TSUKUBA J. MATH.
Vol. 25 No. 2 (2001), 465–479

THE INDEX AND CERTAIN STABILITY OF MINIMAL ANTI INVARIANT SUBMANIFOLDS IN SASAKIAN AND KÄHLER MANIFOLDS

By

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Abstract. The index forms of minimal anti invariant submanifolds in Sasakian and Kähler manifolds are obtained. We give lower bounds for the index of these submanifolds in terms of their intrinsic quantities. Certain stability of the minimal submanifolds is also considered, which is related to eigenspaces of the Laplacian.

0. Introduction

Recently, in [4], Itoh obtains several properties for minimal Legendrian surfaces in five dimensional Sasakian manifolds. In his paper, the index form for a minimal Legendrian surface is obtained, and a lower bound of the index in terms of the genus of the surface is given. On the other hands, for a minimal Lagrangian submanifold in a Kähler manifold, the index form are obtained in [3] and [6]. A lower bound of the index in terms of the first Betti number is given in [3], which is credited to Chen, Loung and Nagano, 1980. In [6], for a minimal Lagrangian submanifold in a Kähler manifold, the notion of the hamiltonian stability is defined and studied. From [5] and [6], the real projective space and the Clifford torus in the complex projective space, which are minimal Lagrangian, are unstable in the usual sense but hamiltonian stable. The main purpose of this paper is to give lower bounds for the index of anti invariant (not necessarily Legendrian or Lagrangian) minimal submanifolds in Sasakian and Kähler manifolds in terms of intrinsic quantities of those submanifolds, and to study certain stability of the minimal submanifolds, which is related to eigenspaces of the Laplacian.

AMS-Subject Classification: Primary 53C42, Secondly 53C25.
KEYWORDS: Anti invariant submanifolds, Index forms, partial stability.
Received July 28, 2000.
Revised January 15, 2001.

We obtain the index forms of minimal anti invariant submanifolds in Sasakian and Kähler manifolds. Using these, lower bounds of the index of these submanifolds in terms of the first Betti number, the index and the nullity of the identity map of the submanifold are given. We define and study i-partial stability for an anti invariant submanifold, which is a generalization of the hamiltonian stability in the case where the submanifold is Lagrangian. There are minimal Legendrian submanifolds unstable in the usual sense but 1- or 2-partial stable.

In Section 1, we will prepare the preliminaries. The index form for a minimal anti invariant submanifold in a Sasakian manifold will be obtained in Section 2. Using this, lower bounds of the index are given and i-partial stability is studied in Section 3. Finally, in the last section, similar theorems are obtained for minimal anti invariant submanifolds in Kähler manifolds.

The author would like to express his sincere gratitude to Professor N. Abe for his helpful advice and to Professor S. Yamaguchi for his constant encouragement.

1. Preliminaries

Let P and M be Riemannian manifolds of dimensions m and $m+p$, respectively. The tangent bundles of P and M are denoted by TP and TM, respectively. Let $f : P \to M$ be an isometric immersion. Around each $x \in P$, there exist a neighborhood $U \subset P$ such that the restriction of f to U is an embedding onto $f(U)$. Therefore, we may identify U with its image under f. We denote the metric on M and the induced metric on P by the same letter g. Hence we may consider the tangent space of P at $x \in P$ as subspace of the tangent space of M at x, and write

$$T_xM = T_xP \oplus T_xP^{\perp},$$

where $T_xP^{\perp}$ is the orthogonal complement of T_xP in T_xM. From this decomposition, we obtain a vector bundle $TP^{\perp} = \bigcup_{x \in P} T_xP^{\perp}$, called the normal bundle. For a vector bundle E, $\Gamma(E)$ denotes the set of smooth sections of E. Let $\bar{\nabla}$ and ∇ be the Levi-Civita connections of M and P, respectively. The Gauss-Weingarten formulae are given by

$$\bar{\nabla}_X Y = \nabla_X Y + h(X, Y) \quad \text{and} \quad \bar{\nabla}_X v = -A_v X + \nabla^{\perp}_X v$$

for $X, Y \in \Gamma(TP)$ and $v \in \Gamma(TP^{\perp})$, where $\nabla^{\perp}$ is the normal connection, h is the second fundamental form and A is the shape operator. Let $\bar{R}$ and R be the curvature tensor of $\bar{\nabla}$ and ∇, respectively. The equation of Gauss is given by

$$g(R(X,Y)Z,W) = g(\bar{R}(X,Y)Z,W) + g(h(X,W),h(Y,Z)) - g(h(X,Z),h(Y,W))$$

for $X, Y, Z, W \in TP$. The mean curvature vector H is defined by $H = (1/m)\,\mathrm{Tr}\,h$. A submanifold P is called minimal if $H = 0$ on P. For $v \in \Gamma(TP^{\perp})$, set

$$\Delta^N v = \sum_{i=1}^{m} (\nabla^{\perp}_{\nabla_{e_i} e_i} v - \nabla^{\perp}_{e_i} \nabla^{\perp}_{e_i} v),$$

$$\tilde{R}(v) = \sum_{i=1}^{m} (\bar{R}(e_i, v)e_i)^{\perp}$$

and

$$\tilde{A}(v) = \sum_{i=1}^{m} h(A_v e_i, e_i),$$

where $e_1, \dots, e_m$ is an orthonormal frame field on P and $(\cdot)^{\perp}$ denotes the normal part of $(\cdot)$. The index form I associated to the second variation formula for a minimal submanifold P given by

$$I(v,v') = \int_P \{g(\Delta^N v, v') + g(\tilde{R}(v), v') - g(\tilde{A}(v), v')\}\, d\mu_P \quad \text{for } v, v' \in \Gamma(TP^{\perp}),$$

where $d\mu_P$ is the volume element of P. See [7], for example.

For a vector field X (resp. 1-form ω) on a Riemannian manifold, its metrically equivalent 1-form (resp. vector field) is denoted by $X^{\flat}$ (resp. $\omega^{\sharp}$). The Laplacian operator acting on k-forms is denoted by Δ_k. For a symmetric $(0,2)$-tensor S, we define a $(1,1)$-tensor $S^{\sharp}$ defined by $g(S^{\sharp}(X), Y) = S(X,Y)$ for vector fields X, Y.

Let M be a $2n+1$ dimensional manifold and φ, ξ, η be a $(1,1)$-tensor field, a vector field, 1-form on M respectively such that

$$\varphi^2(X) = -X + \eta(X)\xi, \quad \varphi(\xi) = 0, \quad \eta(\varphi(X)) = 0 \quad \text{and} \quad \eta(\xi) = 1$$

for any vector field X on M. Then M is said to have an almost contact structure (φ, ξ, η) and is called an almost contact manifold. If a Riemannian metric tensor field g is given on an almost contact manifold M and satisfies

$$g(\varphi(X), \varphi(Y)) = g(X,Y) - \eta(X)\eta(Y) \quad \text{and} \quad \eta(X) = g(\xi, X)$$

for any vector fields X and Y on M, then (φ, ξ, η, g) is called an almost contact metric structure and M is called an almost contact metric manifold. If $d\eta(X, Y) = g(X, \varphi(Y))$ for any vector fields X and Y on M, then an almost contact metric structure is called a contact metric structure. If moreover the structure is normal, that is, $N + d\eta \otimes \xi = 0$, then a contact metric structure is called Sasakian structure and M is called Sasakian manifold, where N is the Nijenhuis torsion for φ. If $(M, \varphi, \xi, \eta, g)$ is a Sasakian manifold, then we have

$$\bar{\nabla}_X \xi = -\varphi(X) \quad \text{and} \quad (\bar{\nabla}_X \varphi)(Y) = g(X, Y)\xi - \eta(Y)X,$$

where $\bar{\nabla}$ is the Levi-Civita connection for g. A Sasakian manifold of constant φ-sectional curvature c is called a Sasakian space form. The curvature tensor $\bar{R}$ of a Sasakian space form of constant φ-sectional curvature c is given by

$$\begin{aligned}\bar{R}(X, Y)Z = {} & \frac{c+3}{4}\{g(Y, Z)X - g(X, Z)Y\} \\ & - \frac{c-1}{4}\{\eta(Y)\eta(Z)X - \eta(X)\eta(Z)Y + g(Y, Z)\eta(X)\xi - g(X, Z)\eta(Y)\xi \\ & - g(\varphi(Y), Z)\varphi(X) + g(\varphi(X), Z)\varphi(Y) + 2g(\varphi(X), Y)\varphi(Z)\}.\end{aligned}$$

Let L be an m-dimensional Riemannian submanifold orthogonal to ξ in M. Then we have

$$TL \perp \varphi(TL)$$

and

$$TM|_L = TL \oplus \varphi(TL) \oplus E \oplus \operatorname{span}\{\xi\} \quad \text{(orthogonal direct sum)},$$

where E is an invariant vector bundle over L with rank $E = 2(n - m) =: p$. Let π be the projection from $TM|_L$ to E. We define $\alpha \in \Gamma(\operatorname{Hom}(TL \otimes TL, E))$ by

$$\alpha(X, Y) = \pi(\nabla_X^{\perp}\varphi(Y)) \quad \text{for } X, Y \in \Gamma(TL).$$

The following equations hold:

$$\nabla_X^{\perp}\varphi(Y) = \varphi(\nabla_X Y) + \alpha(X, Y) + g(X, Y)\xi, \tag{1.1}$$

$$g(h(X, Y), \xi) = g(A_\xi X, Y) = 0 \tag{1.2}$$

and

$$g(h(X, Y), \varphi(Z)) = g(h(X, Z), \varphi(Y)) \tag{1.3}$$

for $X, Y, Z \in \Gamma(TL)$.

Note that if $m = n$, the submanifold L is called Legendrian. Clearly if L is

Legendrian, then we have $\alpha = 0$. Here we give typical examples of minimal Legendrian submanifold. The unit $(2n+1)$-sphere S^{2n+1} with standard metric is a Sasakian space form of constant φ-sectional curvature 1. The unit n-sphere S^n can be embedded as a totally geodesic Legendrian submanifold in S^{2n+1}. Also, an n-dimensional flat torus T^n can be also embedded as a minimal Legendrian submanifold in S^{2n+1} by the similar way to the case of $n = 2$ given in [1] and [2].

2. Index Forms

Let $(M, \eta, \xi, \varphi, g)$ be a $(2n+1)$-dimensional Sasakian manifold and L an m-dimensional compact orientable minimal submanifold orthogonal to ξ in M. In this section, we give the index form associated to the volume for the minimal submanifold L. Set

$$\Delta^T = \sum_{i=1}^{m} (\nabla_{\nabla_{e_i} e_i} - \nabla_{e_i} \nabla_{e_i}), \quad \mathrm{Tr}\, \alpha = \sum_{i=1}^{m} \alpha(e_i, e_i) \quad \text{and} \quad (\delta\alpha)(X) = \sum_{i=1}^{m} (\nabla_{e_i} \alpha)(e_i, X),$$

where $e_1, \dots, e_m$ is an orthonormal frame field on L. We define symmetric $(0,2)$-tensors β and $\bar{r}$ on L by

$$\beta(X, Y) = \sum_{i=1}^{m} g(\alpha(e_i, X), \alpha(e_i, Y)) \quad \text{and} \quad \bar{r}(X, Y) = \sum_{i=1}^{p} g(\bar{R}(v_i, X)v_i, Y),$$

where $v_1, \dots, v_p$ is an orthonormal frame of E. In the case where $p = 0$, that is, L is Legendrian, we set $\bar{r} = 0$.

LEMMA 2.1. *The following equations hold*:

$$\Delta^N(\rho\xi) = (\Delta\rho)\xi + 2\varphi(\mathrm{grad}\, \rho) + \rho m \xi + \rho\, \mathrm{Tr}\, \alpha, \tag{2.1}$$

$$\begin{aligned}\Delta^N(\varphi(X)) = {} & \varphi(\Delta_1 X^\flat)^\sharp - \varphi((Ric_L)^\sharp(X)) - 2\, \mathrm{div}(X)\xi \\ & + \varphi(X) + (\delta\alpha)(X) - 2\sum_{i=1}^{m} \alpha(e_i, \nabla_{e_i} X),\end{aligned} \tag{2.2}$$

$$\begin{aligned}g(\tilde{R}(\varphi(X)), \varphi(Y)) = {} & -Ric_M(X, Y) + Ric_L(X, Y) \\ & + g(\bar{A}(\varphi(X)), \varphi(Y)) + \beta(X, Y) + g(X, Y) - \bar{r}(X, Y),\end{aligned} \tag{2.3}$$

$$g(\tilde{R}(\varphi(X)), \xi) = 0, \tag{2.4}$$

$$g(\tilde{R}(\xi), \xi) = -m \tag{2.5}$$

and

(2.6) $$\tilde{A}(\xi) = 0,$$

where $\rho \in C^{\infty}(L)$ *is a smooth function and* $X, Y \in \Gamma(TL)$.

PROOF. Let $e_1, \dots, e_m$ be an orthonormal frame field on L. Since $\Delta^N \xi = m\xi + \operatorname{Tr} \alpha$, we have

$$\Delta^N(\rho\xi) = (\Delta\rho)\xi + 2\varphi(\operatorname{grad} \rho) + \rho\Delta^N \xi = (\Delta\rho)\xi + 2\varphi(\operatorname{grad} \rho) + \rho m\xi + \rho \operatorname{Tr} \alpha.$$

From (1.1) and $\bar{\nabla}_X \xi = -\varphi(X)$, it follows that

$$\begin{aligned}
\Delta^N \varphi(X) &= \sum_{i=1}^{m} (\nabla^{\perp}_{\nabla_{e_i} e_i} \varphi(X) - \nabla^{\perp}_{e_i} \nabla^{\perp}_{e_i} \varphi(X)) \\
&= \varphi(\Delta^T X) - 2(\operatorname{div} X)\xi + \varphi(X) + (\delta\alpha)(X) - 2\sum_{i=1}^{m} \alpha(e_i, \nabla_{e_i} X) \\
&= \varphi(\Delta_1 X^{\flat})^{\sharp} - \varphi((Ric_L)^{\sharp}(X) - 2(\operatorname{div} X)\xi \\
&\quad + \varphi(X) + (\delta\alpha)(X) - 2\sum_{i=1}^{m} \alpha(e_i, \nabla_{e_i} X),
\end{aligned}$$

where we used the Weitzenböck formula to get the last equality.

Next we consider the curvature part. From the invariance of $\bar{R}$, we have

$$\begin{aligned}
g(\tilde{R}(\varphi(X), \varphi(Y)) &= \sum_{i=1}^{m} g(\bar{R}(e_i, \varphi(X))e_i, \varphi(Y)) = -\sum_{i=1}^{m} g(\bar{R}(\varphi(e_i), X)Y, \varphi(e_i)) \\
&= -\sum_{i=1}^{m} g(\bar{R}(\varphi(e_i), X)Y, \varphi(e_i)) - \sum_{i=1}^{m} g(\bar{R}(e_i, X)Y, e_i) \\
&\quad - \sum_{k=1}^{p} g(\bar{R}(v_i, X)Y, v_i) - g(\bar{R}(\xi, X)Y, \xi) \\
&\quad + \sum_{i=1}^{m} g(\bar{R}(e_i, X)Y, e_i) + \sum_{k=1}^{p} g(\bar{R}(v_i, X)Y, v_i) + g(\bar{R}(\xi, X)Y, \xi) \\
&= -Ric_M(X, Y) + Ric_L(X, Y) + g(\bar{A}(\varphi(X)), \varphi(Y)) \\
&\quad + \beta(X, Y) + g(X, Y) - \bar{r}(X, Y)
\end{aligned}$$

where $v_1, \dots, v_p$ is orthonormal frame section of E. And the equations

$$g(\tilde{R}(\varphi(X)), \xi) = \sum_{i=1}^{m} g(\bar{R}(e_i, \varphi(X))e_i, \xi) = \sum_{i=1}^{m} g(\bar{R}(e_i, \xi)e_i, \varphi(X)) = 0$$

and

$$g(\tilde{R}(\xi), \xi) = \sum_{i=1}^{m} g(\bar{R}(e_i, \xi)e_i, \xi) = -m$$

hold. Finally, from (1.2), it follows that $\tilde{A}(\xi) = 0$. Q.E.D.

We define $\mathscr{F} : \Gamma(TL) \to \Gamma(TL)$ by

$$\mathscr{F}(X) := (\Delta_1 X^\flat)^\sharp + 2X - (Ric_M|_L)^\sharp(X) + \beta^\sharp(X) - \bar{r}^\sharp(X)$$

for $X \in \Gamma(TL)$.

THEOREM 2.2. *Let L be a compact orientable minimal submanifold orthogonal to ξ in a Sasakian manifold M. Then the index form I is given by*

$$I(\rho\xi, \rho'\xi) = \int_L ((\Delta\rho)\rho')\, d\mu_L,$$

$$I(\rho\xi, \varphi(X)) = \int_L 2g(\operatorname{grad} \rho, X)\, d\mu_L = -\int_L 2\rho(\operatorname{div} X)\, d\mu_L$$

and

$$I(\varphi(X), \varphi(Y)) = \int_L g(\mathscr{F}(X), Y)\, d\mu_L,$$

where $X, Y \in \Gamma(TL)$.

The index for a Legendrian minimal surface in a 5-dimensional Sasakian manifold is obtained in [4].

3. The Index and Partial Stability

Let $(M, \eta, \xi, \varphi, g)$ be a $(2n+1)$-dimensional Sasakian manifold and L an m-dimensional compact orientable minimal submanifold orthogonal to ξ in M. The first Betti number of L is denoted by $b_1(L)$. Set $F := (Ric_M)|_L - 2g - \beta + \bar{r}$. The index of I is denoted by $\operatorname{Index}(L)$.

THEOREM 3.1. *Let M be a Sasakian manifold and L a compact orientable minimal submanifold orthogonal to ξ. If F is positive definite, then*

$$\mathrm{Index}(L) \geq b_1(L).$$

PROOF. Take a nontrivial harmonic 1-form α. From Theorem 2.2 and positive definiteness of F, we have

$$I(\varphi(\alpha^\sharp), \varphi(\alpha^\sharp)) = \int_L \{-F(\alpha^\sharp, \alpha^\sharp)\}\, d\mu_L < 0$$

which shows the required lower bound for the index of L. Q.E.D.

If L is a connected compact Riemann surface, then $b_1(L) = 2g$, where g is the genus of L. Hence Theorem 4.1 is a generalization of Theorem 6.1 in [4].

Let E_i be the eigenspace of the i-th eigenvalue $\lambda_i(L)$ of Δ_0. We say that L is *i-partial stable* if i is the minimum natural number satisfying

$$I(\varphi(\mathrm{grad}\, f), \varphi(\mathrm{grad}\, f)) \geq 0 \quad \text{for all } f \in \bigoplus_{l \geq i} E_l.$$

One can define the stability of this kind for an anti invariant minimal submanifold in a Kähler manifold. In the final section, we consider a generalization of the theorem on the hamiltonian stability for a Lagrangian submanifold in an Einstein-Kähler manifold proved in [6]. The unit tangent sphere bundle of L is denoted by UL.

THEOREM 3.2. *Let M be a Sasakian manifold and L a compact orientable minimal submanifold orthogonal to ξ.*

(i) *If $\lambda_i(L) \geq \max_{v \in UL} F(v, v)$, then L is i-partial stable.*

(ii) *If L is i-partial stable, then we have $\lambda_i(L) \geq \min_{v \in UL} F(v, v)$.*

PROOF. Let f_l be an l-th eigen function of Δ_0, and set $F_{\max} := \max_{v \in UL} F(v, v)$ and $F_{\min} := \min_{v \in UL} F(v, v)$. First, assume that $l \geq i$ and $\lambda_i(L) \geq F_{\max}$. For $f = \sum_{l \geq i} f_l$ We have

$$\begin{aligned} &I(\varphi(\mathrm{grad}\, f), \varphi(\mathrm{grad}\, f)) \\ &= \int_L \left\{ \sum_{l \geq i} \lambda_l(L) g(\mathrm{grad}\, f_l, \mathrm{grad}\, f_l) - F(\mathrm{grad}\, f, \mathrm{grad}\, f) \right\} d\mu_L \\ &\geq \int_L \left\{ \sum_{l \geq i} \lambda_l(L) g(\mathrm{grad}\, f_l, \mathrm{grad}\, f_l) - F_{\max} g(\mathrm{grad}\, f, \mathrm{grad}\, f) \right\} d\mu_L \\ &\geq 0. \end{aligned}$$

Hence L is i-partial stable. If L is i-partial stable, then we obtain

$$\begin{aligned} 0 &\leq I(\varphi(\operatorname{grad} f_i), \varphi(\operatorname{grad} f_i)) \\ &\leq \int_L \{\lambda_i(L) g(\operatorname{grad} f_i, \operatorname{grad} f_i) - F_{\min} g(\operatorname{grad} f_i, \operatorname{grad} f_i)\}\, d\mu_L \\ &\leq (\lambda_i(L) - F_{\min}) \int_L g(\operatorname{grad} f_i, \operatorname{grad} f_i)\, d\mu_L. \end{aligned}$$

Q.E.D.

The following corollary can be proved immediately from Theorem 3.2.

COROLLARY 3.3. *Assume that M is an η-Einstein manifold with $Ric_M = ag + b\eta \otimes \eta$ (a, $b \in \mathbf{R}$) and L is Legendrian. Then L is i-partial stable if and only if $\lambda_i(L) \geq a - 2$.*

Minimal submanifolds in the unit sphere are unstable in the usual sense. But, considering values of the first and second eigenvalues of Δ_0 for S^n, a totally geodesic Legendrian submanifold S^n in S^{2n+1} is 1-partial stable (resp. 2-partial stable) if $n = 1, 2$ (resp. $n \geq 3$).

Let X be a vector field on L. Then the 1-parameter group generated by X is a variation of the identity map. It is well-known that the identity map of a Riemannian manifold P is a harmonic map. We consider the relations between the index for the minimal submanifold L and that for the identity map of L. The Jacobi operator $\mathscr{J}_{\mathrm{id}_P}$ associated to the identity map id_P is given by

$$\mathscr{J}_{\mathrm{id}_P}(X) = (\Delta_1 X^\flat)^\sharp - 2(Ric_P)^\sharp(X)$$

for $X \in \Gamma(P)$ ([8]). The index and nullity of id_P are denoted by $\mathrm{Index}(\mathrm{id}_P)$ and $\mathrm{Null}(\mathrm{id}_P)$, respectively. We give lower bounds for $\mathrm{Index}(L)$ in terms of index and nullity of the identity map of L.

LEMMA 3.4. *Let M be a Sasakian manifold and L a compact minimal submanifold orthogonal to ξ. Then the equation*

$$\mathscr{F}(X) = \mathscr{J}_{\mathrm{id}_L}(X) - F^\sharp(X) + 2(Ric_L)^\sharp(X)$$

holds for $X \in \Gamma(L)$.

From Theorem 2.2 and Lemma 3.4, we obtain

THEOREM 3.5. *Let M be a Sasakian manifold and L a compact orientable minimal submanifold orthogonal to ξ.*

(i) *If $F - 2\,Ric_L$ is positive semi-definite, we have*

$$\mathrm{Index}(L) \geq \mathrm{Index}(\mathrm{id}_L).$$

(ii) *If $F - 2\,Ric_L$ is positive definite, we have*

$$\mathrm{Index}(L) \geq \mathrm{Index}(\mathrm{id}_L) + \mathrm{Null}(\mathrm{id}_L).$$

From Theorem 3.5, if $F - 2\,Ric_L$ is positive definite and L is a compact, orientable, stable ($\mathrm{Index}(L) = 0$), minimal submanifold orthogonal to ξ, then the identity map is a local minimum of the energy functional. The space of Killing vector fields on L, which is denoted by $i(L)$, are contained in Ker $\mathscr{J}_{\mathrm{id}_L}$. Therefore if $F - 2\,Ric_L$ is positive definite, then $\mathrm{Index}(L) \geq \dim i(L)$.

Next we consider the case where M is a Sasakian space form with constant φ-sectional curvature c. Then we have

$$Ric_M(X, Y) = \frac{n(c+3)+c-1}{2} g(X, Y),$$

$$Ric_L(X, Y) = \frac{(m-1)(c+3)}{4} g(X, Y) - \beta(X, Y) - g(A_{\varphi(X)}, A_{\varphi(Y)})$$

and

$$\bar{r}(X, Y) = -\frac{(c+3)p}{4} g(X, Y)$$

for $X, Y \in TL$.

From these equations, we obtain the following lemma.

LEMMA 3.6. *For $X, Y \in TL$,*

$$F(X, Y) = \frac{(m+1)c+3m-5}{2} g(X, Y) - \beta(X, Y)$$

and

$$F(X, Y) - 2\,Ric_L(X, Y) = (c-1)g(X, Y) + 2g(A_{\varphi(X)}, A_{\varphi(Y)}) + \beta(X, Y)$$

holds.

PROOF. For $X, Y \in TL$, we have

$$F(X, Y) = \frac{n(c+3)+c-1}{2} g(X, Y) - 2g(X, Y) - \beta(X, Y) - \frac{(c+3)p}{4} g(X, Y)$$

$$= \frac{(m+1)c+3m-5}{2} g(X, Y) - \beta(X, Y)$$

and

$$
\begin{aligned}
F(X, Y) - 2\, Ric_L(X, Y) &= \frac{n(c+3)+c-1}{2} g(X, Y) - 2 \cdot \frac{(m-1)(c+3)}{4} g(X, Y) \\
&\quad + 2g(A_{\varphi(X)}, A_{\varphi(Y)}) + 2\beta(X, Y) \\
&\quad - 2g(X, Y) - \beta(X, Y) - \frac{(c+3)p}{4} g(X, Y) \\
&= (c-1)g(X, Y) + 2g(A_{\varphi(X)}, A_{\varphi(Y)}) + \beta(X, Y)
\end{aligned}
$$

By Theorems 3.1, 3.5 and Lemma 3.6, we obtain

COROLLARY 3.7. *Let M be a Sasakian space form of constant φ-sectional curvature c and L a compact orientable minimal submanifold orthogonal to ξ.*

(i) *If $(m+1)c + 3m - 5 > 0$ and $\beta = 0$, we have*

$$\mathrm{Index}(L) \geq b_1(L)$$

(ii) *If $c \geq 1$, we have*

$$\mathrm{Index}(L) \geq \mathrm{Index}(\mathrm{id}_L).$$

(iii) *If $c > 1$, we have*

$$\mathrm{Index}(L) \geq \mathrm{Index}(\mathrm{id}_L) + \mathrm{Null}(\mathrm{id}_L).$$

Thus we have obtained two lower bounds for Index(L), namely, $b_1(L)$ and Index(id_L). These estimates are independent of each other as the following table shows.

n	$\mathrm{Index}(\mathrm{id}_{S^n})$	$b_1(S^n)$	$\mathrm{Index}(\mathrm{id}_{T^n})$	$b_1(T^n)$
1	0	1	0	1
2	0	0	0	2
$n \geq 3$	$n+1$	0	0	n

4. Minimal Anti Invariant Submanifolds in Kähler Manifolds

In this section, for compact orientable minimal anti invariant submanifolds in Kähler manifolds, we obtain similar results corresponding to Sasakian cases. Since arguments for Kählerian cases are similar to those for Sasakian cases, we omit proofs. Let (M, g, J) be a $2n$-dimensional Kähler manifold with Hermitian metric g and complex structure J, and L an m-dimensional compact orientable minimal anti invariant submanifold. Then we have

$$TM|_L = TL \oplus J(TL) \oplus E \quad \text{(orthogonal direct sum)},$$

where E is an invariant vector bundle over L with rank $E = 2(n-m) =: p$. Let π be the projection from $TM|_L$ to E. For $X, Y \in \Gamma(TL)$, we define

$$\alpha(X, Y) = \pi(\nabla_X^{\perp} J(Y)),$$

$$\beta(X, Y) = \sum_{i=1}^{m} g(\alpha(e_i, X), \alpha(e_i, Y))$$

and

$$\bar{r}(X, Y) = \sum_{k=1}^{p} g(\bar{R}(v_i, X)v_i, Y),$$

where $e_1, \ldots, e_m$ is an orthonormal frame field on L, $v_1, \ldots, v_p$ is an orthonormal frame of E and $\bar{R}$ is the curvature tensor of M. By the similar calculation in Section 2, we obtain

THEOREM 4.1. *For the index form I for L, it holds*

$$I(JX, JY) = \int_L \{g((\Delta_1 X^\sharp)^\flat, Y) - Ric_M(X, Y) + \beta(X, Y) - \bar{r}(X, Y)\}\, d\mu_L,$$

where $X, Y \in \Gamma(TL)$.

For the case where L is Lagrangian, see [3] and [6]. We define

$$\mathscr{F}(X) = (\Delta_1 X^\sharp)^\flat - (Ric_M|_L)^\sharp(X) + \beta^\sharp(X) - \bar{r}^\sharp(X)$$

and obtain

$$\mathscr{F}(X) = \mathscr{J}_{\mathrm{id}_L}(X) + 2(Ric_L)^\sharp(X) - (Ric_M|_L)^\sharp(X) + \beta^\sharp(X) - \bar{r}^\sharp(X)$$

for $X \in \Gamma(TL)$. Set $F := (Ric_M)|_L - \beta + \bar{r}$.

THEOREM 4.2. *Let M be a Kähler manifold and L a compact orientable minimal anti invariant submanifold. If F is positive definite, we have*

$$\mathrm{Index}(L) \geq b_1(L).$$

Theorem 4.2 is a generalization of the inequality mentioned in [3] for a minimal Lagrangian submanifold.

Since the anti invariance of the tangent bundle of L in M, we can define i-partial stability.

THEOREM 4.3. *Let M be a Kähler manifold and L a compact orientable minimal anti invariant submanifold.*

(i) *If $\lambda_i \geq \max_{v \in UL} F(v,v)$, then L is i-partial stable.*

(ii) *If L is i-partial stable, then we have $\lambda_i \geq \min_{v \in UL} F(v,v)$.*

COROLLARY 4.4. *Assume that M is an Einstein-Kähler manifold with $Ric_M = ag$ $(a \in \boldsymbol{R})$ and L is Lagrangian. Then L is i-partial stable if and only if $\lambda_i(L) \geq a$.*

Especially, L is hamiltonian stable, that is, 1-partial stable if and only if $\lambda_1(L) \geq a$. Therefore Theorem 4.3 is a generalization of Theorem 4.4 in [6].

THEOREM 4.5. *Let M be a Kähler manifold and L a compact orientable minimal anti invariant submanifold in M.*

(i) *If $F - 2\,Ric_L$ is positive semi-definite, we have*

$$\mathrm{Index}(L) \geq \mathrm{Index}(\mathrm{id}_L).$$

(ii) *If $F - 2\,Ric_L$ is positive definite, we have*

$$\mathrm{Index}(L) \geq \mathrm{Index}(\mathrm{id}_L) + \mathrm{Null}(\mathrm{id}_L).$$

Next we consider the case where M is a complex space form with constant holomorphic sectional curvature c. Then the curvature tensor $\bar{R}$ of M satisfies

$$\begin{aligned}\bar{R}(X,Y)Z = \frac{1}{4}c\{&g(Y,Z)X - g(X,Z)Y + g(JY,Z)JX \\ &- g(JX,Z)JY + 2g(X,JY)JZ\}\end{aligned}$$

for $X, Y, Z \in TM$. Then we have

$$Ric_L(X,Y) = \frac{c(m-1)}{4}g(X,Y) - g(A_{JX}, A_{JY}) - \beta(X,Y) \quad \text{for } X, Y \in TL,$$

$$\bar{r}(X,Y) = -\frac{cp}{4}g(X,Y) \quad \text{for } X, Y \in TL$$

and

$$Ric_M(X', Y') = \frac{c(n+1)}{2} g(X', Y') \quad \text{for } X', Y' \in TM.$$

Hence the following equations hold:

$$F(X, Y) = \frac{c(m+1)}{2} g(X, Y) - \beta(X, Y)$$

and

$$F(X, Y) - 2\, Ric_L(X, Y) = cg(X, Y) + 2g(A_{JX}, A_{JY}) + \beta(X, Y)$$

for $X, Y \in TL$.

COROLLARY 4.6. *Let M be a complex space form of constant holomorphic sectional curvature c and L a compact orientable minimal anti invariant submanifold.*

(i) *If $c > 0$ and $\beta = 0$, we have*

$$\mathrm{Index}(L) \geq b_1(L)$$

(ii) *If $c > 0$, we have*

$$\mathrm{Index}(L) \geq \mathrm{Index}(\mathrm{id}_L) + \mathrm{Null}(\mathrm{id}_L).$$

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