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HOMOGENEITY OF HYPERSURFACES IN A SPHERE

By

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§ 1. Introduction

A Riemannian manifold $(M, \langle\, , \rangle)$ is called homogeneous if its isometric transformation group acts transitively on M. In general, it is not easy to obtain the isometric transformation group for a given Riemannian manifold. Thus, it is a question whether we can decide the homogeneity or the locally homogeneity of a given Riemannian manifold by more elementary method. For this question, the result of W. Ambrose and I. M. Singer [1] is known.

I. M. Singer defined the concept of the curvature homogeneous Riemannian manifold (further, its higher order version). We here review it. Let M be an n-dimensional Riemannian manifold with the Riemannian connection ∇ and the curvature tensor R. The k-th covariant differential of a tensor field K is denoted by $\nabla^k K$ and $\nabla^0 K = K$, by definition. A linear isomorphism Φ of the tangent space T_pM onto the tangent space T_qM is naturally extended to a linear isomorphism of the tensor algebra $\mathfrak{T}(T_pM)$ onto $\mathfrak{T}(T_qM)$, which is also denoted by Φ. If M is locally homogeneous, i.e., for each p, $q \in M$ there exists a local isometry φ of a neighborhood of p onto a neighborhood of q which maps p to q, then for any integer $k \geq 0$, the following condition $R(k)$ is satisfied:

$$R(k) : \text{For each } p, q \in M, \text{ there exists a linear isometry } \Phi \text{ of } T_pM$$
$$\text{onto } T_qM \text{ such that } \Phi(\nabla^i R)_p = (\nabla^i R)_q, \quad \text{for } i = 0, 1, \ldots, k.$$

In fact, Φ is given by $d\varphi$. A Riemannian manifold M satisfying the condition $R(0)$ (resp. $R(k)$) is called *curvature homogeneous* (resp. *curvature homogeneous up to order* k). I. M. Singer [17] dealt with the converse problem and he proved that if a complete and simply connected Riemannian manifold M satisfies the condition $R(k)$ for a certain k, then M is homogeneous. Following his proof, we see that if a Riemannian manifold M satisfies the condition $R(k)$ for a certain k, then M is locally homogeneous. In his theorem, the minimum of such integers k

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depends on M, but it is not greater than $n(n-1)/2+1$. He also posed the following problem (and also its higher order version): Do there exist curvature homogeneous spaces which are not homogeneous? This problem was solved by K. Sekigawa [13], who constructed 3-dimensional complete simply connected non-homogeneous curvature homogeneous spaces (cf. also [18]). With respect to the first order version, K. Sekigawa [14] proved that a 3-dimensional complete simply connected Riemannian manifold which is curvature homogeneous up to order 1 is homogeneous. It is also known that the similar statement is valid in 4-dimensional case ([16]). However, the higher order version is still unresolved in general.

K. Tsukada [20] studied curvature homogeneous hypersurfaces in real space forms and classified them. In general, a curvature homogeneous hypersurface in a real space form is not necessarily an isoparametric one, where an isoparametric hypersurface is the hypersurface which has constant principal curvatures. By using his result, we see that an $n(\geq 4)$-dimensional complete curvature homogeneous hypersurface in S^{n+1} is isoparametric (cf. Theorem 4.6 and Remark 4.7).

In the present paper, in connection with the I. M. Singer's problem, we consider the homogeneity of hypersurfaces in S^{n+1} and prove the following

THEOREM A. *Let M be an oriented closed hypersurface in S^{n+1} which is curvature homogeneous up to order* 4. *Then, M is homogeneous.*

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§ 2. Preliminaries

Let $\tilde{M}=(\tilde{M}^{n+1}(\tilde{c}),\langle\,,\,\rangle)$ be an $(n+1)$-dimensional real space form of constant curvature $\tilde{c}$ and M an oriented hypersurface immersed in $\tilde{M}$ by an immersion ψ. Since ψ is locally an imbedding, we may identify $x\in M$ with $\psi(x)\in\tilde{M}$ locally, and T_xM with the subspace $(d\psi)_x(T_xM)$ of $T_{\psi(x)}\tilde{M}$. Let ∇ (resp. $\tilde{\nabla}$) be the Riemannian connection on M (resp. $\tilde{M}$) with respect to the induced metric via ψ which is also denoted by $\langle\,,\,\rangle$ (resp. the Riemannian metric $\langle\,,\,\rangle$) and R (resp. $\tilde{R}$) the curvature tensor of M (resp. $\tilde{M}$). The curvature tensor R is defined by

$$R(X,Y)Z=[\nabla_X,\nabla_Y]Z-\nabla_{[X,Y]}Z \tag{2.1}$$

for X, Y, $Z\in\mathfrak{X}(M)$, where $\mathfrak{X}(M)$ denotes the set of all tangential vector fields to M. From now on, we use the notational convention: X, Y, Z, W, $V_i\in\mathfrak{X}(M)$. We

denote by σ the second fundamental form of M in $\tilde{M}$. Let ξ be the unit normal vector field on M. We put

$$\sigma(X, Y) = H(X, Y)\xi. \tag{2.2}$$

The $(0,2)$-type tensor field H on M is called the *second fundamental tensor field.* Then, the Gauss formula and the Weingarten formula are given respectively by

$$\tilde{\nabla}_X Y = \nabla_X Y + H(X, Y)\xi, \tag{2.3}$$

$$\tilde{\nabla}_X \xi = -AX. \tag{2.4}$$

The (1,1)-type tensor field A is called the Weingarten map and is related to the second fundamental tensor field H by

$$H(X, Y) = \langle AX, Y\rangle. \tag{2.5}$$

The Gauss equation and the Codazzi equation are given respectively by

$$\begin{aligned} \langle R(X, Y)Z, W\rangle &= \tilde{c}\{\langle X, W\rangle\langle Y, Z\rangle - \langle X, Z\rangle\langle Y, W\rangle\} \\ &\quad + H(X, W)H(Y, Z) - H(X, Z)H(Y, W) \\ &=: \tilde{c}\langle R_0(X, Y)Z, W\rangle \\ &\quad + H(X, W)H(Y, Z) - H(X, Z)H(Y, W), \end{aligned} \tag{2.6}$$

$$(\nabla_X H)(Y, Z) = (\nabla_Y H)(X, Z). \tag{2.7}$$

We use the following notational convention:

$$\begin{aligned} (\nabla^i R)(V_i, \dots, V_1; X, Y)Z &:= (\nabla^i_{V_i,\dots,V_1} R)(X, Y)Z, \\ (\nabla^i H)(V_i, \dots, V_1; X, Y) &:= (\nabla^i_{V_i,\dots,V_1} H)(X, Y). \end{aligned} \tag{2.8}$$

From (2.7) and (2.8), we have immediately

$$(\nabla^{i+1} H)(V_i, \dots, V_1, X; Y, Z) = (\nabla^{i+1} H)(V_i, \dots, V_1, Y; X, Z), \tag{2.9}$$

for $i \geq 1$. From (2.6) and (2.8), by direct calculation, we have easily the following

$$\begin{aligned} \langle(\nabla^i R)(V_i, \dots, V_1; X, Y)Z, W\rangle &= \{(\nabla^i H)(V_i, \dots, V_1; X, W)H(Y, Z) \\ &\quad - (\nabla^i H)(V_i, \dots, V_1; X, Z)H(Y, W)\} \end{aligned} \tag{2.10}$$

$$+\sum_{j=1}^{i}\Bigg[\sum_{1\le a_1<\cdots<a_j\le i}$$

$$\{(\nabla^{i-j}H)(V_i,\dots,\hat{V}_{a_j},\dots,\hat{V}_{a_1},\dots,V_1;X,W)(\nabla^{j}H)(V_{a_j},\dots,V_{a_1};Y,Z)$$

$$-(\nabla^{i-j}H)(V_i,\dots,\hat{V}_{a_j},\dots,\hat{V}_{a_1},\dots,V_1;X,Z)(\nabla^{j}H)(V_{a_j},\dots,V_{a_1};Y,W)\}\Bigg]$$

for $i\ge 1$.

§ 3. Isoparametric hypersurfaces in S^{n+1}

In the first part of this section, we shall recall some well-known facts about isoparametric hypersurfaces in real space forms. Let M be an oriented isoparametric hypersurface in a real space form $\tilde{M}=(\tilde{M}^{n+1}(\tilde{c}),\langle\ ,\ \rangle)$. From (2.6), an isoparametric hypersurface in a real space form is necessarily curvature homogeneous. We assume that M has g distinct constant principal curvatures, that is, the Weingarten map A has g distinct eigenvalues $\lambda_1>\lambda_2>\cdots>\lambda_g$ at each point, which are constants and have the same multiplicities on M. E. Cartan studied isoparametric hypersurfaces in real space forms. He [3] showed that, if there exists an isoparametric hypersurface in $\tilde{M}$ with $g\ge 3$, it must be $\tilde{c}>0$. By using this fact, he classified completely closed isoparametric hypersurfaces in $\tilde{M}$ with $g=1$ or 2 and showed that all of them are homogeneous. For the case $g=3$, he [4] showed that $m_1=m_2=m_3$ and classified closed isoparametric hypersurfaces in S^{n+1} and checked that all of them are homogeneous. Further, he [6] gave examples of isoparametric hypersurfaces in S^{n+1} with $g=4$ such that $m_1=\cdots=m_4=1$ or 2 and checked their homogeneity. In the mean time, he posed several problems. One of them is the following: Are all of closed isoparametric hypersurfaces in S^{n+1} homogeneous? This problem which is a special case of the I. M. Singer's problem, was solved negatively (cf. [12], [7]).

In the rest of this section, we recall some several results about isoparametric hypersurfaces in S^{n+1}, which will be useful for our arguments in the present paper. Let M be an oriented isoparametric hypersurface in S^{n+1} with g distinct constant principal curvatures. Let $\lambda_i=\cot\theta_i$, $0<\theta_1<\theta_2<\cdots<\theta_g<\pi$ and m_i be the multiplicity of λ_i. H. F. Münzner [9] showed the following results.

PROPOSITION 3.1. *For an isoparametric hypersurface in S^{n+1}, the number g of distinct eigenvalues of A is* 1, 2, 3, 4 *or* 6.

PROPOSITION 3.2. *The following equalities hold,*

(i) $\theta_i = \theta_1 + \dfrac{i-1}{g}\pi$,

(ii) $m_i = m_{i+2}$, *where* $i + g \equiv i$.

REMARK 3.3. Taking account of Proposition 3.1 and Proposition 3.2, we may easily observe the following.

(i) In the case of $g = 2$, 4 or 6, we have $\lambda_i \neq 0$ for any $i = 1, \dots, g$.

(ii) We assume that $\lambda_i = 0$ for some i $(1 \leq i \leq g)$. Then we have $g = 1$ or 3. In the case of $g = 1$, we have $\lambda_1 = 0$, that is, M is totally geodesic in S^{n+1}. In the case of $g = 3$, we have $\lambda_1 = \sqrt{3}$, $\lambda_2 = 0$, $\lambda_3 = -\sqrt{3}$, and hence M is minimal.

H. Takagi [19] studied oriented isoparametric hypersurfaces M in S^{n+1} satisfying the following condition $H(k)$ which is analogous to $R(k)$:

$$H(k) : \text{For each } p, q \in M, \text{ there exists a linear isometry } \Phi \text{ of } T_pM$$
$$\text{onto } T_qM \text{ such that } \Phi(\nabla^i H)_p = (\nabla^i H)_q, \quad \text{for } i = 0, 1, \dots, k.$$

Using the results of H. F. Münzner, he gave a condition for isoparametric hypersurfaces in S^{n+1} to be homogeneous:

THEOREM 3.4. *Let M be a closed isoparametric hypersurface in S^{n+1} with g distinct principal curvatures. Then, M is homogeneous if and only if the condition $H(g-2)$ is satisfied.*

§4. Proof of the Theorem A

Let V be the n-dimensional real vector space with inner product $\langle\, , \rangle$ and S^2V^* the space of $(0,2)$-type symmetric tensors on V and $\tilde{S}^2V^* = \{H \in S^2V^* \mid \operatorname{rank} H \geq 3\}$. Then, we have the following

LEMMA 4.1. *If the equality*

$$G(x,w)H(y,z) - G(x,z)H(y,w) + H(x,w)G(y,z) - H(x,z)G(y,w) = 0$$

holds for $H \in \tilde{S}^2V^*$ and $G \in S^2V^*$, then $G = 0$.

PROOF. We choose an orthonormal basis $\{e_r\}$ $(r = 1, \dots, n)$ of V such that $H(e_r, e_s) = \mu_r \delta_{rs}$. Since $\operatorname{rank} H \geq 3$, we may suppose that $\mu_r \neq 0$ for $r = 1, 2, 3$.

We use the following notational convention: $H_{rs} := H(e_r, e_s)$, $G_{rs} := G(e_r, e_s)$ and the range of indices: r, s, t, $u = 1, \dots, n$; $\alpha, \beta = 4, \dots, n$.

By assumption, we have

$$H_{rs} = H_{sr} = \mu_r \delta_{rs}, \quad G_{rs} = G_{sr}, \tag{4.1}$$

$$H_{11} = \mu_1 \neq 0, \quad H_{22} = \mu_2 \neq 0, \quad H_{33} = \mu_3 \neq 0, \tag{4.2}$$

$$G_{ru}H_{st} - G_{rt}H_{su} + H_{ru}G_{st} - H_{rt}G_{su} = 0. \tag{4.3}$$

For $r = u = 1$, $s = t = 2$, from (4.1) and (4.3), we have

$$G_{11}H_{22} + H_{11}G_{22} = 0. \tag{4.4}$$

Similarly, from (4.1) and (4.3), we have also

$$G_{11}H_{33} + H_{11}G_{33} = 0, \quad G_{22}H_{33} + H_{22}G_{33} = 0. \tag{4.5}$$

From (4.5), we have

$$H_{33}(G_{11}H_{22} - H_{11}G_{22}) = 0.$$

Thus, from (4.2), we have

$$G_{11}H_{22} - H_{11}G_{22} = 0. \tag{4.6}$$

From (4.2), (4.4) $\sim$ (4.6), we have

$$G_{11} = 0, \quad G_{22} = 0, \quad G_{33} = 0. \tag{4.7}$$

For $r = u = 3$, $s = 2$, $t = 1$, from (4.1) $\sim$ (4.3), we have

$$G_{21} = 0. \tag{4.8}$$

Similarly, from (4.1) $\sim$ (4.3), we have also

$$G_{31} = 0, \quad G_{32} = 0, \quad G_{\alpha 1} = 0, \quad G_{\alpha 2} = 0, \quad G_{\alpha 3} = 0, \quad G_{\alpha\beta} = 0, \tag{4.9}$$

for $\alpha \neq \beta$. For $r = u = 1$, $s = t = \alpha$, from (4.1) $\sim$ (4.3) and (4.7), we have

$$G_{\alpha\alpha} = 0. \tag{4.10}$$

Hence, from (4.1) and (4.7) $\sim$ (4.10), we see $G = 0$. □

Let M be an oriented hypersurface in $\tilde{M} = (\tilde{M}^{n+1}(\tilde{c}), \langle\, , \rangle)$. Let $V = T_pM (p \in M)$ and $K (\subset \otimes^4 V^*)$ be the space of curvature-like tensors (for the definition, see [2]). We define the map $F_k : \tilde{S}^2 V^* \times (V^* \otimes S^2 V^*) \times \cdots \times (\otimes^k V^* \otimes S^2 V^*) \to K \times (V^* \otimes K) \times \cdots \times (\otimes^k V^* \otimes K)$ by

$$F_k(H, \nabla H, \dots, \nabla^k H) = (f_0, f_1, \dots, f_k), \tag{4.11}$$

$$(f_0(H))(x, y, z, w) := H(x, w)H(y, z) - H(x, z)H(y, w), \tag{4.12}$$

$$\begin{aligned}(f_i(H, \nabla H, \dots, \nabla^i H))&(v_i, \dots, v_1; x, y, z, w) \\ := \{(\nabla^i H)&(v_i, \dots, v_1; x, w)H(y, z) - (\nabla^i H)(v_i, \dots, v_1; x, z)H(y, w)\} \\ &+ \sum_{j=1}^{i} \Bigg[\sum_{1 \le a_1 < \cdots < a_j \le i} \\ &\{(\nabla^{i-j} H)(v_i, \dots, \hat{v}_{a_j}, \dots, \hat{v}_{a_1}, \dots, v_1; x, w)(\nabla^j H)(v_{a_j}, \dots, v_{a_1}; y, z) \\ &- (\nabla^{i-j} H)(v_i, \dots, \hat{v}_{a_j}, \dots, \hat{v}_{a_1}, \dots, v_1; x, z)(\nabla^j H)(v_{a_j}, \dots, v_{a_1}; y, w)\}\Bigg]\end{aligned} \tag{4.13}$$

for $i \ge 1$ (cf. [2]). We can easily check that F_k is well-defined. Then, by Lemma 4.1, we have the following

LEMMA 4.2. *For each integer $k \ge 0$, the map F_k is injective.*

PROOF. The injectivity of F_0 is easy to see by the similar argument of the classical rigidity theorem (cf. [8, Chapter VII, Theorem 6.2 and Corollary 6.3], [20, Proposition 2.2]).

We shall prove the injectivity of $F_1, \dots, F_k$ by induction.

(I) We suppose that $F_1(H, \nabla H) = F_1(\overline{H}, \nabla \overline{H})$. By injectivity of $F_0 (= f_0)$, we have $H = \overline{H}$. Therefore, from (4.13) and hypothesis, we have

$$\begin{aligned}(\nabla H - \nabla \overline{H})(v_1; x, w)H(y, z) - (\nabla H - \nabla \overline{H})(v_1; x, z)H(y, w) \\ + H(x, w)(\nabla H - \nabla \overline{H})(v_1; y, z) - H(x, z)(\nabla H - \nabla \overline{H})(v_1; y, w) = 0.\end{aligned} \tag{4.14}$$

From (4.14), by Lemma 4.1, we have $\nabla H = \nabla \overline{H}$. Hence, we see that F_1 is injective.

(II) We assume that F_i is injective. We suppose that $F_{i+1}(H, \nabla H, \dots, \nabla^{i+1} H) = F_{i+1}(\overline{H}, \nabla \overline{H}, \dots, \nabla^{i+1} \overline{H})$. By the inductive assumption, we have $H = \overline{H}$, $\nabla H = \nabla \overline{H}, \dots, \nabla^i H = \nabla^i \overline{H}$. Therefore, from (4.13) and hypothesis, we have

$$\begin{aligned}(\nabla^{i+1} H - \nabla^{i+1} \overline{H})&(v_{i+1}, \dots, v_1; x, w)H(y, z) \\ - (\nabla^{i+1} H - \nabla^{i+1} \overline{H})&(v_{i+1}, \dots, v_1; x, z)H(y, w)\end{aligned} \tag{4.15}$$

$$+ H(x,w)(\nabla^{i+1}H - \nabla^{i+1}\overline{H})(v_{i+1},\dots,v_1;y,z)$$
$$- H(x,z)(\nabla^{i+1}H - \nabla^{i+1}\overline{H})(v_{i+1},\dots,v_1;y,w) = 0.$$

From (4.15), by Lemma 4.1, we have $\nabla^{i+1}H = \nabla^{i+1}\overline{H}$. Hence, we see that F_{i+1} is injective.

From (I) and (II), we may conclude that Lemma 4.2 follows. □

By Lemma 4.2, we have the following

LEMMA 4.3. *Let M be an oriented hypersurface in $\tilde{M} = (\tilde{M}^{n+1}(\tilde{c}), \langle\, , \,\rangle)$. If the rank of the second fundamental tensor field H is not less than 3 on M, then the two conditions $R(k)$ and $H(k)$ is equivalent for each integer $k \geq 0$.*

PROOF. From (2.6) and (2.10), we may easily see that $H(k)$ implies $R(k)$. Therefore, it is sufficient to show that $R(k)$ implies $H(k)$. We assume that $R(k)$ holds, i.e., for each p, $q \in M$, there exists a linear isometry Φ of T_pM onto T_qM such that $\Phi(\nabla^i R)_p = (\nabla^i R)_q$ for $i = 0, 1, \dots, k$. Taking account of (2.6), we define the $(0,4)$-type tensor field T on M by

$$T(X,Y,Z,W) = \langle (R - \tilde{c}R_0)(X,Y)Z, W\rangle. \tag{4.16}$$

Since $\Phi(R_p) = R_q$, we have $\Phi(T_p) = T_q$. Therefore, from (2.6), (4.12) and the hypothesis, we have

$$\begin{aligned}(f_0(\Phi H_p))(x,y,z,w) &= (\Phi(T_p))(x,y,z,w)\\ &= (T_q)(x,y,z,w) = (f_0(H_q))(x,y,z,w)\end{aligned} \tag{4.17}$$

for x, y, z, $w \in T_qM$. From (2.10), (4.13) and the hypothesis, we have

$$\begin{aligned}&(f_i(\Phi H_p, \Phi(\nabla H)_p, \dots, \Phi(\nabla^i H)_p))(v_i,\dots,v_1;x,y,z,w)\\ &= \langle (\Phi(\nabla^i R)_p)(v_i,\dots,v_1;x,y)z, w\rangle\\ &= \langle ((\nabla^i R)_q)(v_i,\dots,v_1;x,y)z, w\rangle\\ &= (f_i(H_q, (\nabla H)_q, \dots, (\nabla^i H)_q))(v_i,\dots,v_1;x,y,z,w)\end{aligned} \tag{4.18}$$

for x, y, z, w, $v_i \in T_qM$ and $i = 1, \dots, k$. From (4.17) and (4.18), we have

$$F_k(\Phi H_p, \Phi(\nabla H)_p, \dots, \Phi(\nabla^k H)_p) = F_k(H_q, (\nabla H)_q, \dots, (\nabla^k H)_q).$$

By Lemma 4.2, we have $\Phi(\nabla^i H)_p = (\nabla^i H)_q$ for $i = 0, 1, \dots, k$. Hence, we see that $H(k)$ holds. □

In particular, for an oriented isoparametric hypersurface in S^{n+1}, we have the following

LEMMA 4.4. *Let M be an oriented isoparametric hypersurface in S^{n+1}. The two conditions $R(k)$ and $H(k)$ are equivalent for each integer $k(\geq 0)$.*

PROOF. From (2.6) and (2.10), we may easily see that $H(k)$ implies $R(k)$. Therefore, it suffices to show that $R(k)$ implies $H(k)$. If $n=2$, then, taking account of (2.7), we may easily observe that $\nabla H=0$. Therefore, we have immediately Lemma 4.4 in this case. If $\operatorname{rank} H \geq 3$ on M, it reduces to Lemma 4.3. Thus, we shall show that Lemma 4.4 is also valid for the remaining case where $n \geq 3$ and $\operatorname{rank} H < 3$ on M.

We assume that $n \geq 3$ and $\operatorname{rank} H < 3$ on M. Then, taking account of Remark 3.3 and Proposition 3.2 (ii), one of the following may occur:

(1) $g=1$ and $\lambda_1=0$ ($\operatorname{rank} H=0$)

(2) $g=3$, $\lambda_1=\sqrt{3}$, $\lambda_2=0$, $\lambda_3=-\sqrt{3}$ and $n=3$ ($\operatorname{rank} H=2$)

In the case of (1), M is totally geodesic in S^{n+1}. Hence, we see that M satisfies $H(k)$ for any k.

In the case of (2), for any point $p \in M$, we may choose an orthonormal basis $\{e_r(p)\}$ $(r=1,2,3)$ of T_pM such that $Ae_r(p)=\lambda_r e_r(p)$. Then, for each points p, $q \in M$, we may define a linear isometry $\Phi: T_pM \to T_qM$ by $\Phi(e_r(p))=e_r(q)$. We use the following notational convention:

$$R_{rstu}(p) := \langle R_p(e_r(p), e_s(p))e_t(p), e_u(p)\rangle,$$

$$(\nabla R)_{lrstu}(p) := \langle (\nabla R)_p(e_l(p); e_r(p), e_s(p))e_t(p), e_u(p)\rangle,$$

$$\cdots\cdots$$

$$(\nabla^i R)_{l_i\cdots l_1 rstu}(p) := \langle (\nabla^i R)_p(e_{l_i}(p), \ldots, e_{l_1}(p); e_r(p), e_s(p))e_t(p), e_u(p)\rangle;$$

$$H_{st}(p) := H_p(e_s(p), e_t(p)),$$

$$(\nabla H)_{lst}(p) := (\nabla H)_p(e_l(p); e_s(p), e_t(p)),$$

$$\cdots\cdots$$

$$(\nabla^i H)_{l_i\cdots l_1 st}(p) := (\nabla^i H)_p(e_{l_i}(p), \ldots, e_{l_1}(p); e_s(p), e_t(p))$$

and so on, where the latin indices (except i and k) run over the range 1, 2, 3. By the definition of Φ, we easily see

$$\begin{aligned} (\Phi(\nabla^i R)_p)_{l_i\cdots l_1 rstu}(q) &= (\nabla^i R)_{l_i\cdots l_1 rstu}(p), \\ (\Phi(\nabla^i H)_p)_{l_i\cdots l_1 st}(q) &= (\nabla^i H)_{l_i\cdots l_1 st}(p). \end{aligned} \tag{4.19}$$

Therefore, we see immediately that

(i) $\Phi(\nabla^i R)_p = (\nabla^i R)_q$ if and only if $(\nabla^i R)_{l_i \cdots l_1 rstu}(p) = (\nabla^i R)_{l_i \cdots l_1 rstu}(q)$,

(ii) $\Phi(\nabla^i H)_p = (\nabla^i H)_q$ if and only if $(\nabla^i H)_{l_i \cdots l_1 st}(p) = (\nabla^i H)_{l_i \cdots l_1 st}(q)$,

for each $i \geq 0$.

Since M is isoparametric, M always satisfies $H(0)$. We shall prove by induction that $R(k)$ implies $H(k)$ for $k \geq 1$.

(I) We assume that M satisfies $R(1)$. Since M is minimal by Remark 3.3 (ii), we have

$$H_{11} + H_{22} + H_{33} = 0,$$

$$(\nabla H)_{l11} + (\nabla H)_{l22} + (\nabla H)_{l33} = 0. \tag{4.20}$$

From (2.10), we have

$$\begin{aligned}(\nabla R)_{lrstu} &= (\nabla H)_{lru} H_{st} - (\nabla H)_{lrt} H_{su} + H_{ru}(\nabla H)_{lst} - H_{rt}(\nabla H)_{lsu} \\ &= \lambda_s \delta_{st}(\nabla H)_{lru} - \lambda_s \delta_{su}(\nabla H)_{lrt} + \lambda_r \delta_{ru}(\nabla H)_{lst} - \lambda_r \delta_{rt}(\nabla H)_{lsu}.\end{aligned} \tag{4.21}$$

Since $\lambda_1 \neq 0$, from (4.21), (i) and hypothesis, we have

$$\begin{aligned}(\nabla H)_{l23}(p) &= (\nabla H)_{l23}(q), \\ (\nabla H)_{233}(p) &= (\nabla H)_{233}(q), \\ (\nabla H)_{322}(p) &= (\nabla H)_{322}(q).\end{aligned} \tag{4.22}$$

Since $\lambda_3 \neq 0$, from (4.21), (i) and hypothesis, we have

$$\begin{aligned}(\nabla H)_{l12}(p) &= (\nabla H)_{l12}(q), \\ (\nabla H)_{211}(p) &= (\nabla H)_{211}(q), \\ (\nabla H)_{122}(p) &= (\nabla H)_{122}(q).\end{aligned} \tag{4.23}$$

From (4.20), (4.22) and (4.23), we have

$$(\nabla H)_{222}(p) = (\nabla H)_{222}(q). \tag{4.24}$$

From (4.21), we have also

$$(\nabla R)_{13131} = -\sqrt{3}(\nabla H)_{133} + \sqrt{3}(\nabla H)_{111},$$

$$(\nabla R)_{31313} = \sqrt{3}(\nabla H)_{311} - \sqrt{3}(\nabla H)_{333}.$$

Hence, by the hypothesis, we have

$$
\begin{aligned}
&(\nabla H)_{133}(p) - (\nabla H)_{111}(p) = (\nabla H)_{133}(q) - (\nabla H)_{111}(q), \\
&(\nabla H)_{311}(p) - (\nabla H)_{333}(p) = (\nabla H)_{311}(q) - (\nabla H)_{333}(q).
\end{aligned} \tag{4.25}
$$

On one hand, by (4.20), (4.22) and (4.23), we have

$$
\begin{aligned}
&(\nabla H)_{111}(p) + (\nabla H)_{133}(p) = (\nabla H)_{111}(q) + (\nabla H)_{133}(q), \\
&(\nabla H)_{311}(p) + (\nabla H)_{333}(p) = (\nabla H)_{311}(q) + (\nabla H)_{333}(q).
\end{aligned} \tag{4.26}
$$

Thus, from (4.25) and (4.26), we have

$$
\begin{aligned}
(\nabla H)_{111}(p) &= (\nabla H)_{111}(q), \\
(\nabla H)_{133}(p) &= (\nabla H)_{133}(q), \\
(\nabla H)_{311}(p) &= (\nabla H)_{311}(q), \\
(\nabla H)_{333}(p) &= (\nabla H)_{333}(q).
\end{aligned} \tag{4.27}
$$

Therefore, from (2.7), (4.22) $\sim$ (4.24), (4.27) and (ii), we see that M satisfies $H(1)$.

(II) We assume that $R(i)$ implies $H(i)$. We suppose that M satisfies $R(i+1)$. From (2.10), (i) and (ii), by inductive assumption, we have

$$
\begin{aligned}
0 &= (\nabla^{i+1}R)_{l_{i+1}\cdots l_1 rstu}(p) - (\nabla^{i+1}R)_{l_{i+1}\cdots l_1 rstu}(q) \\
&= \{(\nabla^{i+1}H)_{l_{i+1}\cdots l_1 ru}(p)H_{st}(p) - (\nabla^{i+1}H)_{l_{i+1}\cdots l_1 rt}(p)H_{su}(p) \\
&\quad + H_{ru}(p)(\nabla^{i+1}H)_{l_{i+1}\cdots l_1 st}(p) - H_{rt}(p)(\nabla^{i+1}H)_{l_{i+1}\cdots l_1 su}(p)\} \\
&\quad - \{(\nabla^{i+1}H)_{l_{i+1}\cdots l_1 ru}(q)H_{st}(q) - (\nabla^{i+1}H)_{l_{i+1}\cdots l_1 rt}(q)H_{su}(q) \\
&\quad + H_{ru}(q)(\nabla^{i+1}H)_{l_{i+1}\cdots l_1 st}(q) - H_{rt}(q)(\nabla^{i+1}H)_{l_{i+1}\cdots l_1 su}(q)\} \\
&= \{\mu_s\delta_{st}(\nabla^{i+1}H)_{l_{i+1}\cdots l_1 ru}(p) - \mu_s\delta_{su}(\nabla^{i+1}H)_{l_{i+1}\cdots l_1 rt}(p) \\
&\quad + \mu_r\delta_{ru}(\nabla^{i+1}H)_{l_{i+1}\cdots l_1 st}(p) - \mu_r\delta_{rt}(\nabla^{i+1}H)_{l_{i+1}\cdots l_1 su}(p)\} \\
&\quad - \{\mu_s\delta_{st}(\nabla^{i+1}H)_{l_{i+1}\cdots l_1 ru}(q) - \mu_s\delta_{su}(\nabla^{i+1}H)_{l_{i+1}\cdots l_1 rt}(q) \\
&\quad + \mu_r\delta_{ru}(\nabla^{i+1}H)_{l_{i+1}\cdots l_1 st}(q) - \mu_r\delta_{rt}(\nabla^{i+1}H)_{l_{i+1}\cdots l_1 su}(q)\}.
\end{aligned} \tag{4.28}
$$

Thus, by the similar procedure as in (I), we may easily show that M satisfies $H(i+1)$.

From (I) and (II), by the induction, we can conclude that $R(k)$ implies $H(k)$ for each $k \geq 0$. This completes the proof of Lemma 4.4. $\square$

From the above Lemma 4.4, taking account of Theorem 3.4 and Proposition 3.1, we have the following

LEMMA 4.5. *Let M be an oriented closed isoparametric hypersurface in S^{n+1}. If M satisfies the condition $R(4)$, then M is homogeneous.*

In general, a curvature homogeneous hypersurface in a real space form is not necessarily an isoparametric one. Concerning this, K. Tsukada proved the following ([20, Theorem B])

THEOREM 4.6. *Let M be an $n(\geq 4)$-dimensional curvature homogeneous space and ψ an isometric immersion of M into S^{n+1}. Then one of the following may occur:*

(i) *M is a Riemannian manifold of constant curvature* 1.

(ii) *The immersion ψ has constant principal curvatures.*

REMARK 4.7. In the case (i) of the above Theorem 4.6, if the hypersurface M is complete, by the result [11, Theorem 2], we see that M is totally geodesic (and hence, M is isoparametric).

If $n \geq 4$, we see immediately that Theorem A follows from Lemma 4.5 and Theorem 4.6 with Remark 4.7. Further, we can easily show that a 3-dimensional oriented closed hypersurface in S^{n+1} satisfying the condition $R(1)$ is homogeneous by taking account of the results of K. Sekigawa ([15, §2]) and E. Cartan ([3] and [4]). Therefore, we have finally Theorem A.

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