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REPRESENTATIONS OF A LINK GROUP

By

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In this paper we shall consider the link L illustrated in the Figure 1, and construct representations of the link group of L.

The link group $\pi(L)$ has the following presentations:

$$\pi(L) \simeq \langle x_0, x_1, x_2, x_3, x_4, x_5 | [x_i, x_{i-1}x_{i+1}^{-1}] = 1,\ i = 0, 1, \dots 5 \text{ (mod. 6)}\rangle$$

$$\simeq \langle x_0, x_1, x_2, x_3, x_4, x_5, y_0, y_1, y_2, y_3, y_4, y_5 |$$

$$[x_i, y_i] = 1,\ x_{i-1}x_{i+1}^{-1}y_i^{-1} = 1,\ i = 0, 1, \dots, 5 \text{ (mod. 6)}\rangle,$$

where $[x, y] = xyx^{-1}y^{-1}$. Note that, in the last presentation, it holds that $y_0y_2y_4 = 1$ and $y_1y_3y_5 = 1$.

Now, representations of $\pi(L)$ to $\mathrm{PSL}(2, \boldsymbol{C})$ can be constructed using the following theorem. We set $c(z) = (z + 1/z)/2$ and $s(z) = (z - 1/z)/2$.

THEOREM. Let λ_0, λ_2, λ_4, μ_0, μ_2, μ_4 be complex numbers not equal to $0, \pm 1$. Suppose that we can take complex numbers λ_1, λ_3, λ_5, μ_1, μ_3, μ_5 not equal to $0, \pm 1$, satisfying the following conditions (i) and (ii): for $i = 1, 3, 5$ (mod. 6)

(i) $c(\lambda_i) = c(\mu_{i-1})c(\mu_{i+1})$
$+\{c(\lambda_{i-1})c(\lambda_{i+1}) - c(\lambda_{i+3})\}s(\mu_{i-1})s(\mu_{i+1})/\{s(\lambda_{i-1})s(\lambda_{i+1})\}$,

(ii) $s(\lambda_i)c(\mu_i)/s(\mu_i) = -c(\lambda_{i-1})c(\mu_{i+1})s(\mu_{i-1})/s(\lambda_{i-1})$
$-c(\mu_{i-1})c(\mu_{i+1})s(\mu_{i+1})/s(\lambda_{i+1})$
$-c(\mu_{i+3})s(\lambda_{i+3})s(\mu_{i-1})s(\mu_{i+1})/\{s(\mu_{i+3})s(\mu_{i-1})s(\mu_{i+1})\}$.

Then, we can take $A_i \in \mathrm{SL}(2, \boldsymbol{C})$ $(i = 0, 1, \dots, 5 \text{ (mod.6)})$ to construct a non-abelian representation of $\pi(L)$ to $\mathrm{PSL}(2, \boldsymbol{C})$ by corresponding

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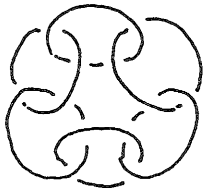

Figure 1

$$y_i \to A_i \begin{pmatrix} \lambda_i & 0 \\ 0 & 1/\lambda_i \end{pmatrix} A_i^{-1},$$

$$x_i \to A_i \begin{pmatrix} \mu_i & 0 \\ 0 & 1/\mu_i \end{pmatrix} A_i^{-1},$$

$i = 0, 1, \ldots, 5$ (mod. 6).

REMARK. The conditions (i) & (ii) are equivalent to the following (iii) & (iv), for (i)+(ii) makes (iii) and (i)−(ii) makes (iv):

$$\text{(iii)} \quad \frac{s(\mu_i/\lambda_i)}{s(\mu_i)} = \frac{s(\mu_{i+1}/\lambda_{i+1})s(\mu_{i-1}/\lambda_{i-1})}{s(\lambda_{i+1})s(\lambda_{i-1})} - \frac{s(\lambda_{i+3}\mu_{i+3})s(\mu_{i+1})s(\mu_{i-1})}{s(\mu_{i+3})s(\lambda_{i+1})s(\lambda_{i-1})},$$

$$\text{(iv)} \quad \frac{s(\lambda_i\mu_i)}{s(\mu_i)} = \frac{s(\lambda_{i+1}\mu_{i+1})s(\lambda_{i-1}\mu_{i-1})}{s(\lambda_{i+1})s(\lambda_{i-1})} - \frac{s(\mu_{i+3}/\lambda_{i+3})s(\mu_{i+1})s(\mu_{i-1})}{s(\mu_{i+3})s(\lambda_{i+1})s(\lambda_{i-1})}.$$

REMARK 2. For $i = 1, 3, 5$ (mod. 6), $c(\lambda_i)$ is determined by (i). Let the value of the right-hand side of (i) be a, then λ_i is determined by the equation $\lambda_i + 1/\lambda_i = 2a$, or the quadratic equation $\lambda_i^2 - 2a\lambda_i + 1 = 0$. In some special cases the two roots of the equation $x^2 - 2ax + 1 = 0$ may be among 0, ± 1. Except these cases $\lambda_i (\neq 0, \pm 1)$ exists.

Suppose that λ_i exists. Then μ_i is determined by (ii). Let the value of the right-hand side of (ii) be b. Then μ_i is determined by the equation $(\mu_i + 1/\mu_i)/(\mu_i - 1/\mu_i) = b/s(\lambda_i)$, so

$$\mu_i = \pm\sqrt{-(s(\lambda_i) + b/(s(\lambda_i) - b)}.$$

In some special cases these values are among 0, ± 1. Except these cases μ_i $(\neq 0, \pm 1)$ exists.

REMARK 3. In the theorem we have assumed that $\lambda_i \neq \pm 1$. But in the case $\lambda_i = \pm 1$ (parabolic case), where $0 \leq i \leq 5$, if we modify (i), (ii) by replacing $s(\mu_i)/s(\lambda_i)$ by a_i and by replacing $y_i \to A_i\begin{pmatrix}\lambda_i & 0\\ 0 & 1/\lambda_i\end{pmatrix}A_i^{-1}$ by $y_i \to A_i\begin{pmatrix}\pm 1 & 1\\ 0 & \pm 1\end{pmatrix}A_i^{-1}$ and by replacing $x_i \to A_i\begin{pmatrix}\mu_i & 0\\ 0 & 1/\lambda_i\end{pmatrix}A_i^{-1}$ by $x_i \to A_i\begin{pmatrix}\pm 1 & a_i\\ 0 & \pm 1\end{pmatrix}A_i^{-1}$. Then we also have a representation.

REMARK 4. By using this theorem, I would like to construct a bi-rational representation (i.e. a homorphism into the group of bi-rational transformations of a certain algebraic variety) of the mapping class group of the closed orientable surface of genus 2. This will be carried out in the subsequent paper.

Roughly speaking, it follows from this theorem that the space of the non-equivalent representations of $\pi(L)$ to $\mathrm{PSL}(2, C)$ has the complex dimension at least 6.

PROOF OF THE THEOREM. For $i = 0, 2, 4 \pmod{6}$, let

$$U_i = \begin{pmatrix} u_{i1} & u_{i2}\\ u_{i3} & u_{i4}\end{pmatrix},$$

where

$$u_{i1} = \frac{\lambda_{i+2} - \lambda_i\lambda_{i-2}}{\lambda_{i+2}(\lambda_{i-2}^2 - 1)}, \quad u_{i2} = \lambda_0\lambda_2\lambda_4 - 1,$$

$$u_{i3} = \frac{\lambda_i - \lambda_{i+2}\lambda_{i-2}}{\lambda_i(\lambda_{i+2}^2 - 1)(\lambda_{i-2}^2 - 1)}, \quad u_{i4} = \frac{\lambda_{i+2}(\lambda_{i-2} - \lambda_i\lambda_{i+2})}{\lambda_i(\lambda_{i+2}^2 - 1)}.$$

Then, it is easy to check that $U_0U_2U_4 = E$ and

$$\begin{pmatrix}\lambda_0 & 0\\ 0 & 1/\lambda_0\end{pmatrix}U_4\begin{pmatrix}\lambda_2 & 0\\ 0 & 1/\lambda_2\end{pmatrix}U_0\begin{pmatrix}\lambda_4 & 0\\ 0 & 1/\lambda_4\end{pmatrix}U_2 = E,$$

where

$$E = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}.$$

Moreover it holds that, for $i = 0, 2, 4 \pmod{6}$,

$$u_{i1}u_{i4} = \frac{c(\lambda_i) - c(\lambda_{i+2})c(\lambda_{i-2}) + s(\lambda_{i+2})s(\lambda_{i-2})}{2s(\lambda_{i+2})s(\lambda_{i-2})} \tag{1}$$

and

$$u_{i2}u_{i3} = \frac{c(\lambda_i) - c(\lambda_{i+2})c(\lambda_{i-2}) - s(\lambda_{i+2})s(\lambda_{i-2})}{2s(\lambda_{i+2})s(\lambda_{i-2})}.$$

So, by putting $A_0 = E$, $A_2 = U_4$, $A_4 = U_2^{-1}$, we have, for $i = 0, 2, 4$ (mod. 6), $U_i = A_{i+2}^{-1}A_{i-2}$, and

$$\left\{A_0\begin{pmatrix}\lambda_0 & 0 \\ 0 & 1/\lambda_0\end{pmatrix}A_0^{-1}\right\}\left\{A_2\begin{pmatrix}\lambda_2 & 0 \\ 0 & 1/\lambda_2\end{pmatrix}A_2^{-1}\right\}\left\{A_4\begin{pmatrix}\lambda_4 & 0 \\ 0 & 1/\lambda_4\end{pmatrix}A_4^{-1}\right\} = E.$$

For $i = 0, 2, 4$ (mod. 6), let

$$Y_i = A_i\begin{pmatrix}\lambda_i & 0 \\ 0 & 1/\lambda_i\end{pmatrix}A_i^{-1}, \quad X_i = A_i\begin{pmatrix}\mu_i & 0 \\ 0 & 1/\mu_i\end{pmatrix}A_i^{-1}.$$

Then, we have $Y_0Y_2Y_4 = E$, which corresponds the relator $y_0y_2y_4 = 1$.

Next we compute the trace of $X_{i-1}X_{i+1}^{-1}$, for $i = 1, 3, 5$ (mod. 6).

$$\begin{aligned}
&\mathrm{Tr}(X_{i-1}X_{i+1}^{-1}) \\
&= \mathrm{Tr}\left(A_{i-1}\begin{pmatrix}\mu_{i-1} & 0 \\ 0 & 1/\mu_{i-1}\end{pmatrix}A_{i-1}^{-1}A_{i+1}\begin{pmatrix}1/\mu_{i+1} & 0 \\ 0 & \mu_{i+1}\end{pmatrix}A_{i+1}^{-1}\right) \\
&= \mathrm{Tr}\left(\begin{pmatrix}\mu_{i-1} & 0 \\ 0 & 1/\mu_{i-1}\end{pmatrix}A_{i-1}^{-1}A_{i+1}\begin{pmatrix}1/\mu_{i+1} & 0 \\ 0 & \mu_{i+1}\end{pmatrix}A_{i+1}^{-1}A_{i-1}\right) \\
&= \mathrm{Tr}\left(\begin{pmatrix}\mu_{i-1} & 0 \\ 0 & 1/\mu_{i-1}\end{pmatrix}U_{i+3}\begin{pmatrix}1/\mu_{i+1} & 0 \\ 0 & 1/\mu_{i+1}\end{pmatrix}U_{i+3}^{-1}\right) \\
&= \mu_{i-1}(u_{i+3,1}u_{i+3,4}/\mu_{i+1} - \mu_{i+1}u_{i+3,2}u_{i+3,3}) \\
&\quad + (\mu_{i+1}u_{i+3,1}u_{i+3,4} - u_{i+3,2}u_{i+3,3}/\mu_{i+1})/\mu_{i-1} \\
&= \left(\frac{\mu_{i-1}}{\mu_{i+1}} + \frac{\mu_{i+1}}{\mu_{i-1}}\right)u_{i+3,1}u_{i+3,4} - \left(\mu_{i-1}\mu_{i+1} + \frac{1}{\mu_{i-1}\mu_{i+1}}\right)u_{i+3,2}u_{i+3,3} \\
&= \left(\frac{\mu_{i-1}}{\mu_{i+1}} + \frac{\mu_{i+1}}{\mu_{i-1}}\right)\frac{c(\lambda_{i+3}) - c(\lambda_{i+1})c(\lambda_{i-1}) + s(\lambda_{i+1})s(\lambda_{i-1})}{2s(\lambda_{i+1})s(\lambda_{i-1})} \\
&\quad - \left(\mu_{i-1}\mu_{i+1} + \frac{1}{\mu_{i-1}\mu_{i+1}}\right)\frac{c(\lambda_{i+3}) - c(\lambda_{i+1})c(\lambda_{i-1}) - s(\lambda_{i+1})s(\lambda_{i-1})}{2s(\lambda_{i+1})s(\lambda_{i-1})}.
\end{aligned}$$

by (1). Now,

$$\frac{\mu_{i-1}}{\mu_{i+1}}+\frac{\mu_{i+1}}{\mu_{i-1}}=2(c(\mu_{i-1})c(\mu_{i+1})-s(\mu_{i-1})s(\mu_{i+1})),$$

$$\mu_{i-1}\mu_{i+1}+\frac{1}{\mu_{i-1}\mu_{i+1}}=2(c(\mu_{i-1})c(\mu_{i+1})+s(\mu_{i-1})s(\mu_{i+1})).$$

So,

$$\begin{aligned}
&\mathrm{Tr}(X_{i-1}X_{i+1}^{-1})\\
&\quad=2(c(\mu_{i-1})c(\mu_{i+1})-s(\mu_{i-1})s(\mu_{i+1}))\\
&\qquad\cdot(c(\lambda_{i+3})-c(\lambda_{i+1})c(\lambda_{i-1})+s(\lambda_{i+1})s(\lambda_{i-1}))/(2s(\lambda_{i+1})s(\lambda_{i-1}))\\
&\qquad-2(c(\mu_{i-1})c(\mu_{i+1})+s(\mu_{i-1})s(\mu_{i+1}))\\
&\qquad\cdot(c(\lambda_{i+3})-c(\lambda_{i+1})c(\lambda_{i-1})-s(\lambda_{i+1})s(\lambda_{i-1}))/(2s(\lambda_{i+1})s(\lambda_{i-1}))\\
&\quad=2(c(\mu_{i-1})c(\mu_{i+1})+(c(\lambda_{i-1})c(\lambda_{i+1})-c(\lambda_{i+3}))s(\mu_{i-1})s(\mu_{i+1}))/(s(\lambda_{i-1})s(\lambda_{i+1}))\\
&\quad=2c(\lambda_i).
\end{aligned}$$

Here we have used the hypothesis of the theorem. Thus,

$$\mathrm{Tr}(X_{i-1}X_{i+1}^{-1})=\lambda_i+\frac{1}{\lambda_i}.$$

$X_{i-1}X_{i+1}^{-1}\in \mathrm{SL}(2,\boldsymbol{C})$ and $\lambda_i\neq\pm 1$. So, there exists an $A_i\in \mathrm{SL}(2,\boldsymbol{C})$ such that

$$X_{i-1}X_{i+1}^{-1}=A_i\begin{pmatrix}\lambda_i & 0\\ 0 & 1/\lambda_i\end{pmatrix}A_i^{-1}.$$

Now we put, for $i=1,3,5$ (mod. 6),

$$Y_i=A_i\begin{pmatrix}\lambda_i & 0\\ 0 & 1/\lambda_i\end{pmatrix}A_i^{-1},\quad X_i=A_i\begin{pmatrix}\mu_i & 0\\ 0 & 1/\mu_i\end{pmatrix}A_i^{-1}.$$

Then, we have $Y_i=X_{i-1}X_{i+1}^{-1}$, which corresponds to the relator $y_i=x_{i-1}x_{i+1}^{-1}$, for $i=1,3,5$ (mod. 6).

Obviously, $[X_i,Y_i]=E$, for $i=0,1,\dots,5$ (mod. 6). It remains to prove that $Y_i\sim X_{i-1}X_{i+1}^{-1}$, for $i=0,2,4$ (mod. 6), where $A\sim B$ means that $A=\xi B$, for some scalar $\xi\neq 0$.

Now $Y_i \sim X_{i-1}X_{i+1}^{-1}$ is equivalent to

$$A_i\begin{pmatrix}\lambda_i & 0\\ 0 & 1/\lambda_i\end{pmatrix}A_i^{-1} \sim \left\{A_{i-1}\begin{pmatrix}\mu_{i-1} & 0\\ 0 & 1/\mu_{i-1}\end{pmatrix}A_{i-1}^{-1}\right\}\left\{A_{i+1}\begin{pmatrix}1/\mu_{i+1} & 0\\ 0 & \mu_{i+1}\end{pmatrix}A_{i+1}^{-1}\right\}$$

or

$$\begin{pmatrix}\lambda_i & 0\\ 0 & 1/\lambda_i\end{pmatrix}V_{i+1}\begin{pmatrix}\mu_{i+1} & 0\\ 0 & 1/\mu_{i+1}\end{pmatrix}V_{i+1}^{-1} \sim V_i^{-1}\begin{pmatrix}\mu_{i-1} & 0\\ 0 & 1/\mu_{i-1}\end{pmatrix}V_i, \tag{2}$$

where $V_i = A_{i-1}^{-1}A_i$.

Now, for $i = 1, 3, 5$ (mod. 6),

$$X_{i-1}X_{i+1}^{-1} = A_i\begin{pmatrix}\lambda_i & 0\\ 0 & 1/\lambda_i\end{pmatrix}A_i^{-1}. \tag{3}$$

So,

$$A_{i-1}^{-1}X_{i-1}X_{i+1}^{-1}A_{i-1} = A_{i-1}^{-1}A_i\begin{pmatrix}\lambda_i & 0\\ 0 & 1/\lambda_i\end{pmatrix}A_i^{-1}A_{i-1} = V_i\begin{pmatrix}\lambda_i & 0\\ 0 & 1/\lambda_i\end{pmatrix}A_i^{-1}.$$

Also,

$$A_{i-1}^{-1}X_{i-1}X_{i+1}^{-1}A_{i-1} = \begin{pmatrix}\mu_{i-1} & 0\\ 0 & 1/\mu_{i-1}\end{pmatrix}U_{i+3}\begin{pmatrix}1/\mu_{i+1} & 0\\ 0 & \mu_{i+1}\end{pmatrix}U_{i+3}^{-1}.$$

Hence we have

$$V_i\begin{pmatrix}\lambda_i & 0\\ 0 & 1/\lambda_i\end{pmatrix}V_i^{-1} = \begin{pmatrix}\mu_{i-1} & 0\\ 0 & 1/\mu_{i-1}\end{pmatrix}U_{i+3}\begin{pmatrix}1/\mu_{i+1} & 0\\ 0 & \mu_{i+1}\end{pmatrix}U_{i+3}^{-1}.$$

Let

$$V_i = \begin{pmatrix}v_{i1} & v_{i2}\\ v_{i3} & v_{i4}\end{pmatrix}.$$

Then,

$$V_i\begin{pmatrix}\lambda_i & 0\\ 0 & 1/\lambda_i\end{pmatrix}V_i^{-1} = \begin{pmatrix}\lambda_i v_{i1}v_{i4} - v_{i2}v_{i3}/\lambda_i & (1/\lambda_i - \lambda_i)v_{i1}v_{i2}\\ (\lambda_i - 1/\lambda_i)v_{i3}v_{i4} & v_{i1}v_{i4}/\lambda_i - \lambda_i v_{i2}v_{i3}\end{pmatrix}.$$

$$\begin{pmatrix}\mu_{i-1} & 0\\ 0 & 1/\mu_{i-1}\end{pmatrix}U_{i+3}\begin{pmatrix}1/\mu_{i+1} & 0\\ 0 & \mu_{i+1}\end{pmatrix}U_{i+3}^{-1}$$

$$= \begin{pmatrix}\mu_{i-1}(u_{i+3,1}u_{i+3,4}/\mu_{i+1} - u_{i+3,2}u_{i+3,3}\mu_{i+1}) & \mu_{i-1}(\mu_{i+1} - 1/\mu_{i+1})u_{i+3,1}u_{i+3,2}\\ (1/\mu_{i+1} - \mu_{i+1})u_{i+3,3}u_{i+3,4}/\mu_{i-1} & (\mu_{i+1}u_{i+3,1}u_{i+3,4} - u_{i+3,2}u_{i+3,3}/\mu_{i+1})/\mu_{i-1}\end{pmatrix}$$

Hence

$$v_{i1}v_{i2} = -\frac{s(\mu_{i+1})\mu_{i-1}(\lambda_{i-1} - \lambda_{i+3}\lambda_{i+1})(\lambda_0\lambda_2\lambda_4 - 1)}{s(\lambda_i)\lambda_{i-1}(\lambda_{i+1}^2 - 1)},$$

$$v_{i3}v_{i4} = -\frac{s(\mu_{i+1})\lambda_{i-1}(\lambda_{i+3} - \lambda_{i-1}\lambda_{i+1})(\lambda_{i+1} - \lambda_{i-1}\lambda_{i+3})}{\mu_{i-1}s(\lambda_i)\lambda_{i+3}^2(\lambda_{i-1}^2 - 1)^2(\lambda_{i+1}^2 - 1)},$$

$$v_{i1}v_{i4} + v_{i2}v_{i3} = \frac{c(\mu_{i-1})s(\mu_{i+1})\{c(\lambda_{i+1})c(\lambda_{i-1}) - c(\lambda_{i+3})\}}{s(\lambda_i)s(\lambda_{i+1})s(\lambda_{i-1})} + \frac{s(\mu_{i-1})c(\mu_{i+1})}{s(\lambda_i)},$$

for $i = 1, 3, 5$ (mod. 6). Hence, for $i = 0, 2, 4$ (mod. 6),

$$\begin{aligned} v_{i+1,1}v_{i+1,2} &= -\frac{s(\mu_{i+2})\mu_i(\lambda_i - \lambda_{i-2}\lambda_{i+2})(\lambda_0\lambda_2\lambda_4 - 1)}{s(\lambda_{i+1})\lambda_i(\lambda_{i+2}^2 - 1)}, \\ v_{i+1,3}v_{i+1,4} &= -\frac{s(\mu_{i+2})\lambda_i(\lambda_{i-2} - \lambda_i\lambda_{i+2})(\lambda_{i+2} - \lambda_i\lambda_{i-2})}{\mu_i s(\lambda_{i+1})\lambda_{i-2}^2(\lambda_i^2 - 1)(\lambda_{i+2}^2 - 1)}, \\ v_{i+1,1}v_{i+1,4} + v_{i+1,2}v_{i+1,3} &= \frac{c(\mu_i)s(\mu_{i+2})\{c(\lambda_{i+2})c(\lambda_i) - c(\lambda_{i-2})\}}{s(\lambda_{i+1})s(\lambda_{i+2})s(\lambda_i)} + \frac{s(\mu_i)c(\mu_{i+2})}{s(\lambda_{i+1})} \end{aligned} \tag{4}$$

Next, for $i = 1, 3, 5$ (mod. 6), by (3),

$$\begin{aligned} A_{i+1}^{-1}X_{i-1}X_{i+1}^{-1}A_{i+1} &= A_{i+1}^{-1}A_i\begin{pmatrix} \lambda_i & 0 \\ 0 & 1/\lambda_i \end{pmatrix}A_i^{-1}A_{i+1} \\ &= V_{i+1}^{-1}\begin{pmatrix} \lambda_i & 0 \\ 0 & 1/\lambda_i \end{pmatrix}V_{i+1}. \end{aligned}$$

Also,

$$\begin{aligned} A_{i+1}^{-1}X_{i-1}X_{i+1}^{-1}A_{i+1} &= A_{i+1}^{-1}A_{i-1}\begin{pmatrix} \mu_{i-1} & 0 \\ 0 & 1/\mu_{i-1} \end{pmatrix}A_{i-1}^{-1}A_{i+1}\begin{pmatrix} 1/\mu_{i+1} & 0 \\ 0 & \mu_{i+1} \end{pmatrix} \\ &= U_{i+3}^{-1}\begin{pmatrix} \mu_{i-1} & 0 \\ 0 & 1/\mu_{i-1} \end{pmatrix}U_{i+3}\begin{pmatrix} 1/\mu_{i+1} & 0 \\ 0 & \mu_{i+1} \end{pmatrix} \end{aligned}$$

Hence we have

$$V_{i+1}^{-1}\begin{pmatrix} \lambda_i & 0 \\ 0 & 1/\lambda_i \end{pmatrix}V_{i+1} = U_{i+3}^{-1}\begin{pmatrix} \mu_{i-1} & 0 \\ 0 & 1/\mu_{i-1} \end{pmatrix}U_{i+3}\begin{pmatrix} 1/\mu_{i+1} & 0 \\ 0 & \mu_{i+1} \end{pmatrix}.$$

Hence by (1),

$$v_{i+1,2}v_{i+1,4}=\frac{s(\mu_{i-1})\mu_{i+1}\lambda_{i-1}(\lambda_{i+1}-\lambda_{i-1}\lambda_{i+3})(\lambda_0\lambda_2\lambda_4-1)}{s(\lambda_i)\lambda_{i+3}(\lambda_{i-1}^2-1)},$$

$$v_{i+1,1}v_{i+1,3}=\frac{s(\mu_{i-1})(\lambda_{i-1}-\lambda_{i+3}\lambda_{i+1})(\lambda_{i+3}-\lambda_{i-1}\lambda_{i+1})}{s(\lambda_i)\mu_{i+1}\lambda_{i-1}\lambda_{i+3}(\lambda_{i-1}^2-1)(\lambda_{i+1}^2-1)^2},$$

$$v_{i+1,1}v_{i+1,4}+v_{i+1,2}v_{i+1,3}=\frac{c(\mu_{i+1})s(\mu_{i-1})\{c(\lambda_{i+3})-c(\lambda_{i+1})c(\lambda_{i-1})\}}{s(\lambda_i)s(\lambda_{i+1})s(\lambda_{i-1})}-\frac{s(\mu_{i+1})c(\mu_{i-1})}{s(\lambda_i)},$$

for $i=1,3,5$ (mod. 6). Hence, for $i=0,2,4$ (mod. 6), we have

$$v_{i2}v_{i4}=\frac{s(\mu_{i-2})\mu_i\lambda_{i-2}(\lambda_i-\lambda_{i-2}\lambda_{i+2})(\lambda_0\lambda_2\lambda_4-1)}{s(\lambda_{i-1})\lambda_{i+2}(\lambda_{i-2}^2-1)},$$

$$v_{i1}v_{i3}=\frac{s(\mu_{i-2})(\lambda_{i-2}-\lambda_{i+2}\lambda_i)(\lambda_{i+2}-\lambda_{i-2}-\lambda_i)}{s(\lambda_{i-1})\mu_i\lambda_{i-2}\lambda_{i+2}(\lambda_{i-2}^2-1)(\lambda_i^2-1)^2}, \qquad (5)$$

$$v_{i1}v_{i4}+v_{i2}v_{i3}=\frac{c(\mu_i)s(\mu_{i-2})\{c(\lambda_{i+2})-c(\lambda_i)c(\lambda_{i-2})\}}{s(\lambda_{i-1})s(\lambda_i)s(\lambda_{i-2})}-\frac{s(\mu_i)c(\mu_{i-2})}{s(\lambda_{i-1})}.$$

Now we prove (2). Let

$$\begin{pmatrix}\lambda_i & 0\\ 0 & 1/\lambda_i\end{pmatrix}V_{i+1}\begin{pmatrix}\mu_{i+1} & 0\\ 0 & 1/\mu_{i+1}\end{pmatrix}v_{i+1}^{-1}=\begin{pmatrix}a_1 & b_1\\ c_1 & d_1\end{pmatrix},$$

and

$$V_i^{-1}\begin{pmatrix}\mu_{i-1} & 0\\ 0 & 1/\mu_{i-1}\end{pmatrix}v_i=\begin{pmatrix}a_2 & b_2\\ c_2 & d_2\end{pmatrix},$$

($i=0,2,4$ mod. 6) and we prove that

$$\begin{pmatrix}a_1 & b_1\\ c_1 & d_1\end{pmatrix}\sim\begin{pmatrix}a_2 & b_2\\ c_2 & d_2\end{pmatrix}.$$

Now,

$$a_1=\lambda_i(v_{i+1,1}v_{i+1,4}\mu_{i+1}-v_{i+1,2}v_{i+1,3}/\mu_{i+1}),$$

$$b_1=\lambda_i(1/\mu_{i+1}-\mu_{i+1})v_{i+1,1}v_{i+1,2},$$

$$c_1=(\mu_{i+1}^{-1}/\mu_{i+1})v_{i+1,3}v_{i+1,4}/\lambda_i,$$

$$d_1 = (v_{i+1,1}v_{i+1,4}/\mu_{i+1} - v_{i+1,2}v_{i+1,3}\mu_{i+1})/\lambda_i,$$

$$a_2 = v_{i1}v_{i4}\mu_{i-1} - v_{i2}v_{i3}/\mu_{i-1},$$

$$b_2 = \mu_{i-1} - 1/\mu_{i-1})v_{i2}v_{i4},$$

$$c_2 = (1/\mu_{i-1} - \mu_{i-1})v_{i1}v_{i3},$$

$$d_2 = v_{i1}v_{i4}/\mu_{i-1} - v_{i2}v_{i3}\mu_{i-1}.$$

By (4) and (5),

$$\begin{aligned} b_1 &= \lambda_1(-2s(\mu_{i+1}))\frac{-s(\mu_{i+2})\mu_i(\lambda_i - \lambda_{i-2}\lambda_{i+2})(\lambda_0\lambda_2\lambda_4 - 1)}{s(\lambda_{i+1})\lambda_i(\lambda_{i+2}^2 - 1)} \\ &= \frac{s(\mu_{i+1})s(\mu_{i+2})}{s(\lambda_{i+1})s(\lambda_{i+2})} \cdot \frac{\mu_i(\lambda_i - \lambda_{i-2}\lambda_{i+2})(\lambda_0\lambda_2\lambda_4 - 1)}{\lambda_{i+2}}, \\ b_2 &= 2s(\mu_{i-1})\frac{s(\mu_{i-2})\mu_i\lambda_{i-2}(\lambda_i - \lambda_{i-2}\lambda_{i+2})(\lambda_0\lambda_2\lambda_4 - 1)}{s(\lambda_{i-1})\lambda_{i+2}(\lambda_{i-2}^2 - 1)} \\ &= \frac{s(\mu_{i-1})s(\mu_{i-2})}{s(\lambda_{i-1})s(\lambda_{i-2})} \cdot \frac{\mu_i(\lambda_i - \lambda_{i-2}\lambda_{i+2})(\lambda_0\lambda_2\lambda_4 - 1)}{\lambda_{i+2}}. \end{aligned}$$

Let

$$b_3 = \mu_i(\lambda_i - \lambda_{i-2}\lambda_{i+2})(\lambda_0\lambda_2\lambda_4 - 1)/\lambda_{i+2}.$$

Then,

$$b_1 = \frac{s(\mu_{i+1})s(\mu_{i+2})}{s(\lambda_{i+1})s(\lambda_{i+2})}\, b_3, \quad b_2 = \frac{s(\mu_{i-1})s(\mu_{i-2})}{s(\lambda_{i-1})s(\lambda_{i-2})}\, b_3. \tag{6}$$

Similarly, if we put

$$c_3 = -\frac{(\lambda_{i-2} - \lambda_i\lambda_{i+2})(\lambda_{i+2} - \lambda_i\lambda_{i-2})}{\mu_i\lambda_{i-2}^2\lambda_{i+2}(\lambda_i^2 - 1)^2},$$

then, we have

$$c_1 = \frac{s(\mu_{i+1})s(\mu_{i+2})}{s(\lambda_{i+1})s(\lambda_{i+2})}\, c_3, \quad c_2 = \frac{s(\mu_{i-1})s(\mu_{i-2})}{s(\lambda_{i-1})s(\lambda_{i-2})}\, c_3. \tag{7}$$

Next we compute a_1 and a_2.

$$
\begin{aligned}
a_1 &= \lambda_i(v_{i+1,1}v_{i+1,4}\mu_{i+1} - v_{i+1,2}v_{i+1,3}/\mu_{i+1}) \\
&= \lambda_i\{(v_{i+1,1}v_{i+1,4} + v_{i+1,2}v_{i+1,3})s(\mu_{i+1}) + (v_{i+1,1}v_{i+1,4} - v_{i+1,2}v_{i+1,3})c(\mu_{i+1})\} \\
&= \lambda_i\{(v_{i+1,1}v_{i+1,4} + v_{i+1,2}v_{i+1,3})s(\mu_{i+1}) + c(\mu_{i+1})\} \\
&= \lambda_i s(\mu_{i+1})\{(v_{i+1,1}v_{i+1,4} + v_{i+1,2}v_{i+1,3}) + c(\mu_{i+1})/s(\mu_{i+1})\} \\
&= \lambda_i s(\mu_{i+1})\left\{\frac{c(\mu_i)s(\mu_{i+2})\{c(\lambda_{i+2})c(\lambda_i) - c(\lambda_{i-2})\}}{s(\lambda_{i+1})s(\lambda_{i+2})s(\lambda_i)} + \frac{s(\mu_i)c(\mu_{i+2})}{s(\lambda_{i+1})} + \frac{c(\mu_{i+1})}{s(\mu_{i+1})}\right\} \\
&= \lambda_i\frac{s(\mu_{i+1})}{s(\lambda_{i+1})}\left\{\frac{c(\mu_{i+1})s(\mu_{i+2})\{c(\lambda_{i+2})c(\lambda_i) - c(\lambda_{i-2})\}}{s(\lambda_{1+2})s(\lambda_i)}\right. \\
&\qquad \left. + s(\mu_i)c(\mu_{i+2}) + \frac{s(\lambda_{i+1})c(\mu_{i+1})}{s(\mu_{i+1})}\right\}.
\end{aligned}
$$

Here we have used (4).

Now by the hypothesis of the theorm,

$$
\begin{aligned}
& s(\lambda_{i+1})c(\mu_{i+1})/s(\mu_{i+1}) \\
&\quad = -s(\mu_i)c(\lambda_i)c(\mu_{i+2}) - s(\mu_{i+2})c(\mu_i)c(\lambda_{i+2})/s(\lambda_{i+2}) \\
&\qquad - s(\lambda_{i-2})s(\mu_i)s(\mu_{i+2})c(\mu_{i-2})/\{s(\mu_{i-2})s(\lambda_i)s(\lambda_{i+2})\}.
\end{aligned}
$$

So,

$$
\begin{aligned}
a_1 &= \lambda_i\frac{s(\mu_{i+1})}{s(\lambda_{i+1})}\left\{\frac{c(\mu_i)s(\mu_{i+2})\{c(\lambda_{i+2})c(\lambda_i) - c(\lambda_{i-2})\}}{s(\lambda_{i+2})s(\lambda_i)}\right. \\
&\qquad + s(\mu_i)c(\mu_{i+2}) - \frac{s(\mu_i)}{s(\lambda_i)}\,c(\lambda_i)c(\mu_{i+2}) - \frac{s(\mu_{i+2})}{s(\lambda_{i+2})}\,c(\mu_i)c(\lambda_{i+2}) \\
&\qquad \left. - \frac{s(\lambda_{i-2})s(\mu_i)s(\mu_{i+2})}{s(\mu_{i-2})s(\lambda_i)s(\lambda_{i+2})}\,c(\mu_{i-2})\right\} \\
&= \frac{s(\mu_{i+1})s(\mu_{i+2})\lambda_i}{s(\lambda_{i+1})s(\lambda_{i+2})s(\lambda_i)}\left\{c(\mu_i)c(\lambda_{i+2})c(\lambda_i) - c(\mu_i)c(\lambda_{i-2})\right. \\
&\qquad + s(\mu_i)\frac{c(\mu_{i+2})s(\lambda_{i+2})s(\lambda_i)}{s(\mu_{i+2})} - \frac{s(\mu_i)c(\lambda_i)c(\mu_{i+2})s(\lambda_{i+2})}{s(\mu_{i+2})} \\
&\qquad \left. - c(\mu_i)c(\lambda_{i+2})s(\lambda_i) - \frac{s(\lambda_{i-2})s(\mu_i)c(\mu_{i-2})}{s(\mu_{i-2})}\right\}
\end{aligned}
$$

$$= \frac{s(\mu_{i+1})s(\mu_{i+2})\lambda_i}{s(\lambda_{i+1})s(\lambda_{i+2})s(\lambda_i)}\left\{c(\mu_i)c(\lambda_{i+2})\{c(\lambda_i) - s(\lambda_i)\}\right.$$

$$+\frac{s(\mu_i)c(\mu_{i+2})s(\lambda_{i+2})\{s(\lambda_i) - c(\lambda_i)\}}{s(\mu_{i+2})} - c(\mu_i)c(\lambda_{i-2})$$

$$\left.-\frac{s(\lambda_{i-2})s(\mu_i)c(\mu_{i-2})}{s(\mu_{i-2})}\right\}$$

$$= \frac{s(\mu_{i+1})s(\mu_{i+2})\lambda_i}{s(\lambda_{i+1})s(\lambda_{i+2})s(\lambda_i)}\left\{\frac{c(\mu_i)c(\lambda_{i+2})}{\lambda_i} - \frac{s(\mu_i)c(\mu_{i+2})s(\lambda_{i+2})}{s(\mu_{i+2})\lambda_i}\right.$$

$$\left.- c(\mu_i)c(\lambda_{i-2}) - \frac{s(\lambda_{i-2})s(\mu_i)c(\mu_{i-2})}{s(\mu_{i-2})}\right\}$$

$$= \frac{s(\mu_{i+1})s(\mu_{i+2})}{s(\lambda_{i+1})s(\lambda_{i+2})s(\lambda_i)}\left\{c(\mu_i)c(\lambda_{i+2}) - \frac{s(\mu_i)c(\mu_{i+2})s(\lambda_{i+2})}{s(\mu_{i+2})}\right.$$

$$\left.- \lambda_i c(\mu_i)c(\lambda_{i-2}) - \frac{\lambda_i s(\lambda_{i-2})s(\mu_i)c(\mu_{i-2})}{s(\mu_{i-2})}\right\}.$$

So, if we put

$$a_3 = \frac{1}{s(\lambda_i)}\left\{c(\mu_i)c(\lambda_{i+2}) - \frac{s(\mu_i)c(\mu_{i+2})s(\lambda_{i+2})}{s(\mu_{i+2})}\right.$$

$$\left.- \lambda_i c(\mu_i)c(\lambda_{i-2}) - \frac{\lambda_i s(\lambda_{i-2})s(\mu_i)c(\mu_{i-2})}{s(\mu_{i-2})}\right\},$$

then, we have

$$a_1 = \frac{s(\mu_{i+1})s(\mu_{i+2})}{s(\lambda_{i+1})s(\lambda_{i+2})}\, a_3. \tag{8}$$

Also,

$$a_2 = v_{i1}v_{i4}\mu_{i-1} - v_{i2}v_{i3}/\mu_{i-1}$$

$$= (v_{i1}v_{i4} + v_{i2}v_{i3})s(\mu_{i-1}) + (v_{i1}v_{i4} - v_{i2}v_{i3})c(\mu_{i-1})$$

$$= (v_{i1}v_{i4} + v_{i2}v_{i3})s(\mu_{i-1}) + c(\mu_{i-1})$$

$$= s(\mu_{i-1})\{(v_{i1}v_{i4} + v_{i2}v_{i3}) + c(\mu_{i-1})/s(\mu_{i-1})\}$$

$$= s(\mu_{i-1})\left\{\frac{c(\mu_i)s(\mu_{i-2})\{c(\lambda_{i+2}) - c(\lambda_i)c(\lambda_{i-2})\}}{s(\lambda_{i-1})s(\lambda_i)s(\lambda_{i-2})} - \frac{s(\mu_i)c(\mu_{i-1})}{s(\lambda_{i-1})} + \frac{c(\mu_{i-1})}{s(\mu_{i-1})}\right\}$$

$$= \frac{s(\mu_{i-1})}{s(\lambda_{i-1})}\left\{\frac{c(\mu_i)s(\mu_{i-2})\{c(\lambda_{i+2}) - c(\lambda_i)c(\lambda_{i-2})\}}{s(\lambda_i)s(\lambda_{i-2})} - s(\mu_i)c(\mu_{i-2}) + \frac{c(\mu_{i-1})s(\lambda_{i-1})}{s(\mu_{i-1})}\right\}.$$

Here we have used (5). Now, by the hypothesis of the theorem,

$$\frac{c(\mu_{i-1})s(\lambda_{i-1})}{s(\mu_{i-1})} = -\frac{s(\mu_{i-2})}{s(\lambda_{i-2})}c(\lambda_{i-2})c(\mu_i) - \frac{s(\mu_i)}{s(\lambda_i)}c(\mu_{i-2})c(\lambda_i)$$
$$- \frac{s(\lambda_{i+2})s(\mu_{i-2})c(\mu_i)}{s(\mu_{i+2})s(\lambda_{i-2})s(\lambda_i)}c(\mu_{i+2}).$$

Hence,

$$a_2 = \frac{s(\mu_{i-1})}{s(\lambda_{i-1})}\left\{\frac{c(\mu_i)s(\mu_{i-2})\{c(\lambda_{i+2}) - c(\lambda_i)c(\lambda_{i-2})\}}{s(\lambda_i)s(\lambda_{i-2})}\right.$$
$$- s(\mu_i)c(\mu_{i-2}) - \frac{s(\mu_{i-2})}{s(\lambda_{i-2})}c(\lambda_{i-2})c(\mu_i) - \frac{s(\mu_i)}{s(\lambda_i)}c(\mu_{i-2})c(\lambda_i)$$
$$\left.- \frac{s(\lambda_{i+2})s(\mu_{i-2})s(\mu_i)}{s(\mu_{i+2})s(\lambda_{i-2})s(\lambda_i)}c(\mu_{i+2})\right\}$$

$$= \frac{s(\mu_{i-1})s(\mu_{i-2})}{s(\lambda_{i-1})s(\lambda_{i-2})s(\lambda_i)}\left\{c(\mu_i)c(\lambda_{i+2}) - c(\mu_i)c(\lambda_i)c(\lambda_{i-2})\right.$$
$$- \frac{s(\mu_i)c(\mu_{i-2})s(\lambda_{i-2})s(\lambda_i)}{s(\mu_{i-2})} - s(\lambda_i)c(\lambda_{i-2})c(\mu_i)$$
$$\left.- \frac{s(\mu_i)s(\lambda_{i-2})}{s(\mu_{i-2})}c(\mu_{i-2})c(\lambda_i) - \frac{s(\lambda_{i+2})s(\mu_i)c(\mu_{i+2})}{s(\mu_{i+2})}\right\}$$

$$= \frac{s(\mu_{i-1})s(\mu_{i-2})}{s(\lambda_{i-1})s(\lambda_{i-2})s(\lambda_i)}\left\{c(\mu_i)c(\lambda_{i+2}) - c(\mu_i)c(\lambda_{i-2})\{s(\lambda_i) + c(\lambda_i)\}\right.$$
$$\left.- \frac{c(\mu_{i-2})s(\lambda_{i-2})s(\mu_i)\{s(\lambda_i) + c(\lambda_i)\}}{s(\mu_{i-2})} - s(\lambda_i)c(\lambda_{i-2})c(\mu_i) - \frac{s(\lambda_{i+2})s(\mu_i)c(\mu_{i+2})}{s(\mu_{i+2})}\right\}$$

$$= \frac{s(\mu_{i-1})s(\mu_{i-2})}{s(\lambda_{i-1})s(\lambda_{i-2})s(\lambda_i)}\left\{c(\mu_i)c(\lambda_{i+2}) - \frac{s(\mu_i)c(\mu_{i+2})s(\lambda_{i+2})}{s(\mu_{i+2})}\right.$$
$$\left.- \lambda_i c(\mu_i)c(\lambda_{i-2}) - \frac{\lambda_i s(\lambda_{i-2})s(\mu_i)c(\mu_{i-2})}{s(\mu_{i-2})}\right\}.$$

Hence

$$a_2 = \frac{s(\mu_{i-1})s(\mu_{i-2})}{s(\lambda_{i-1})s(\lambda_{i-2})}\, a_3. \tag{9}$$

Similarly, if we put

$$d_3 = -\frac{c(\mu_i)c(\lambda_{i+2})s(\mu_{i+2}) + s(\mu_i)s(\lambda_{i+2})c(\mu_{i+2})}{s(\lambda_i)s(\mu_{i+2})} + \frac{c(\mu_i)c(\lambda_{i-2})s(\mu_{i-2}) - s(\mu_i)s(\lambda_{i-2})c(\mu_{i-2})}{\lambda_i s(\lambda_i)s(\mu_{i-2})},$$

we have

$$d_1 = \frac{s(\mu_{i+1})s(\mu_{i+2})}{s(\lambda_{i+1})s(\lambda_{i+2})}\, d_3, \quad d_2 = \frac{s(\mu_{i-1})s(\mu_{i-2})}{s(\lambda_{i-1})s(\lambda_{i-2})}\, d_3. \tag{10}$$

By (6), (7), (8), (9), (10),

$$\begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} = \frac{s(\mu_{i+1})s(\mu_{i+2})}{s(\lambda_{i+1})s(\lambda_{i+2})} \begin{pmatrix} a_3 & b_3 \\ c_3 & d_3 \end{pmatrix}$$

and

$$\begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix} = \frac{s(\mu_{i-1})s(\mu_{i-2})}{s(\lambda_{i-1})s(\lambda_{i-2})} \begin{pmatrix} a_3 & b_3 \\ c_3 & d_3 \end{pmatrix}.$$

Hence

$$\begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} \sim \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix}.$$

Thus, (2) is proved. (Q.E.D)