebra.

vii) The element $\Lambda = \Sigma x_{11}^{s} \overbrace{x_{22}x_{11}x_{22}\cdots}^{t}$ $(1 \leqq s \leqq 2N, 0 \leqq t \leqq L-1)$ is non-zero by i).

Recall that Λ is called a left (resp. right) integral if $a\Lambda$ (resp. Λa) $= \varepsilon(a)\Lambda$ for all $a \in A_{NL}^{(\nu\lambda)}$.

It is enough to check on the subset $\{x_{ij}\}$. Observe the following.

$$x_{12}\Lambda = x_{21}\Lambda = 0$$
$$= \varepsilon(x_{12})\Lambda = \varepsilon(x_{21})\Lambda.$$

$$x_{11}\Lambda = \Sigma x_{11}^{s+1} \overbrace{x_{22}x_{11}x_{22}\cdots}^{t} = \Sigma x_{11}^{s} \overbrace{x_{22}x_{11}x_{22}\cdots}^{t} = \Lambda$$
$$= \varepsilon(x_{11})\Lambda.$$

$$x_{22}\Lambda = \Sigma x_{22}x_{11}^{s} \overbrace{x_{22}x_{11}x_{22}\cdots}^{t}$$
$$= \Sigma_{s:\mathrm{even}} x_{11}^{s} x_{22} \overbrace{x_{22}x_{11}x_{22}\cdots}^{t} + \Sigma_{s:\mathrm{odd}} x_{11}^{s-1} x_{22}x_{11} \overbrace{x_{22}x_{11}x_{22}\cdots}^{t}$$
$$= \Sigma_{s:\mathrm{even}, t=0} x_{11}^{s} x_{22} + \Sigma_{s:\mathrm{even}, t=1} x_{11}^{s+2} + \Sigma_{s:\mathrm{even}, t\geqq 2} x_{11}^{s+3} \overbrace{x_{22}x_{11}x_{22}\cdots}^{t-2}$$
$$+ \Sigma_{s:\mathrm{odd}, t\leqq L-3} x_{11}^{s-1} \overbrace{x_{22}x_{11}x_{22}\cdots}^{t+2} + \Sigma_{s:\mathrm{odd}, t=L-2} x_{11}^{s} \overbrace{x_{22}x_{11}x_{22}\cdots}^{L-1}$$
$$+ \Sigma_{s:\mathrm{odd}, t=L-1} x_{11}^{s+2} \overbrace{x_{22}x_{11}x_{22}\cdots}^{L-2}$$
$$= \Sigma_{s:\mathrm{even}} x_{11}^{s} x_{22} + \Sigma_{s:\mathrm{even}} x_{11}^{s} + \Sigma_{s:\mathrm{odd}, 0\leqq t\leqq L-3} x_{11}^{s} \overbrace{x_{22}x_{11}x_{22}\cdots}^{t}$$
$$+ \Sigma_{s:\mathrm{even}, 2\leqq t\leqq L-1} x_{11}^{s} \overbrace{x_{22}x_{11}x_{22}\cdots}^{t} + \Sigma_{s:\mathrm{odd}} x_{11}^{s} \overbrace{x_{22}x_{11}x_{22}\cdots}^{L-1}$$
$$+ \Sigma_{s:\mathrm{odd}} x_{11}^{s} \overbrace{x_{22}x_{11}x_{22}\cdots}^{L-2}$$
$$= \Lambda$$
$$= \varepsilon(x_{22})\Lambda.$$

Thus Λ is a left integral. It is similarly shown that Λ is a right integral. Therefore Λ is a non-zero two-sided integral.

viii) It follows that $\varepsilon(\Lambda) = 2NL \neq 0$ iff $chk \nmid NL$. □

REMARK 3.2. For the multiplication relations of $A_{NL}^{(\nu\lambda)}$, we note the following.

- x_{ij}^2 is central.
- $x_{ii}^{2N+1} = x_{ii}$, and $x_{i,i+1}^{2N+1} = \nu x_{i,i+1}$.
- $x_{11}^{4N} + x_{12}^{4N} = 1$.
- $(x_{11}^{2s} + \mu x_{12}^{2s})^{-1} = x_{11}^{2(2N-s)} + \mu x_{12}^{2(2N-s)}$ for $1 \leqq s \leqq N, \mu = \pm 1$.

Set $h_{\pm} = x_{11}^2 \pm x_{12}^2$ and $g = \overbrace{x_{11}x_{22}x_{11}\cdots}^{L} + \sqrt{\lambda}\overbrace{x_{12}x_{21}x_{12}\cdots}^{L}$ for a fixed $\sqrt{\lambda}$. C_m denotes the cyclic group of order m.

PROPOSITION 3.3. i) *The subgroup* $\langle h_+, h_- \rangle$ *of* G *is central in* $A_{NL}^{(\nu\lambda)}$, *and the order is* $2N$. *As groups*

$$\langle h_+, h_- \rangle \simeq \begin{cases} C_N \times C_2, & \text{if } (N, \nu) = (\text{even}, +1); \\ C_{2N}, & \text{otherwise.} \end{cases}$$

ii) $G \subset Z(A_{NL}^{(\nu\lambda)})$, *the center of* $A_{NL}^{(\nu\lambda)}$, *iff* $g \in Z(A_{NL}^{(\nu\lambda)})$ *iff* $(L, \lambda) = (\text{even}, +1)$.

PROOF. i) The order of $\langle h_+, h_- \rangle$ is $2N$ by Theorem 3.1.
If $(N, \nu) =$

$$\begin{cases} (\text{even}, +1), & \text{then } \langle h_+, h_- \rangle = \langle h_+ \rangle \times \langle x_{11}^{2N} - x_{12}^{2N} \rangle, \\ (\text{even}, -1), & \text{then } \langle h_+, h_- \rangle = \langle h_+ \rangle = \langle h_- \rangle, \\ (\text{odd}, +1), & \text{then } \langle h_+, h_- \rangle = \langle h_- \rangle, \\ (\text{odd}, -1), & \text{then } \langle h_+, h_- \rangle = \langle h_+ \rangle. \end{cases}$$

ii) Note that $G = \langle h_+, h_- \rangle \cup \langle h_+, h_- \rangle g$. So it follows that $G \subset Z(A_{NL}^{(\nu\lambda)})$ iff $g \in Z(A_{NL}^{(\nu\lambda)})$.
It is easy to see that

$$g \text{ is central} \Leftrightarrow \begin{cases} x_{ii} \cdot \overbrace{x_{11}x_{22}\cdots}^{L} = \overbrace{x_{11}x_{22}\cdots}^{L} \cdot x_{ii}, & \\ x_{i,i+1} \cdot \overbrace{x_{12}x_{21}\cdots}^{L} = \overbrace{x_{12}x_{21}\cdots}^{L} \cdot x_{i,i+1}, & \text{for } i = 1, 2. \end{cases}$$ □

REMARK 3.4.
i) The dimension of a simple subcoalgebra of $A_{NL}^{(\nu\lambda)}$ is either 1 or $2^2 = 4$.
ii) The simple subcoalgebra C_{01} generates $A_{NL}^{(\nu\lambda)}$ as an algebra.

iii) For the YB-coalgebra $(C,\sigma_{\alpha\beta})$, $C \simeq C_{01} \subset A_{NL}^{(\nu\lambda)}$, $X_{ij} \mapsto x_{ij}$, is a coalgebra $\sigma_{\alpha\beta}$-map.

We identify C and C_{01}.

iv) $A_{12}^{(+-)}$ ($\simeq A_{12}^{(--)}$, see Prop.3.12 below) is the "non-trivial" semisimple Hopf algebra of dimension 8 ([Mas2]). The ideal decomposition is given as follows:

$$\begin{aligned} A_{12}^{(+-)} &= k(x_{11}+x_{22}+x_{11}^2+x_{11}x_{22}) \oplus k(x_{11}-x_{22}-x_{11}^2+x_{11}x_{22}) \\ &\oplus k(x_{11}-x_{22}+x_{11}^2-x_{11}x_{22}) \oplus k(x_{11}+x_{22}-x_{11}^2-x_{11}x_{22}) \\ &\oplus span_k\{x_{12},x_{21},x_{12}^2,x_{12}x_{21}\}. \end{aligned}$$

v) Since the subHopf algebra $K = k\langle h_+, h_-\rangle$ is normal, $A_{NL}^{(\nu\lambda)}K^+$ is a Hopf ideal, where $K^+ = \operatorname{Ker}\varepsilon_K$. So $A_{NL}^{(\nu\lambda)}/A_{NL}^{(\nu\lambda)}K^+ = \bar{A}$ is a Hopf algebra of dimension $2L$. It is easy to see that the elements $\bar{x}_{11} = a$, $\bar{x}_{22} = b \in \bar{A}$ are grouplike and generate $\bar{A}$ as an algebra. This means that $\bar{A}$ is a group-algebra. Moreover let $ab = c$, then the order of c is L. Then,

$$\begin{aligned} \bar{A} &= k\langle a,b \mid a^2 = 1 = b^2, \overbrace{baba\cdots}^{L} = \overbrace{abab\cdots}^{L}\rangle \\ &= k\langle a,c \mid a^2 = 1, c^L = 1, aca^{-1} = c^{-1}\rangle \\ &= kD_L, \quad \text{where } D_L \text{ is the dihedral group of order } 2L. \end{aligned}$$

Thus we obtain a short exact sequence by means of [Mas1, Definition 1.3]

$$1 \to K \hookrightarrow A_{NL}^{(\nu\lambda)} \to kD_L \to 1.$$

vi) As bialgebras

$$\begin{aligned} B/J_2^\lambda &= B/(X_{11}X_{22}-X_{22}X_{11}, X_{12}X_{21}-\lambda X_{21}X_{12}) \\ &= k\langle X_{ij}\rangle/(X_{11}^2-X_{22}^2, X_{12}^2-X_{21}^2, X_{ij}X_{lm}\ (i+j+l+m\equiv 1), \\ &\qquad X_{11}X_{22}-X_{22}X_{11}, X_{12}X_{21}-\lambda X_{21}X_{12}) \\ &= B^{(\lambda)}/(X_{11}^2-X_{22}^2, X_{12}^2-X_{21}^2). \end{aligned}$$

Thus $A_{N2}^{(\nu\lambda)}$ is furthermore a quotient bialgebra of $B^{(\lambda)}$:

$$A_{N2}^{(\nu\lambda)} = B^{(\lambda)}/(X_{11}^2-X_{22}^2, X_{12}^2-X_{21}^2, 1-(X_{11}^{2N}+\nu X_{12}^{2N})).$$

We note that $\{X_{11}^2-X_{22}^2, X_{12}^2-X_{21}^2\}$ spans a coideal of $B^{(\lambda)}$ and that $\{1-(X_{11}^{2N}+\nu X_{12}^{2N})\}$ spans a coideal modulo the coideal $span_k\{X_{11}^2-X_{22}^2, X_{12}^2-X_{21}^2\}$.

Recall that C_{st} denotes a simple subcoalgebra of dimension 4 of $A_{NL}^{(\nu\lambda)}$ for $0 \leqq s \leqq N-1$, $1 \leqq t \leqq L-1$. Let $\langle C_{st} \rangle$ denote the subHopf algebra generated by C_{st}. It is easy to see that $\langle C_{st} \rangle$ is commutative iff either t is even or $(L, \lambda) = (2t, +1)$. So it follows that t is odd if $\langle C_{st} \rangle$ is non-commutative.

We show that $\langle C_{st} \rangle$ is a member of the family $\{A_{NL}^{(\nu\lambda)}\}$ if t is odd.

Set

$$GCD(L, t) = m_L, \quad GCD(N, 2s+t) = m_N,$$
$$L/m_L = L_0, \quad N/m_N = N_0, \quad t/m_L = t_0, \quad (2s+t)/m_N = (s,t)_0,$$
$$(2 \leqq L_0 \leqq L, 1 \leqq N_0 \leqq N).$$

THEOREM 3.5. *Assume that t is odd and $C_{st} \subset A_{NL}^{(\nu\lambda)}$. Then*

$$\langle C_{st} \rangle \simeq A_{N_0 L_0}^{(\nu\lambda)} \quad \textit{as Hopf algebras.}$$

PROOF. Let t be odd, and fix $0 \leqq s \leqq N-1$ and $1 \leqq t \leqq L-1$. We note that integers $2s+t$, t_0, $(s,t)_0$, m_L and m_N are also odd.

Set

$$z_{11} = x_{11}^{2s} \overbrace{x_{11}x_{22}\cdots x_{11}}^{t}, \quad z_{12} = x_{12}^{2s} \overbrace{x_{12}x_{21}\cdots x_{12}}^{t},$$
$$z_{21} = x_{12}^{2s} \overbrace{x_{21}x_{12}\cdots x_{21}}^{t}, \quad z_{22} = x_{11}^{2s} \overbrace{x_{22}x_{11}\cdots x_{22}}^{t}.$$

The map $\omega : A_{N_0 L_0}^{(\nu\lambda)} \to \langle C_{st} \rangle$, $x_{ij} \mapsto z_{ij}$, is a (well-defined) surjective Hopf algebra map. This is easily verified.

We show that the map ω is injective.

Recall and set that

$$G_0 = G(A_{N_0 L_0}^{(\nu\lambda)})$$
$$= \{x_{11}^{2u} \pm x_{12}^{2u}, x_{11}^{2u} \cdot \overbrace{x_{11}x_{22}x_{11}\cdots}^{L_0} \pm \sqrt{\lambda} x_{12}^{2u} \cdot \overbrace{x_{12}x_{21}x_{12}\cdots}^{L_0} \mid 1 \leqq u \leqq N_0\},$$
$$(C_{uv})_0 = C_{uv} \subset A_{N_0 L_0}^{(\nu\lambda)}.$$

Then it follows that

$$A_{N_0 L_0}^{(\nu\lambda)} = kG_0 \oplus \Sigma (C_{uv})_0.$$

Thus it is enough to show that ω is injective on kG_0 and on $\Sigma (C_{uv})_0$.

It is easy to see that ω is injective on kG_0.

So we show that ω is injective on $\Sigma(C_{uv})_0$.

First we examine $\omega((C_{uv})_0)$ for $0 \leqq u \leqq N_0 - 1$, $1 \leqq v \leqq L_0 - 1$.

Let $tv = qL + r$, for some q, $0 \leqq r \leqq L - 1$. It is easy to see that $r \neq 0$, so it follows that $1 \leqq r$, $L - r \leqq L - 1$.

For $x_{11}^{2u}\overbrace{x_{11}x_{22}x_{11}\cdots}^{v} \in (C_{uv})_0$, observe that

$$
\begin{aligned}
&\omega(x_{11}^{2u}\overbrace{x_{11}x_{22}x_{11}\cdots}^{v}) \\
&= z_{11}^{2u}\overbrace{z_{11}z_{22}z_{11}\cdots}^{v} \\
&= (x_{11}^{2s}\overbrace{x_{11}x_{22}\cdots x_{11}}^{t})^{2u}\cdot \overbrace{(x_{11}^{2s}\cdot x_{11}x_{22}\cdots x_{11})(x_{11}^{2s}\cdot x_{22}x_{11}\cdots x_{22})\cdots}^{v} \\
&= x_{11}^{2(2s+t)u}x_{11}^{2sv}\overbrace{x_{11}x_{22}x_{11}\cdots}^{tv} \\
&= x_{11}^{2(2s+t)u}x_{11}^{2sv} \\
&\quad\times \begin{cases} x_{11}^{qL}\cdot\overbrace{x_{11}x_{22}x_{11}\cdots}^{r}, & \text{if } q \text{ is even,} \\ x_{11}^{(q-1)L}\cdot\overbrace{x_{11}x_{22}x_{11}\cdots}^{L+r}, & \text{if } q \text{ is odd} \end{cases} \\
&= \begin{cases} x_{11}^{2\{(2s+t)u+sv+(q/2)L\}}\cdot\overbrace{x_{11}x_{22}x_{11}\cdots}^{r}, & \text{if } q \text{ is even,} \\ x_{11}^{2\{(2s+t)u+sv+((q-1)/2)L+r\}}\cdot\overbrace{x_{22}x_{11}x_{22}\cdots}^{L-r}, & \text{if } q \text{ is odd} \end{cases} \\
&\neq 0.
\end{aligned}
$$

Let

$$
(a,b) = \begin{cases} ((2s+t)u + sv + \frac{q}{2}L \bmod N, r), & \text{if } q \text{ is even,} \\ ((2s+t)u + sv + \frac{(q-1)}{2}L + r \bmod N, L-r), & \text{if } q \text{ is odd,} \\ (0 \leqq a \leqq N-1, 1 \leqq b \leqq L-1). \end{cases}
$$

So we have that

$$
0 \neq \omega(x_{11}^{2u}\overbrace{x_{11}x_{22}x_{11}\cdots}^{v}) \in \omega((C_{uv})_0) \cap C_{ab}.
$$

Since C_{ab} is a simple subcoalgebra, it follows that

$$\omega((C_{uv})_0) = C_{ab} \subset A_{NL}^{(\nu\lambda)}.$$

Thus ω is injective on $(C_{uv})_0$.

Next assume that there are $0 \leqq u,\ u' \leqq N_0 - 1,\ 1 \leqq v,\ v' \leqq L_0 - 1$ such that $\omega((C_{uv})_0) = \omega((C_{u'v'})_0)$.

Let $tv' = q'L + r',\ 1 \leqq r' \leqq L - 1$.

It is easy to see that $q \equiv q' \mod 2$ implies $u = u'$ and $v = v'$.

So let q be even and q' odd. This implies that $q + q' + 1$ is even and that $L = r + r'$.

We have that $t(v + v') = (q + q' + 1)L$, so it follows that $L_0 \mid v + v'$.

It follows that $L_0 = v + v'$, since $1 \leqq v,\ v' \leqq L_0 - 1$.

So we have $t = (q + q' + 1)m_L$, and this means that t is even. A contradiction.

Thus $\omega((C_{uv})_0) = \omega((C_{u'v'})_0)$ iff $u = u',\ v = v'$, so ω is injective on $\Sigma(C_{uv})_0$. Therefore we have the injectivity of ω.

This completes the proof of the theorem. □

It is easy to see that the following lemma holds.

LEMMA 3.6. *Assume that A_1 and A_2 are bialgebras over an algebraically closed field. If the bialgebra $A_1 \otimes A_2$ is generated by a simple subcoalgebra as an algebra, then so is A_i, $i = 1, 2$. Moreover if any simple subcoalgebra of $A_1 \otimes A_2$ has dimension 1 or n^2, then either A_1 or A_2 is pointed.*

COROLLARY 3.7.

i) *Assume that $A_{NL}^{(\nu\lambda)}$ is non-commutative, i.e. $(L, \lambda) \neq (2, +1)$, and $C_{st} \subset A_{NL}^{(\nu\lambda)}$. Then*

$$\langle C_{st} \rangle = A_{NL}^{(\nu\lambda)} \quad \textit{iff } t \textit{ is odd}, (L, t) = 1 \textit{ and } (N, 2s + t) = 1.$$

ii) *Assume simply that t is odd and $C_{st} \subset A_{NL}^{(\nu\lambda)}$. Then*

$$\langle C_{st} \rangle = A_{NL}^{(\nu\lambda)} \quad \textit{iff } (L, t) = 1, (N, 2s + t) = 1.$$

iii) *Let N be $2^n m$, and m odd. Then*

$$A_{NL}^{(\nu\lambda)} \simeq A_{2^n, L}^{(\nu\lambda)} \otimes kC_m \quad \textit{as Hopf algebras.}$$

iv) *If $A_{2^n, L}^{(\nu\lambda)}$ is non-commutative, then it is indecomposable as the tensor product of its subHopf algebras.*

PROOF. i), ii) These follow from the dimensionality.

iii) Let N be $2^n m$ and m odd. We may assume that $m \geqq 3$. Now let $s = (m-1)/2$, $t = 1$, then it follows that $2s + t = m$, $N_0 = 2^n$, $L_0 = L$, and $\langle C_{st} \rangle \simeq A^{(\nu\lambda)}_{2^n, L}$.

Let $f = x_{11}^{2 \cdot 2^n} + \nu x_{12}^{2 \cdot 2^n}$. Then f is a central grouplike element with order m, and $C_{st} \cdot f = C_{s't}$, where $s' = 2^n + (m-1)/2 \leqq N - 1$.

For such s, s' and t, it follows that

$$\begin{aligned}(2s' + t, N) &= \left(2\left\{2^n + \frac{m-1}{2}\right\} + 1, 2^n m\right) \\ &= (2^{n+1} + m, 2^n m) \\ &= 1.\end{aligned}$$

Thus the simple subcoalgebra $C_{st} \cdot f = C_{s't}$ generates $A^{(\nu\lambda)}_{NL}$ as an algebra by ii). Therefore we have that

$$A^{(\nu\lambda)}_{2^n m, L} \simeq A^{(\nu\lambda)}_{2^n, L} \otimes kC_m, \quad \text{as Hopf algebras.}$$

iv) Let $2^n = N$. Applying Lemma 3.6 to $A^{(\nu\lambda)}_{NL}$, we may assume

$$A^{(\nu\lambda)}_{NL} = \langle C_{st} \rangle \otimes kF,$$

for some $0 \leqq s \leqq N-1$, $1 \leqq t \leqq L-1$, (abelian)subgroup $F \subset G(A^{(\nu\lambda)}_{NL})$.

Since $A^{(\nu\lambda)}_{NL}$ is non-commutative, so is $\langle C_{st} \rangle$. This means that t is odd. By Theorem 3.5, $\langle C_{st} \rangle \simeq A^{(\nu\lambda)}_{N_0 L_0}$.

Comparing the dimensions, we have that $|F| = m_N m_L$.

Counting the number of 4-dimensional simple subcolagebras, we have the following:

$$\begin{aligned}N(L-1) &= N_0(L_0 - 1) \cdot |F| \\ &= N_0(L_0 - 1) m_N m_L \\ &= N(L - m_L).\end{aligned}$$

Thus we have that $m_L = 1$.

On the other hand, it follows that $m_N = 1$ since $2s + t$ is odd and N is a power of 2.

Thus we have that $F = \langle 1 \rangle$. □

Next we show that we can obtain all braidings on $A^{(\nu\lambda)}_{NL}$. See [GW], [G]. We identifiy $C < A^{(\nu\lambda)}_{NL}$ as in Remark 3.4. Note that any braiding on $A^{(\nu\lambda)}_{NL}$ is determined on $C \otimes C$. If a bilinear map τ on C extends to a braiding on $A_{NL}^{(\nu\lambda)}$, we denote the braiding by $\tilde{\tau}$.

Recall YB-forms $\sigma_{\alpha\beta}$, $\tau_{\alpha\beta}^{(\lambda)}$ on C.

CLAIM 3.8. *Let σ be a braiding on $A_{NL}^{(\nu\lambda)}$.*

i) *If $L \geqq 3$, $\sigma|_{C\otimes C}$ coincides with $\sigma_{\alpha\beta}$ for some $\alpha, \beta \in k^{\times}$ such that $(\alpha\beta)^N = \nu$, $(\alpha\beta^{-1})^L = \lambda$.*

ii) *If $L = 2$, $\sigma'|_{C\otimes C}$ coincides with either $\sigma_{\alpha\beta}$ for some $\alpha, \beta \in k^{\times}$ such that $(\alpha\beta)^N = \nu$, $(\alpha\beta^{-1})^2 = \lambda$ or $\tau_{\gamma\delta}^{(\lambda)}$ for some $\gamma, \delta \in k^{\times}$ such that $\delta^2 = \gamma^2$, $\gamma^{2N} = 1$.*

PROOF. i) Assume that $L \geqq 3$.

The subcoalgebra $C \cdot C$ of $A_{NL}^{(\nu\lambda)}$ has a basis

$$\{x_{11}^2, x_{12}^2, x_{11}x_{22}, x_{22}x_{11}, x_{12}x_{21}, x_{21}x_{12}\}.$$

We have similarly as in Proposition 2.9,

$$\sigma|_{C\otimes C} = \sigma_{\alpha\beta} \quad \text{for some } \alpha, \beta \in k^{\times}.$$

Moreover σ satisfies the following:

$$\begin{aligned} 0 &= \sigma(1 - (x_{11}^{2N} + \nu x_{12}^{2N}), x_{11}) \\ &= 1 - \nu\{\sigma_{\alpha\beta}(x_{12}, x_{12})\sigma_{\alpha\beta}(x_{12}, x_{21})\}^N \\ &= 1 - \nu(\alpha\beta)^N. \end{aligned}$$

Thus it follows that $(\alpha\beta)^N = \nu$.

Observe that when L is even,

$$\begin{aligned} 0 &= \sigma(\overbrace{x_{21}x_{12}\cdots x_{12}}^{L} - \lambda\overbrace{x_{12}x_{21}\cdots x_{21}}^{L}, x_{22}) \\ &= \alpha^L - \lambda\beta^L, \end{aligned}$$

and that when L is odd,

$$\begin{aligned} 0 &= \sigma(\overbrace{x_{21}x_{12}\cdots x_{21}}^{L} - \lambda\overbrace{x_{12}x_{21}\cdots x_{12}}^{L}, x_{21}) \\ &= \alpha^L - \lambda\beta^L. \end{aligned}$$

Thus in either case, it follows that $\alpha^L = \lambda\beta^L$, or $(\alpha\beta^{-1})^L = \lambda$.

ii) Assume that $L = 2$.

The subcoalgebra $C \cdot C$ of $A_{N2}^{(\nu\lambda)}$ has a basis

$$\{x_{11}^2, x_{12}^2, x_{11}x_{22}, x_{12}x_{21}\}.$$

As in the proof of Proposition 2.9, we have the following:

$$\sigma(x_{ij}, x_{lm})x_{jj}x_{mm} = x_{ll}x_{ii}\sigma(x_{ij}, x_{lm}),$$

$$\sigma(x_{i,j+1}, x_{l,m+1})x_{j+1,j}x_{m+1,m} = x_{l,l+1}x_{i,i+1}\sigma(x_{i+1,j}, x_{l+1,m}).$$

Using these relations, we have the following with α, β, γ, $\delta \in k$,

σ	X_{11}	X_{12}	X_{21}	X_{22}
X_{11}	γ	0	0	δ
X_{12}	0	α	β	0
X_{21}	0	β	α	0
X_{22}	$\lambda\delta$	0	0	γ.

Moreover σ satisfies the following equations:

$$0 = \sigma(x_{11}x_{12}, x_{12}) = \gamma\alpha,$$

$$0 = \sigma(x_{11}x_{21}, x_{12}) = \gamma\beta,$$

$$0 = \sigma(x_{11}x_{12}, x_{21}) = \delta\beta,$$

$$0 = \sigma(x_{11}x_{21}, x_{21}) = \delta\alpha.$$

So it follows that either $\gamma = 0 = \delta$ or $\alpha = 0 = \beta$.
Thus $\sigma|_{C\otimes C}$ is either $\sigma_{\alpha\beta}$ or $\tau_{\gamma\delta}^{(\lambda)}$, for α, β, γ, $\delta \in k^\times$.
If $\sigma|_{C\otimes C} = \sigma_{\alpha\beta}$, then the relations on α, β follow similarly as in the proof of i).
Let $\sigma|_{C\otimes C} = \tau_{\gamma\delta}^{(\lambda)}$. Observe that

$$\begin{aligned} 0 &= \sigma(x_{11}^2 - x_{22}^2, x_{22}) \\ &= \tau_{\gamma\delta}^{(\lambda)}(x_{11}, x_{22})^2 - \tau_{\gamma\delta}^{(\lambda)}(x_{22}, x_{22})^2 \\ &= \delta^2 - \gamma^2, \\ 0 &= \sigma(1 - (x_{11}^{2N} - \nu x_{12}^{2N}), x_{11}) \\ &= 1 - \tau_{\gamma\delta}^{(\lambda)}(x_{11}, x_{11})^{2N} \\ &= 1 - \gamma^{2N}. \end{aligned}$$

Thus it follows that $\delta^2 = \gamma^2$, $\gamma^{2N} = 1$. □

Claim 3.9.
i) *The YB-form* $\sigma_{\alpha\beta}$ *extends to a braiding on* $A_{NL}^{(\nu\lambda)}$ *if* $(\alpha\beta)^N = \nu$, $(\alpha\beta^{-1})^L = \lambda$.
ii) *The YB-form* $\tau_{\gamma\delta}^{(\lambda)}$ *extends to a braiding on* $A_{N2}^{(\nu\lambda)}$ *if* $\delta^2 = \gamma^2$, $\gamma^{2N} = 1$.

Proof. Recall that B has braidings $\{\tilde{\sigma}_{\alpha\beta} \mid \alpha, \beta \in k^\times\}$ and that $B^{(\lambda)}$ has braidings $\{\tilde{\tau}_{\gamma\delta}^{(\lambda)} \mid \gamma, \delta \in k^\times\}$.

i) It is easy to see by Proposition 1.1 that $\tilde{\sigma}_{\alpha\beta} : B \otimes B \to k$ induces a braiding on $A_{NL}^{(\nu\lambda)}$ iff

$$\begin{cases} (\alpha\beta)^N = \nu, \\ (\alpha\beta^{-1})^L = \lambda. \end{cases}$$

ii) Recall that $A_{N2}^{(\nu\lambda)} = B^{(\lambda)}/(X_{11}^2 - X_{22}^2, X_{12}^2 - X_{21}^2, 1 - (X_{11}^{2N} + \nu X_{12}^{2N}))$.

It follows that $\tau_{\gamma\delta}^{(\lambda)}$ induces a braiding on $B^{(\lambda)}/(X_{11}^2 - X_{22}^2, X_{12}^2 - X_{21}^2)$ iff $\delta^2 = \gamma^2$, and that $\tau_{\gamma\delta}^{(\lambda)}$ induces a braiding on $A_{N2}^{(\nu\lambda)}$ iff $\delta^2 = \gamma^2$, $\gamma^{2N} = 1 = \delta^{2N}$. □

Proposition 3.10.
i) *The set of braidings on* $A_{NL}^{(\nu\lambda)}$ *is given as follows*:

$$\{\tilde{\sigma}_{\alpha\beta} \mid (\alpha\beta)^N = \nu, (\alpha\beta^{-1})^L = \lambda\}, \quad \text{if } L \geqq 3,$$

$$\{\tilde{\sigma}_{\alpha\beta}, \tilde{\tau}_{\gamma\delta}^{(\lambda)} \mid (\alpha\beta)^N = \nu, (\alpha\beta^{-1})^2 = \lambda, \delta^2 = \gamma^2, \gamma^{2N} = 1\}, \quad \text{if } L = 2.$$

ii) $A_{NL}^{(\nu\lambda)}$ *is, in fact, a braided Hopf algebra.*
If $\mathrm{ch}\, k \nmid NL$, *the number of braidings on* $A_{NL}^{(\nu\lambda)}$ *is*

$$\begin{cases} 2NL, & \text{if } L \geqq 3, \\ 8N, & \text{if } L = 2. \end{cases}$$

iii) *The number of symmetric braidings on* $A_{NL}^{(\nu\lambda)}$ *is given as follows*;
When $L \geqq 3$,

N	L	(ν, λ)	$\tilde{\sigma}$
odd	*odd*	$(\pm 1, \pm 1)$	2
		$(\pm 1, \mp 1)$	0
odd	*even*	$(\nu, +1)$	2
		$(\nu, -1)$	0
even	*odd*	$(+1, \lambda)$	2
		$(-1, \lambda)$	0
even	*even*	$(+1, +1)$	4
		otherwise	0.

When $L=2$,

N	(ν,λ)	$\tilde{\sigma}$	$\tilde{\tau}^{(\lambda)}$
odd	$(\nu,+1)$	2	4
	$(\nu,-1)$	0	0
even	$(+1,+1)$	4	4
	$(+1,-1)$	0	0
	$(-1,+1)$	0	4
	$(-1,-1)$	0	0.

PROOF. i) This follows from Claim 3.8 and 3.9.

ii) There is a surjective map

$$\{(p,q)\in k\times k\mid p^{2N}=\nu, q^{2L}=\lambda\}\to\{(\alpha,\beta)\in k\times k\mid(\alpha\beta)^N=\nu,(\alpha\beta^{-1})^L=\lambda\},$$
$$(p,q)\mapsto(pq,pq^{-1}).$$

Set $(p,q)\sim(p',q')\Leftrightarrow(p,q)=\pm(p',q')$. It is an equivalence relation, which induces the bijection

$$\{(p,q)\mid p^{2N}=\nu,q^{2L}=\lambda\}/\sim\ \approx\{(\alpha,\beta)\mid(\alpha\beta)^N=\nu,(\alpha\beta^{-1})^L=\lambda)\}.$$

Let $chk\nmid NL$. Then it follows that $|\{\tilde{\sigma}\}|=2N\cdot 2L\cdot\frac{1}{2}=2NL$. For $\tilde{\tau}^{(\lambda)}$, since $\gamma^{2N}=1$ and $\delta^2=\gamma^2$, it follows that $|\{\tilde{\tau}^{(\lambda)}\}|=2N\cdot 2=4N$.

iii) Recall that $chk\neq 2$. On $A_{NL}^{(\nu\lambda)}$, $\tilde{\sigma}_{\alpha\beta}$ is symmetric iff $\alpha^2=1=\beta^2$ and $(\alpha\beta)^N=\nu$, $(\alpha\beta^{-1})^L=\lambda$.

On $A_{N2}^{(\nu\lambda)}$, $\tilde{\tau}_{\gamma\delta}^{(\lambda)}$ is symmetric iff $\gamma^2=1$, $\delta^2=\lambda$ and $\gamma^{2N}=1$, $\delta^2=\gamma^2$. □

REMARK 3.11. The algebra map $\theta: A_{NL}^{(\nu\lambda)}\to A_{NL}^{(\nu\lambda)cop}$, $x_{ij}\mapsto x_{ji}$, is a bijective Hopf algebra map. Define $\langle a,b\rangle=\tilde{\sigma}_{\alpha\beta}(\theta(a),b)$ for $a,b\in A_{NL}^{(\nu\lambda)}$.

The linear map $\langle\ ,\ \rangle: A_{NL}^{(\nu\lambda)}\otimes A_{NL}^{(\nu\lambda)}\to k$ is a non-trivial Hopf paring.

Using Proposition 3.10, we have the following indispensable proposition.

PROPOSITION 3.12. *$A_{N_1L_1}^{(\nu_1\lambda_1)}\simeq A_{N_2L_2}^{(\nu_2\lambda_2)}$ if and only if both $(N_1,L_1)=(N_2,L_2)$ and*

$$\begin{cases}(\nu_2,\lambda_2)=\pm(\nu_1,\lambda_1), & (\textit{case } N_1,L_1 \textit{ odd});\\ \lambda_2=\lambda_1, & (\textit{case } N_1 \textit{ odd}, L_1 \textit{ even});\\ \nu_2=\nu_1, & (\textit{case } N_1 \textit{ even}, L_1 \textit{ odd});\\ (\nu_2,\lambda_2)=(\nu_1,\lambda_1), & (\textit{case } N_1,L_1 \textit{ even}).\end{cases}$$

PROOF. For a fixed $\sqrt{-1}$, we can define a bialgebra map $\xi : B \to B$,

$$\begin{aligned} \xi : X_{ii} &\mapsto X_{ii}, \\ X_{12} &\mapsto \sqrt{-1}X_{12}, \\ X_{21} &\mapsto -\sqrt{-1}X_{21}. \end{aligned}$$

Let

$$\check{A}_{NL}^{(\nu\lambda)} = \begin{cases} A_{NL}^{(-\nu,-\lambda)}, & \text{if } N, L \text{ are odd,} \\ A_{NL}^{(-\nu,\lambda)}, & \text{if } N \text{ is odd, } L \text{ is even,} \\ A_{NL}^{(\nu,-\lambda)}, & \text{if } N \text{ is even, } L \text{ is odd,} \\ A_{NL}^{(\nu\lambda)}, & \text{if } N, L \text{ are even.} \end{cases}$$

Then the following diagram commutes:

$$\begin{array}{ccc} B & \xrightarrow{\xi} & B \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\pi} \\ A_{NL}^{(\nu\lambda)} & \xrightarrow{\simeq} & \check{A}_{NL}^{(\nu\lambda)}. \end{array}$$

Thus by Proposition 3.10.iii), if N or L is odd, then the statement follows. Assume that both N and L are even. Then

$$(\nu_1, \lambda_1) = \begin{cases} (++) \Rightarrow \text{by Prop. 3.10.iii)}, (\nu_2, \lambda_2) = (++). \\ (-+) \Rightarrow \text{by Prop. 3.3.ii)}, G(A_{N_1L_1}^{(\nu_1\lambda_1)}) \text{ is central so } \lambda_2 = +1. \\ \qquad \text{By Prop. 3.10.iii)}, \nu_2 = -1 \text{ so} (\nu_2, \lambda_2) = (-+). \\ (+-) \Rightarrow \text{by Prop. 3.3.ii)}, kG(A_{N_1L_1}^{(\nu_1\lambda_1)}) \cap Z(A_{N_1L_1}^{(\nu_1\lambda_1)}) = K \text{ so } \lambda_2 = -1. \\ \qquad \text{By Prop. 3.3.i)}, \nu_2 = +1 \text{ so} (\nu_2, \lambda_2) = (+-). \\ (--) \Rightarrow \text{it follows that} (\nu_2, \lambda_2) = (--). \end{cases}$$

This completes the proof. □

REMARK 3.13 ([Mas2], [F]). The "non-trivial" 8-dimensional semisimple Hopf algebra is given by

$$A_{1,2}^{(+-)} \simeq A_{1,2}^{(--)}.$$

Let $chk \neq 3$. The two "non-trivial" 12-dimensional semisimple Hopf algebras are given by

$$A_{1,3}^{(++)} \simeq A_{1,3}^{(--)} \quad \text{and} \quad A_{1,3}^{(+-)} \simeq A_{1,3}^{(-+)}.$$

Recall that H is a Hopf closure of B and that $A_{NL}^{(\nu\lambda)}$ is a Hopf algebra which is a quotient of B through π. So there is a Hopf algebra map $\tilde{\pi} : H \to A_{NL}^{(\nu\lambda)}$ such that $\tilde{\pi} = \pi|_B$.

H is a right $A_{NL}^{(\nu\lambda)}$-comodule algebra via $\tilde{\pi}$. See [DT]. Then

PROPOSITION 3.14. *H is a cleft $A_{NL}^{(\nu\lambda)}$-comodule algebra. Namely there is an invertible comodule map $\phi : A_{NL}^{(\nu\lambda)} \to H$.*

PROOF. Recall the basis $\{x_{11}^s \cdot \overbrace{x_{22}x_{11}\cdots}^{t}, x_{12}^s \cdot \overbrace{x_{21}x_{12}\cdots}^{t} \mid 1 \leqq s \leqq 2N,\ 0 \leqq t \leqq L-1\}$. This can be written as follows:

$$\begin{pmatrix} & x_{11}^{2(s+1)} & & x_{12}^{2(s+1)} \\ x_{11}^{2s} \cdot x_{11} & x_{11}^{2s} \cdot x_{22} & x_{12}^{2s} \cdot x_{12} & x_{12}^{2s} \cdot x_{21} \\ x_{11}^{2s} \cdot x_{11}x_{22} & x_{11}^{2s} \cdot x_{22}x_{11} & x_{12}^{2s} \cdot x_{12}x_{21} & x_{12}^{2s} \cdot x_{21}x_{12} \\ \vdots & \vdots & \vdots & \vdots \\ x_{11}^{2s} \cdot \overbrace{x_{11}x_{22}\cdots}^{L-1} & x_{11}^{2s} \cdot \overbrace{x_{22}x_{11}\cdots}^{L-1} & x_{12}^{2s} \cdot \overbrace{x_{12}x_{21}\cdots}^{L-1} & x_{12}^{2s} \cdot \overbrace{x_{21}x_{12}\cdots}^{L-1} \\ x_{11}^{2s} \cdot \overbrace{x_{11}x_{22}\cdots x_{LL}}^{L} & & x_{12}^{2s} \cdot \overbrace{x_{12}x_{21}\cdots x_{L,L+1}}^{L} & \end{pmatrix}$$

for $0 \leqq s \leqq N-1$.

We use it. Define, for example, a linear map $\phi : A_{NL}^{(\nu\lambda)} \to B \to H$ by the small letters to its capital letters, i.e., x_{ij} to X_{ij}, etc. Then ϕ is a right $A_{NL}^{(\nu\lambda)}$-comodule map.

We define another linear map $\psi : A_{NL}^{(\nu\lambda)} \to H$ as follows:

On the bottom row,

$$\psi : x_{11}^{2s} \cdot \overbrace{x_{11}x_{22}\cdots x_{LL}}^{L} \mapsto (\overbrace{X_{LL}\cdots X_{22}X_{11}}^{L} \cdot X_{11}^{2s})\left(\frac{1}{d_+}\right)^{2s+L},$$

$$x_{12}^{2s} \cdot \overbrace{x_{12}x_{21}\cdots x_{L,L+1}}^{L} \to \lambda(\overbrace{X_{L,L+1}\cdots X_{21}X_{12}}^{L} \cdot X_{12}^{2s})\left(\frac{1}{d_+}\right)^{2s+L},$$

and on the other rows,

$$\psi = S \circ \phi.$$

Then we have $\psi = \phi^{-1}$, so ϕ is invertible.
Therefore H is a cleft $A_{NL}^{(\nu\lambda)}$-comodule algebra. □

Added in Proof

The group $G = G(A_{NL}^{(\nu\lambda)})$ is abelian, and the type is given as follows. The case that L is even:

$$G = \langle h_+, h_- \rangle \times \langle h_+^{-L/2} g \rangle$$
$$= \begin{cases} (C_N \times C_2) \times C_2, & \text{if } (N, \nu) = (even, +1); \\ (C_{2N}) \times C_2, & \text{otherwise.} \end{cases}$$

The case that L is odd:

$$G = \begin{cases} \langle h_\lambda^{(1-L)/2} g \rangle = C_{4N} & \text{if } \nu = -\lambda^N; \\ \langle h_\lambda^{(1-L)/2} g \rangle \times \langle h_+^{-1} h_- \rangle = C_{2N} \times C_2, & \text{if } \nu = \lambda^N. \end{cases}$$

Proposition 3.12 follows from this and Proposition 3.3.

References

[D] Y. Doi, Braided bialgebras and quadratic bialgebras, Comm. Algebra **21(5)**, 1731–1749.

[DT] Y. Doi and M. Takeuchi, Cleft comodule algebras for a bialgebra, Comm. Algebra **14** (1986), 801–817.

[F] N. Fukuda, Semisimple Hopf algebras of dimension 12 (to appear).

[GW] S. Gelaki and S. Westreich, On the quasitriangularity of $U_q(sl_n)'$, preprint.

[G] S. Gelaki, Quantum groups of dimension pq^2, preprint.

[H] T. Hayashi, Quantum groups and quantum determinants, J. Algebra **152** (1992), 146–165.

[Man] Yu. Manin, Quantum groups and non-commutative geometry, U. of Montreal Lectures, 1988.

[Mas1] A. Masuoka, Coideal subalgebras in finite Hopf algebras, J. Algebra **163** (1994), 819–831.

[Mas2] ———, Semisimple Hopf algebras of dimension 6, 8, Israel J. Math. **92** (1995), 361–373.

[M] S. Montgomery, Hopf algebras and their actions on rings, American Mathematical Society, Prividence, 1993.

[S] M. Sweedler, Hopf Algebras, Benjamin, New York, 1969.

[T1] M. Takeuchi, A two-parameter quantization of $GL(n)$, Proc. Japan Acad. **66. Ser. A** (1990), 112–114.

[T2] ———, Matric bialgebras and quantum groups, Israel J. Math. **72** (1990), 232–251.

Institute of Mathematics
University of Tsukuba
Ibaraki 305, Japan