

HAAR NONMEASURABLE PARTITIONS OF COMPACT GROUPS

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Abstract. We prove that a compact connected nonmetrizable group contains a proper dense ω -bounded subgroup. This is then used to show that every compact nonmetrizable group G can be partitioned into $|G|$ -many pairwise disjoint dense topologically homogeneous ω -bounded subsets each of cardinal $|G|$ and each Haar nonmeasurable with full Haar outermeasure. This allows us to then generalize an observation of Kakutani and Oxtoby and to conclude that each infinite compact group G may be partitioned into a collection of $|G|$ -many pairwise disjoint dense subsets of full Haar outermeasure. A corollary of these results is that the Stone-Cech compactification βX of an infinite discrete space X may be partitioned into $|\beta X|$ pairwise disjoint ω -bounded subsets each of size $|\beta X|$.

1. Introduction

In [16] Kakutani and Oxtoby (see also [8, 16.7]) showed that each compact infinite metric group G contains a family $\{X_\alpha : \alpha \in A\}$ of subsets, where the cardinal number of A , $|A| = c = \text{continuum}$, the subsets X_α are pairwise disjoint, and $\mu^*(X_\alpha) = 1$ where μ^* is Haar outermeasure of G . As we shall see in Section 2 this implies that the $c = |G| = |A|$ disjoint subsets are each dense in G , and by

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adjoining $G \setminus \bigcup_{\alpha} X_{\alpha}$ to one of the X_{β} , $\beta \in A$, we see that each compact infinite metric group may be partitioned into $|G|$ dense pairwise disjoint nonmeasurable subsets of full Haar outermeasure. It is the purpose of this paper to extend this fact to all compact groups thus showing that all infinite compact groups G may be partitioned into $|G|$ dense nonmeasurable subsets of Haar outermeasure 1. As we shall see, in the case of nonmetrizable groups, surprisingly this fact may be strengthened so that in addition the sets of the partition are each homogeneously invariant ω -bounded subsets (see Theorem 7).

In recent years a number of authors (see [2]–[4]) have investigated the existence of dense proper pseudocompact subgroups in compact nonmetrizable groups. Recently we have shown that if $2^{\omega} < 2^{\omega_1}$ then every nonmetrizable compact group contains a proper dense countably compact subgroup. It should be noted that the method of proof employed showed that every compact group of cardinal greater than the continuum contained such a proper subgroup. This should be contrasted with an old result of Rajagopalan and Subrahmanian [19] in which it is shown that in the category of locally compact groups there are groups which admit no proper dense subgroups. In this paper we will show that the cardinality restriction may be dropped for nonmetrizable compact connected groups. In fact we will show that each compact connected nonmetrizable group contains a dense ω -bounded subgroup (this parallels the result in [4] for compact nonmetrizable Abelian groups) and as a corollary we obtain our Theorem 7 which implies our main theorem.

THEOREM 7. *Let G be a nonmetrizable compact group. Then G can be partitioned into $|G|$ -many pairwise disjoint dense homogeneously invariant ω -bounded subsets each of cardinal $|G|$ and each Haar nonmeasurable with full Haar outermeasure.*

All groups will be T_0 and therefore completely regular, while all topological spaces will be assumed completely regular. A pseudocompact topological space is a space on which every continuous real valued function is bounded. A countable compact space X is one on which every infinite subset contains an accumulation point in X , and an ω -bounded space Y is one on which every countable subset has compact closure in Y . It is well known that X ω -bounded $\Rightarrow X$ countably compact $\Rightarrow X$ pseudocompact. We use $|X|$ to denote the cardinal number of the set X , and we use $w(X)$ to denote the weight of X , which is the smallest cardinal number for the size of a base for the open sets in

X . We use ω to denote the first infinite cardinal, ω_1 to denote the first uncountable cardinal and 0 as the neutral element of our compact groups.

We make use of the following fact found in Engelking [6, 3.7.2]. If $f : X \rightarrow Y$ is a perfect map, then for each compact subspace $Z \subset Y$, $f^{-1}(Z)$ is compact. (A map $f : X \rightarrow Y$ is perfect if f is continuous, X is Hausdorff, f is closed, and all fibers $f^{-1}(y)$ are compact.). Note that if f is a continuous surjective map and if X is compact then trivially f is perfect. Immediate consequences of this fact are:

FACT 1. Let f be a continuous surjective map of the compact set X onto Y . Then $S \subset Y$ is ω -bounded iff $U = f^{-1}(S)$ is ω -bounded.

FACT 2. The continuous image of an ω -bounded set is ω -bounded.

FACT 3. Let $X_i, i \in I$, be a collection of ω -bounded Hausdorff spaces, then $X = \prod_{i \in I} X_i$ is ω -bounded.

2. Topological Homogeneous Spaces.

We now show that the collection of cosets of a topological group G is a collection of topologically homogeneous invariant subspaces of the group. In fact, the stronger condition, that given a coset A of the group and point $x \neq y$ in A there is a homeomorphism f of the group for which A is an invariant set under f and $f(x) = y$, holds. From this we conclude that if X is a topological space homeomorphic to a compact topological group G with a proper dense ω -bounded subgroup H then X can be partitioned into a collection of dense pairwise disjoint homogeneously invariant ω -bounded subsets.

DEFINITION. A topological space X is *topologically homogeneous* if for each pair of points $x \neq y$ in X there is a homeomorphism f of X onto itself such that $f(x) = y$. A subspace A of a topologically homogeneous space X is *homogeneously invariant* if given any $x \neq y$ in A there is a homeomorphism f of X onto itself such that $f(x) = y$ and $f(A) = A$.

LEMMA 1. Let G be a topological group and let A be a coset of a subgroup H of G . Then A is homogeneously invariant

PROOF. First let $A = H$ and let $x \neq y$ be points of H . Note that the map f defined by $f(z) = yx^{-1}z$ satisfies $f(x) = y$. Since translations and inversions are

homeomorphisms of G , f is a homeomorphism. Since H is a group, if $z \in H$ then $f(z) \in H$, so that $f(H) \subset H$. f is onto because if $w \in H$ the point $z = xy^{-1}w \in H$ satisfies $f(z) = w$. Thus H is invariant under f .

Now let $A = aH$ so that $a^{-1}A = H$. Let $x \neq y$ be in A . Then $a^{-1}x \neq a^{-1}y$ are in H . By the previous argument, there is a homeomorphism g of G such that H is an invariant set for g and $g(a^{-1}x) = a^{-1}y$. Let $\tau_a(x) = ax$, $x \in G$. Then let $f = \tau_a \circ g \circ \tau_{a^{-1}}$. It is easy to check that f is the desired homeomorphism. A similar argument holds when $A = Ha$. \square

COROLLARY 1. *If G is a compact nonmetric group and if H is a dense proper ω -bounded subgroup of G then each coset of H is a dense proper ω -bounded homogeneously invariant subset of G .*

In [11] it was noted that in a compact group G , a subset of G has left Haar outermeasure one iff the subset meets every closed G_δ in G of positive Haar measure. A proof of this fact was given in [13] since Comfort in his 1984 survey article [2] noted that no proof of this fact had been published. It is immediate from this fact that in a compact group G a subset of outermeasure 1 must be a dense subset of G (If $U \subset G$ is open it has positive Haar measure and so by [8, 19.30] it contains a Baire set D with the same measure as U . By [13, Th.1], D meets every set in G of outermeasure 1 and so does U).

It is an observation of A. Todd that if μ^* is left Haar outermeasure on a compact group G and if A and B are disjoint subsets of G such that $\mu^*(A) + \mu^*(B) > 1$ then both A and B are nonmeasurable. [If A is measurable then $1 = \mu(G) = \mu(A) + \mu(A^C) = \mu^*(A) + \mu^*(A^C) \geq \mu^*(A) + \mu^*(B) > 1$, a contradiction. Similarly, if B is measurable.]. In their classic paper, Comfort and Ross [5] showed that a dense subgroup of a compact group G is pseudocompact iff it meets every closed G_δ in G . Putting these facts together, we see that every dense proper pseudocompact subgroup of a compact group G (which now has outermeasure 1 in G) and hence every proper ω -bounded dense subgroup and its cosets in a compact group must have Haar outermeasure 1 and be nonmeasurable. Thus we have the following theorem.

THEOREM 1. *Suppose a compact nonmetrizable group G contains a proper dense ω -bounded subgroup H . Then the collection of cosets of H partitions G into a collection of pairwise disjoint homogeneously invariant dense ω -bounded Haar nonmeasurable subsets each of full outermeasure.*

LEMMA 2. *Any compact topological space X homeomorphic to a compact topological group G with a dense ω -bounded proper subgroup H can be partitioned into a collection Ψ of pairwise disjoint proper dense homogeneously invariant ω -bounded subsets. The cardinal of Ψ is equal to the number of cosets of H .*

PROOF. Let $h : X \rightarrow G$ be the homeomorphism. If A is an ω -bounded coset of H then $A^* = h^{-1}(A)$ is an ω -bounded dense subset of X . Furthermore, by Lemma 1 if x and y are distinct elements of A^* there is a homeomorphism g of G onto itself such that $g(A) = A$ and $g(h(x)) = h(y)$. Let $f = h^{-1} \circ g \circ h$, then it is easy to see that f is a homeomorphism of X into itself which satisfies $f(A^*) = A^*$ and $f(x) = y$. It now follows from Theorem 1 that the collection of sets $\{h^{-1}(aH)\}$ is the desired partition. \square

In [15, Lemma 5] the folklore result is noted that a dense subset (not subgroup!) of a compact group that meets every closed G_δ in the group is pseudocompact. It then follows from Gillman and Jerison [7, 6I], and the fact that every compactification of a topological space X is a continuous image of βX , that a dense pseudocompact subset of a compact group G meets every G_δ in the group. Thus we obtain the folk theorem that a dense subset of a compact group is pseudocompact iff it meets every closed G_δ in the group. Hence all dense pseudocompact subsets in a compact group have full outermeasure in the group. This allows us to state the following theorem.

THEOREM 2. *If a compact group G can be partitioned into a non-trivial collection of pairwise disjoint (homogeneously invariant) pseudocompact (countably compact, ω -bounded) dense subsets then each of these subsets are Haar nonmeasurable and are of full Haar outermeasure.*

3. Product-like Group Partitions

We will now show that in the category of nonmetrizable compact product groups and product-like groups it is always possible to partition the groups into "large" collections of dense ω -bounded topologically homogeneous subsets. Along the way we will show that compact nonmetrizable connected groups are product-like. We will use this fact to conclude that such groups always contain proper dense ω -bounded subgroups. This fact then enables us to get another characterization of locally compact metric groups.

LEMMA 3. Let $H = \prod_{\alpha \in A} H_\alpha$, where $|A| \geq \omega_1$, and where each H_α is a nontrivial compact metric group. Let $J = \{h = \{h_\alpha\} \in H : |\{\alpha \in A : h_\alpha \neq 0\}| \leq \omega\}$ be the Σ -product in H . Then $|H/J| = |H|$.

PROOF. Without loss of generality we may assume that $A = B \times \omega_1$ with $|B| = |A|$, i.e. that $H = \prod\{H_{b,\alpha} : b \in B, \alpha \in \omega_1\}$ and $J = \{h = \{h_{b,\alpha}\} \in H : |\{(b, \alpha) \in B \times \omega_1 : h_{b,\alpha} \neq 0\}| \leq \omega\}$. Since each $H_{b,\alpha}$ is non-trivial, we can fix $h_{b,\alpha} \in H_{b,\alpha}$ where $h_{b,\alpha} \neq 0$. For every $f \in \{0, 1\}^B$ define $x_f \in H$ by

$$x_f = \begin{cases} h_{b,\alpha} & \text{if } f(b) = 1 \\ 0 & \text{if } f(b) = 0. \end{cases}$$

Let $\pi : H \rightarrow H/J$ be the quotient map. Define the map $\psi : \{0, 1\}^B \rightarrow H/J$ by $\psi(f) = \pi(x_f)$. We claim that ψ is an injection. Indeed, suppose $f, f' \in \{0, 1\}^B$ and that $f \neq f'$. Then $f(b) \neq f'(b)$ for some $b \in B$. Now for every $\alpha \in \omega_1$ one has $x_f(b, \alpha) \neq x_{f'}(b, \alpha)$, which together with the definition of J implies that $\psi(f) = \pi(x_f) \neq \pi(x_{f'}) = \psi(f')$.

Since ψ is an injection, we have $|H/J| \geq 2^{|B|}$. Since each $H_{b,\alpha}$ is a compact metric group, $|H_{b,\alpha}| \leq c$. Therefore $|H| \leq c^{|B \times \omega_1|} = 2^{|B \times \omega_1|} = 2^{|B|} \leq |H/J| \leq |H|$, which implies $|H| = |H/J|$. \square

THEOREM 3. Let $H = \prod_{\alpha \in A} H_\alpha$, where $|A| \geq \omega_1$, and where each H_α is a nontrivial compact metric group. Then there exists a proper dense ω -bounded subgroup $J' \subset H$ such that $|J'| = |H|$ and J' has $|H|$ distinct cosets.

PROOF. Write $A = B \cup D$ where $|A| = |B| = |D|$. Then $H = \prod_{\alpha \in B} H_\alpha \times \prod_{\beta \in D} H_\beta$. Choose $J \in \prod_{\alpha \in B} H_\alpha$ as in Lemma 1 and define $J' = J \times \prod_{\beta \in D} H_\beta$. Then J' is dense in H and ω -bounded, and it is clear that J' has $|H|$ distinct cosets because $|H/J'| = |H|$.

DEFINITION. A compact group G is *product-like* if there is a continuous surjective homomorphism $\psi : G \rightarrow \prod_{i \in I} H_i$, where each H_i is a nontrivial compact metric group and $|I| = w(G)$.

COROLLARY 2. Let G be a nonmetrizable compact group which is product-like. Then G may be partitioned into $|G|$ dense pairwise disjoint topologically homogeneous ω -bounded cosets each of cardinal $|G|$.

PROOF. There is a continuous surjective homomorphism $f : G \rightarrow H = \prod_{i \in I} H_i$ where each H_i is a compact metric group and where $|I| = w(G) > \omega$. By Theorem 3, H can be partitioned into a collection Ψ^* of pairwise disjoint dense ω -bounded cosets where $|\Psi^*| = |G|$ and where each set $A \in \Psi^*$ satisfies $|A| = |H| = |G|$. (Note that $|G| = 2^{w(G)} = |H|$, (see [9, 28.58]).). The collection $\Psi = \{f^{-1}(A) : A \in \Psi^*\}$ is the desired collection of cosets by Fact 1. \square

NOTE 1. In view of Corollary 2 it is important for us to determine those compact groups which are product-like. Comfort and Robertson [4, 4.10 (d)] showed that not all compact groups are product-like. In fact they showed that for every infinite cardinal α , there exists a compact group K_α , with $w(K_\alpha) = \alpha$, which admits no group homomorphism $\Psi : K_\alpha \rightarrow G \times H$ (even a non-continuous one!) onto a product $G \times H$ of two nontrivial groups. However our Theorem 5 shows that the class of compact product-like groups is large.

The authors have shown in [14] that the following theorem is a consequence of the Pontryagin duality theorem for locally compact Abelian groups.

THEOREM 4. *For a compact Abelian group G of uncountable weight α at least one of the following conditions is satisfied:*

(i) *There is a continuous group homomorphism of G onto T^α , where T is the circle.*

(ii) *There exists a sequence $\{p : p \in P\}$ where P is the prime numbers, a corresponding sequence $\{\alpha(p) : p \in P\}$ of cardinal numbers such that $\alpha = \sup\{\alpha(p) : p \in P\}$, and a continuous group homomorphism of G onto the product $\prod\{Z_p^{\alpha(p)} : p \in P\}$, where Z_p denotes the group of integers mod p .*

NOTATION. Let G be a topological group, then $Z(G)$ is the center of G and G_0 is the component of the neutral element 0 of G .

THEOREM 5. *A compact nonmetrizable group G is product like if it is either Abelian or connected. Thus each such group may be partitioned into $|G|$ dense pairwise disjoint homogeneously invariant ω -bounded cosets, each of size $|G|$.*

PROOF. Theorem 4 shows that each compact Abelian group is product-like. Thus we assume that G is connected. It is a corollary of the Pontryagin-Weil Structure Theorem for compact connected groups (see [3]) that the following sandwich structure theorem for compact connected groups holds:

“If G is a compact connected group, there exist continuous surjective homomorphisms ψ and η such that the following holds:

$$A \times \prod_{i \in I} K_i \xrightarrow{\psi} G \xrightarrow{\eta} G/Z(G) \cong K/Z(K) \cong \prod_{i \in I} H_i,$$

where A is a compact connected Abelian group, $\ker \psi$ is totally disconnected, $K = \prod_{i \in I} K_i$ where each K_i is a compact connected simply connected, non-Abelian Lie group with simple Lie algebra and finite center, and each H_i is a compact connected non-Abelian Lie group which is algebraically simple.”

It is easy to see that the index set in the two Cartesian products, $K = \prod_{i \in I} K_i$ and $H = \prod_{i \in I} H_i$, is the same. Thus there are two cases to consider. If $|I| = w(G) \equiv \alpha$ then clearly G is product-like. If $|I| < \alpha$, we have $w(K) = |I| < \alpha = w(G)$. Since the homomorphism $\psi : A \times K \rightarrow G$ is continuous we have $w(J) \leq w(K) < w(G)$ where $J = \psi(K)$. Thus if we show that $w(G/J) = w(G)$ we are done because G/J is a compact connected Abelian group and therefore G/J and hence G is product-like. This fact follows from our next lemma.

LEMMA 4. *Let G be a compact connected group and let ψ, K be as in Theorem 5. If $J = \psi(K)$ and if $w(J) < w(G)$ then $w(G/J) = w(G)$.*

PROOF. It is shown in [3, 4.3] that if $w(Z(G)_0) < w(G)$ then $w(G/Z(G)) = w(G)$. Since $G/Z(G) \cong \prod_{i \in I} H_i$, and $\prod_{i \in I} H_i$ is a continuous surjective homomorphic image of J we must have $w(G/Z(G)) \leq w(J) < w(G)$. Thus $w(Z(G)_0) = w(G)$.

Let $\phi : G \rightarrow G/J$ be the natural map. Then $\phi|_{Z(G)_0}$ is 1-1. To see this, note that if $\phi(a) = \phi(b)$, where $a, b \in Z(G)_0$, then $a = bh$ where $h \in \ker \phi = J$. Thus $b^{-1}a = h \in J$. However $b^{-1}a \in Z(G)_0$, since $Z(G)_0$ is a compact subgroup of G . Since the center of K is totally disconnected so is $\psi(Z(K))$ since ψ is open by [8, 5.29]. Thus $J \cap Z(G)_0 = \{0\}$ so that $b^{-1}a = 0$, i.e. $a = b$. Since $Z(G)_0$ is compact and ϕ is continuous and 1-1 on $Z(G)_0$, $Z(G)_0$ is topologically isomorphic to $\phi(Z(G)_0) \subset G/J$. But then $w(G) = w(Z(G)_0) = w(\phi(Z(G)_0)) \leq w(G/J) \leq w(G)$, so that $w(G/J) = w(G)$. \square

NOTE 2. It was shown in [15] that a connected locally compact group is metrizable iff every element of the group is a metric element. An element x of a group is a metric element if the closure of the group generated by x is a metric group. Now a compact group that is metrizable has no proper pseudocompact

or ω -bounded noncompact subgroups (in a metric space pseudocompactness is equivalent to compactness) and all compact nonmetrizable connected groups have proper dense ω -bounded subgroups by our theorem. Furthermore, the Iwasawa Decomposition Theorem (see [15] for an exact statement of this theorem) contains the statement that each locally compact connected group K is topologically homeomorphic to $R^n \times G$ where G is a maximal compact connected invariant subgroup of K . Thus we see in fact that the following theorem is true.

THEOREM 6. *Let K be a connected locally compact group. Then the following are equivalent:*

- (a) K is a metric group.
- (b) Every element of K is a metric element.
- (c) K contains no proper ω -bounded noncompact subgroups.

4. ω -bounded Partitions

We are now ready to prove our main results. We can now show that each compact nonmetrizable group can be partitioned into a collection of dense homogeneously invariant ω -bounded subsets. We will also derive an application of this fact to general topology. One immediate application is the conclusion that the Stone-Cech compactification βX of each infinite discrete space X may be partitioned into many proper "large" ω -bounded subsets (though not dense).

COROLLARY 3. *Every totally disconnected nonmetrizable compact group G may be partitioned into a collection Ψ of pairwise disjoint homogeneously invariant dense ω -bounded subsets, where $|\Psi| = |G|$, and where each $A \in \Psi$ satisfies $|A| = |G|$.*

PROOF. By [8, 9.15] every such group G is topologically homeomorphic to $H = \{-1, 1\}^{w(G)}$. By Theorem 3, there is a partition Ψ^* of H into pairwise disjoint topologically homogeneous dense ω -bounded cosets of H , where $|\Psi^*| = |H| = |G|$, and where each $A^* \in \Psi^*$ satisfies $|A^*| = |H|$. If $f : G \rightarrow H$ is the homeomorphism then $\Psi = \{f^{-1}(A^*) : A^* \in \Psi^*\}$ is the desired partition of G by Lemma 2. \square

NOTE 3. At this point one might be tempted to use a theorem of Kuzminov [16] which states that every infinite compact group G is dyadic with G a continuous image of $\{0, 1\}^{w(G)}$. While it is true that the continuous image of an

ω -bounded set is an ω -bounded set there is no way of knowing whether such an ω -bounded set in $\{-1, 1\}^{w(G)}$ maps onto all of G or even whether such a continuous map preserves partitions.

THEOREM 7. *Let G be a nonmetrizable compact group. Then G can be partitioned into $|G|$ -many pairwise disjoint dense homogeneously invariant ω -bounded subsets each of cardinal $|G|$ and each Haar nonmeasurable with full Haar outermeasure.*

PROOF. There is an observation based on a result of Mostert [18] in the thesis of J. Cleary [1] that tells us that considered only as topological spaces, G and $G_0 \times G/G_0$ are homeomorphic. Thus one of the following hold: (a) $w(G) = w(G_0)$ and $w(G/G_0) < w(G)$, (b) $w(G) = w(G/G_0)$ and $w(G_0) < w(G)$, or (c) $w(G) = w(G_0) = w(G/G_0)$. Let $f : G \rightarrow G_0 \times G/G_0$ be the homeomorphism. If (a) holds, there is a partition $\Psi^\#$ of G_0 into $|G|$ dense pairwise disjoint topologically homogeneous ω -bounded cosets each of cardinal $|G|$. Let $\Psi^* = \{B \times G/G_0 : B \in \Psi^\#\}$. If (b) holds, then G/G_0 is totally disconnected so that by Corollary 3, G/G_0 may be partitioned into a collection Ψ^\wedge of $|G|$ pairwise disjoint dense topologically homogeneous ω -bounded subsets each of $|G|$. Let $\Psi^* = \{G_0 \times C : C \in \Psi^\wedge\}$. Finally if (c) holds, construct Ψ^* as in (a). Thus in each case the partition Ψ^* consists of a collection of $|G|$ dense pairwise disjoint topologically homogeneous ω -bounded subsets each of cardinal $|G|$. Now let $\Psi = \{f^{-1}(A) : A \in \Psi^*\}$. It is clear that Ψ has the desired properties. \square

THEOREM 8. *Every infinite compact group G can be partitioned into $|G|$ pairwise disjoint dense nonmeasurable sets of full Haar outermeasure.*

NOTE 4. At the present time an attempt is being made to characterize resolvent topological spaces. A topological space is resolvent if it can be written as a disjoint union of two proper dense subsets. A subtopic of this investigation is to characterize topological spaces that are maximally resolvent. A space X is maximally resolvent if it can be partitioned by a collection $\{X_\alpha : \alpha \in A\}$ of dense subsets where $|A| = |X|$. We have therefore shown that all infinite compact groups are maximally resolvable into a collection of nonmeasurable sets each of full Haar outermeasure. Furthermore, each compact nonmetric group is maximally resolvable into a collection of ω -bounded homogeneously invariant subsets.

PROBLEM 1. Characterize the topological homogeneous spaces that are maximally resolvable into homogeneously invariant subsets.

EXAMPLE. Let A be an infinite discrete topological space. Then its Stone-Cech compactification βA may be partitioned into a collection of size $|\beta A|$ pairwise disjoint ω -bounded sets each of which is itself of size $|\beta A|$. To see this, let $G = T^\alpha$, where T is the circle and $\alpha = 2^{|A|}$. Then G can be partitioned into a collection Ψ^* of $|G|$ dense pairwise disjoint ω -bounded cosets each of cardinal $|G|$. It is shown in [12] that G has a dense subset D for which $|D| = |A|$. Let $f : A \rightarrow D$ be any surjective 1-1 function, then f is continuous. The Stone extension of f , $f^\beta : \beta A \rightarrow T^\alpha$, is surjective and continuous. Thus $\Psi = \{(f^\beta)^{-1}(B) : B \in \Psi^*\}$ is the desired partition of βA . Note that each of the sets $C = (f^\beta)^{-1}(C^*)$, $C^* \in \Psi^*$, cannot be compact since if C were, it would then follow that $f^\beta(C) = C^*$ would be compact and so would be all of G .

This example is the motivation for the following theorem.

THEOREM 9. *If a compact space X admits a continuous map $f : X \rightarrow G$ onto a compact group G such that $w(G) = w(X) \geq \omega_1$, then there exists a partition $X = \bigcup\{X_\alpha : \alpha < |X|\}$ of size $|X|$ into ω -bounded subsets X_α where $|X_\alpha| = |G|$. If in addition f is open, then each of the ω -bounded subsets X_α is dense in X .*

PROOF. By Theorem 6, G admits a decomposition into dense pairwise disjoint ω -bounded subsets H_α , $\alpha < |G|$. Thus each set $X_\alpha = f^{-1}(H_\alpha)$ is ω -bounded and $|X_\alpha| \geq |H_\alpha| = |G|$. However it is a classical theorem (see [10, Th. 3.1] and [9]) that $|X| \leq 2^{w(X)} = 2^{w(G)} = |G|$. Since f is surjective $|X| \geq |G|$, so that in fact $|X_\alpha| = |X| = |G|$. It now follows as in the previous example that the sets X_α , $\alpha < |X|$ are ω -bounded but not compact sets. The last statement of the theorem is clear. \square

As a corollary to the proof we get:

COROLLARY 4. *If a compact space X admits a continuous map $f : X \rightarrow G$ onto a compact group G and if $w(X) = w(G)$ then $|X| = |G| = 2^{w(X)} = 2^{w(G)}$.*

PROBLEM 2. Characterize those compact spaces that are maximally resolvable into ω -bounded subsets.

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