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ON FOURIER COEFFICIENTS OF MAASS WAVE FORMS OF HALF INTEGRAL WEIGHT BELONGING TO KOHNEN'S SPACES

By

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Introduction

Waldspurger [15] proved that the squares of Fourier coefficients $a(n)$ at a square free integer n of a modular form $f(z)=\sum_{n=1}^{\infty}a(n)e[nz]$ of half integral weight are essentially proportional to the critical value of the zeta function at a certain integer attached to the modular form F of even integral weight if f corresponds to F by the Shimura correspondence Ψ and f is an eigen-function of Hecke operators. Kohnen-Zagier [2], [4] determined explicitly the constant of the proportionality in the case of modular forms of belonging to Kohnen's spaces $S_{(2k+1)/2}(N,\chi)$ of weight $(2k+1)/2$ and of square free level N with character χ which is a subspace of $S_{(2k+1)/2}(4N,\chi_1)$, where $S_{(2k+1)/2}(4N,\chi_1)$ means the space of modular cusp forms of half integral weight given in [10]. Kohnen-Zagier [2] (resp. Kohnen [4]) treated the case where $N=1$ (resp. N is an odd square free integer and χ is the trivial character of level N) (cf. Kojima [5] and [7]).

In [10], Shimura intended to generalize such formulas to the case of Hilbert modular forms f of half integral weight and succeeded in obtaining many general interesting formulas. Among these, some explicit and useful formulas about the proportionality constant were formulated under assumptions that f satisfies the multiplicity one theorem. K. K-Makdisi [1] gave a generalization of these to the case of Hilbert-Maass wave forms. For modular forms belonging to Kohnen's spaces, they did not obtain the same explicit formulas as those of Kohnen and Zagier [2], [4].

In [8], we derived such explicit formulas concerning the proportionality constant in some cases of modular forms f of half integral weight whose multiplicity are two and generalized results of Kohnen and Zagier in [2], [4] to

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the case of modular forms of half integral weight belonging to Kohnen's spaces of an arbitrary odd level N and of an arbitrary primitive character modulo N.

The purpose of this paper is to generalize the results in [8] to the case of Maass wave forms of half integral weight belonging to Kohnen's spaces. We derive an explicit relation between the square of a Fourier coefficient $a(4n)$ at a fundamental discriminant $4n$ of Maass wave forms $f(z) = \sum_{n \in \mathbf{Z}-\{0\}} a(n) \cdot W_{\alpha',\beta'}(ny)e[nx]$ belonging to the Kohnen's space $\mathscr{S}_{(2k+1)/2,\lambda'}(N,\chi)$ of half integral weight $(2k+1)/2$ and of arbitrary odd level N of an arbitrary primitive character χ, and the critical value of the zeta function of the modular form F which is the image of f under the Shimura correspondence Ψ. The assumptions on χ and the fundamental discriminant $4n$ are technical conditions. Our methods of the proof are the same as those of Kojima [8]. To obtain our results we need to modify slightly the method.

Section 0 is a preliminary section. In Section 1, we shall summarize some results concerning Maass wave forms of half integral weight, Kohnen's spaces and Hecke operators of Kohnen's spaces of Maass wave forms of half integral weight. In Section 2, using these, we shall determine explicitly the image of Maass wave forms of half integral weight belonging to Kohnen's spaces under the Shimura correspondence Ψ. We show that $\Psi^{4N,\chi(\frac{4\varepsilon}{\tau})}_{(2k+1)/2,\lambda',\tau}(f)(w)$ coincides with $a(4\tau)g(2w)$ for every $f(z) = \sum_{n \in \mathbf{Z}, n \neq 0} a(n) W_{\alpha',\beta'}(ny)e[nx] \in \mathscr{S}_{(2k+1)/2,\lambda'}(N,\chi)$, where τ is a positive square free integer satisfying $\tau \equiv 2, 3 \pmod 4$ and $g(w)$ is an element of $\mathscr{S}_{2k,4\lambda'}(N,\chi^2)$. Moreover, by a method similar to those of Shimura [14] and Kojima [8], we shall verify an integral formula which shows that a modular form f of half integral weight is expressed as the inner product of a theta function and the image $\Psi(f)$ of f by the Shimura correspondence Ψ. Under some assumption about multiplicities, adapting the operator $U(4)$, we shall verify that $\langle \Theta(z,w;\zeta^{\mathbf{Z}}), g(2w) \rangle = \tilde{c}_1 L_\tau(f)(z) + \tilde{c}_2 L_\tau(f)|U(4)(z)$ for a modular form $f \in \mathscr{S}_{(2k+1)/2,\lambda'}(N,\chi)$ with some constants $\tilde{c}_1$ and $\tilde{c}_2$, where $\Theta(z,w;\zeta^{\mathbf{Z}})$ is a theta function and $L_\tau(f)(z) = f(\tau z)\tau^k$. Furthermore, applying the above formulas, we obtain $\tilde{c}_1 = \overline{a(4\tau)}\tilde{c}_3$ and $\tilde{c}_2 = \overline{a(4\tau)}\tilde{c}_4$ with explicit constants $\tilde{c}_3$ and $\tilde{c}_4$. These formulas are keys for our later treatments. In Section 3, using the results of Section 1, Section 2, the computation of the image of a product of theta serires and Eisenstein series by Shimura correspondence and the method of the Rankin's convolution, under some assumptions on the multiplicity property of Hecke operators, we shall derive an explicit connection between the square of Fourier coefficients of a Maass wave form f of half integral weight and the critical value of zeta functions associated with the image $\Psi(f)$ of f by the Shimura correspondence Ψ.

We mention that our results give a generalization of some results in Kojima [8].

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§0. Notation and Preliminaries

We denote by $\boldsymbol{Z}, \boldsymbol{Q}, \boldsymbol{R}$ and $\boldsymbol{C}$ the ring of rational integers, the rational number field, the real number field and the complex number field, respectively. For $z \in \boldsymbol{C}$, we put $e[z] = \exp(2\pi i z)$ and we define $\sqrt{z} = z^{1/2}$ so that $-\pi/2 < \arg z^{1/2} \leqq \pi/2$. Further, we set $z^{k/2} = (\sqrt{z})^k$ for every $k \in \boldsymbol{Z}$. We denote by $\Im(z)$ (resp. $\Re(z)$) the imaginary (resp. real) part of $z \in \boldsymbol{C}$. Let $SL(2, \boldsymbol{R})$ denote the group of all real matrices of degree 2 with determinant one and $\mathfrak{H}$ the complex upper half plane, i.e.,

$$SL(2, \boldsymbol{R}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle| a, b, c \text{ and } d \in \boldsymbol{R} \text{ and } ad - bc = 1 \right\}$$

and

$$\mathfrak{H} = \{ z = x + iy \,|\, x, y \in \boldsymbol{R} \text{ and } y > 0 \}.$$

Define an action of $SL(2, \boldsymbol{R})$ on $\mathfrak{H}$ by

$$z \to \gamma(z) = \frac{az + b}{cz + d} \quad \text{for all } \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \boldsymbol{R}) \quad \text{and for all } z \in \mathfrak{H}.$$

For positive integers M and M', put

$$\Gamma_0(M) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \boldsymbol{R}) \middle| a, b, c \text{ and } d \in \boldsymbol{Z} \text{ and } c \equiv 0 \pmod{M} \right\},$$

$$SL(2, \boldsymbol{Z}) = \Gamma_0(1) \quad \text{and} \quad \Gamma[M, M'] = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(M') \middle| b \equiv 0 \pmod{M} \right\}.$$

We introduce an automorphic factor $j_0(\gamma, z)$ of $\Gamma_0(4)$ determined by $j_0(\gamma, z) = \vartheta_0(\gamma(z))/\vartheta_0(z)$ for every $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(4)$ and for every $z \in \mathfrak{H}$ with $\vartheta_0(z) = \sum_{n=-\infty}^{\infty} e[n^2 z]$ and $\vartheta(z) = \vartheta_0(z/2)$.

§1. Maass Wave Forms of Half Integral Weight

This section is devoted to summarizing several fundamental facts which we need later (cf. [11]). For a $l \in \boldsymbol{R}$, we put

(1-1) $\eta f(z)=-y^2\frac{\partial f}{\partial \bar{z}}(z),\quad \delta^l f(z)=y^{-l}\frac{\partial(y^l f)}{\partial \bar{z}}(z),\quad D^l f(z)=-lf(z)+4\eta\delta^l f(z)$

for every C^∞-function $f(z)$ on $\mathfrak{H}$. For a function f on $\mathfrak{H}$ and $n \in \boldsymbol{Z}(>0)$, we put

$$f\|_n\gamma(z) = (cz+d)^{-n} f\left(\frac{az+b}{cz+d}\right) \quad \text{for every } \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\boldsymbol{R}). \tag{1-2}$$

Let ω be a Dirichlet character modulo M. We denote by $\mathscr{M}_{n,\lambda}(M,\omega)$ the set of all C^∞-functions $f : \mathfrak{H} \to \boldsymbol{C}$ satisfying conditions

(1-3) (i) $f\|_n\gamma(z) = \omega(d)f$ for every $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(M)$,

(ii) $D^n f(z) = \lambda f(z)$ and for every $\sigma \in SL(2,\boldsymbol{Z})$, there exist positive numbers a, b and c depending on the choice of f such that

(iii) $|(\Im(z))^{n/2} f\|_n\sigma(z)| \leqq ay^c$ if $\Im(z) \geqq b$.

Now we take a $(\alpha,\beta) \in \boldsymbol{C}^2$ such that $\alpha\beta = \lambda$, $\alpha+\beta = 1-n$ and define a function $W_{\alpha,\beta}(y)$ on $\boldsymbol{R}^\times$ determined by

$$W_{\alpha,\beta}(y) = \begin{cases} V(4\pi y;\alpha,\beta) & \text{if } y>0, \\ (-4\pi y)^{\alpha+\beta-1} V(-4\pi y; 1-\alpha, 1-\beta) & \text{if } y<0, \end{cases} \tag{1-4}$$

where

$$V(y;l,m) = e^{-y/2}y^l\Gamma(l)^{-1}\int_0^\infty e^{-yt}(1+t)^{-m}t^{l-1}\,dt \quad \text{for } (l,m) \in \boldsymbol{C}^2 \quad (\Re(l)>0)$$

and $y>0$ and $V(y;l,m)$ is the function given by means of the analytic continuation with respect to l for $(l,m) \in \boldsymbol{C}^2$ $(\Re(l) \leqq 0)$ and $y>0$. We denote by $\mathscr{S}_{n,\lambda}(M,\omega)$ the set of all cusp forms f belonging to $\mathscr{M}_{n,\lambda}(M,\omega)$. We have the following lemma (cf. [1], [11]).

LEMMA 1.1. *The function $f(z) \in \mathscr{S}_{n,\lambda}(M,\omega)$ has a Fourier expansion of the form*

$$f(z) = \sum_{\mu \in \boldsymbol{Z}-\{0\}} b(\mu)W_{\alpha,\beta}(\mu y)e[\mu x] \quad (z = x+iy). \tag{1-5}$$

For $n \in \boldsymbol{Z}$ $(n \geqq 0)$ and $z \in \boldsymbol{C}$, put $(z)_n = z(z+1)\cdots(z+n-1)$ $(n>0)$ and $(z)_0 = 1$. We call $f(z) \in \mathscr{S}_{n,\lambda}(M,\omega)$ even if n is even and

$$f(z) = \sum_{\mu \in \boldsymbol{Z}-\{0\}} b(\mu)W_{\alpha,\beta}(\mu y)e[\mu x] \quad \text{and} \tag{1-6}$$

$$b(\mu) = (-1)^{n/2}(\alpha)_{n/2}(\beta)_{n/2}b(|\mu|) \quad (\mu<0).$$

We may define Hecke operators $\{T^M_{n,\lambda,\omega}(m)\}_{m=1}^{\infty}$ acting on $\mathscr{S}_{n,\lambda}(M,\omega)$ satisfying

$$f|T^M_{n,\lambda,\omega}(m)(z) = \sum_{\mu\in \boldsymbol{Z}-\{0\}} b'(\mu)W_{\alpha,\beta}(\mu y)e[\mu x] \tag{1-7}$$

and

$$b'(\mu) = \sum_{d|(m,\mu),\, d>0} \omega(d)\, d^{n-1} b(m\mu/d^2),$$

where

$$f(z) = \sum_{\mu\in \boldsymbol{Z}-\{0\}} b(\mu)W_{\alpha,\beta}(\mu y)e[\mu x] \ (cf.\ [1]).$$

Let k be a positive integer. Let N denote a positive integer and ψ_0 a Dirichlet character modulo $4N$. We denote by $\mathscr{M}_{(2k+1)/2,\lambda'}(4N,\psi_0)$ the set of all C^∞-functions $f:\mathfrak{H}\to \boldsymbol{C}$ such that

(1-8)
(i) $f(\gamma(z)) = \psi_0(d)j_0(\gamma,z)^{2k+1}f(z)$ for every $\gamma = \begin{pmatrix} * & * \\ * & d\end{pmatrix} \in \Gamma_0(4N)$ and $z\in\mathfrak{H}$,

(ii) $D^{(2k+1)/2}f(z) = \lambda' f(z)$ and for every $\begin{pmatrix} a & b \\ c & d\end{pmatrix} \in SL(2,\boldsymbol{Z})$, there exist positive numbers a', b' and c' depending on the choice of f satisfying the condition

(iii) $(\Im(z))^{k/2+1/4}\left|f\left(\dfrac{az+b}{cz+d}\right)(cz+d)^{-(2k+1)/2}\right| \leqq a'(\Im(z))^{c'}$ if $\Im(z)\geqq b'$.

We denote by $\mathscr{S}_{(2k+1)/2,\lambda'}(4N,\psi_0)$ the set of all cusp forms f belonging to $\mathscr{M}_{(2k+1)/2,\lambda'}(4N,\psi_0)$. The form $f\in\mathscr{S}_{(2k+1)/2,\lambda'}(4N,\psi_0)$ has the following Fourier expansion

$$f(z) = \sum_{\mu\in \boldsymbol{Z}-\{0\}} a(\mu)W_{\alpha',\beta'}(\mu y)e[\mu x] \quad (z = x+iy), \tag{1-9}$$

where $a(0)=0$ and α',β' are complex numbers such that $\alpha'\beta'=\lambda'$, $\alpha'+\beta' = 1-(2k+1)/2$ (*cf.* [1]). We may define Hecke operators $\{T^{4N}_{(2k+1)/2,\lambda',\psi_0}(p^2)\}_{m=1}^{\infty}$ acting on $\mathscr{S}_{(2k+1)/2,\lambda'}(4N,\psi_0)$ satisfying conditions

$$f|T^{4N}_{(2k+1)/2,\lambda',\psi_0}(p^2)(z) = \sum_{\mu\in \boldsymbol{Z}-\{0\}} a'(\mu)W_{\alpha',\beta'}(\mu y)e[\mu x], \tag{1-10}$$

$$a'(\mu) = a(p^2\mu) + \psi_0(p)\left(\frac{-1}{p}\right)^k\left(\frac{\mu}{p}\right)p^{k-1}a(\mu) + \psi_0(p^2)p^{2k-1}a(\mu'/p^2)$$

and

$$f(z)=\sum_{\mu\in Z-\{0\}} a(\mu)W_{\alpha',\beta'}(\mu y)e[\mu x],$$

where $a(\mu/p^2)=0$ if $p^2\nmid\mu$ (cf. [1], [6] and [10]).

Let $\mathfrak{b}$ and $\mathfrak{b}'$ denote integral ideals of $\boldsymbol{Q}$ and ψ a Hecke character of $\boldsymbol{Q}$ whose conductor divides $4\mathfrak{b}\mathfrak{b}'$. Let $\mathscr{M}_{(2k+1)/2,\lambda'}(\mathfrak{b},\mathfrak{b}';\psi)$ (resp. $\mathscr{S}_{(2k+1)/2,\lambda'}(\mathfrak{b},\mathfrak{b}';\psi)$) be the space of modular forms (resp. modular cusp forms) of half integral weight $(2k+1)/2$ given in [1] and [14]. Let ψ_0 be a Dirichlet character modulo $4N$ such that $\psi_0(-1)=1$. We choose the Hecke character ψ of $\boldsymbol{Q}$ such that

$$\text{(1-11)}\qquad \prod_{p|4N}\psi_p(a)=\left(\frac{-1}{a}\right)^k\psi_0(a)^{-1}\quad\text{for every } a\in(\boldsymbol{Z}/4N\boldsymbol{Z})^\times,\ \psi_p(\boldsymbol{Z}_p^\times)=1$$

for every $p\nmid 4N$ and $\psi_a(x)=(\operatorname{sgn}(x))^k$ $(x\in\boldsymbol{R})$, where ψ_p (resp. ψ_a) means the restriction of ψ to the p-component (resp. the archimedean factor) of the idele group $\boldsymbol{Q}_A^\times$.

For $f'\in\mathscr{S}_{(2k+1)/2,\lambda'}(\boldsymbol{Z},N\boldsymbol{Z};\psi)$, put $L(f')(z)=f'(2z)$. Then the following mapping is bijective.

$$\text{(1-12)}\qquad L:\mathscr{S}_{(2k+1)/2,\lambda'}(\boldsymbol{Z},N\boldsymbol{Z};\psi)\to\mathscr{S}_{(2k+1)/2,\lambda'}(4N,\psi_0).$$

Let τ be a positive integer. We consider a mapping L_τ of $\mathscr{S}_{(2k+1)/2,\lambda'}(\boldsymbol{Z},N\boldsymbol{Z};\psi)$ into $\mathscr{S}_{(2k+1)/2,\lambda'}(\boldsymbol{Z},N\tau\boldsymbol{Z};\psi\rho_\tau)$ defined by

$$\text{(1-13)}\qquad L_\tau(f)(z)=f(\tau z)\tau^k\quad\text{for every } f(z)\in\mathscr{S}_{(2k+1)/2,\lambda'}(\boldsymbol{Z},N\boldsymbol{Z};\psi),$$

where ρ_τ is the Hecke character associated with the quadratic field $\boldsymbol{Q}(\sqrt{\tau})$.

We put $h(z)=L_\tau(f)(z)$. The following lemma can be proved by [14].

LEMMA 1.2. *The notation being as above, the mapping L_τ gives a bijection of $\mathscr{S}_{(2k+1)/2,\lambda'}(\boldsymbol{Z},N\boldsymbol{Z};\psi)$ onto the set*

$$\text{(1-14)}\qquad \left\{h(z)=\sum_{n\in Z-\{0\}}a'(n)W_{\alpha',\beta'}(ny/2)e[nx/2]\right.$$
$$\left.\in\mathscr{S}_{(2k+1)/2,\lambda'}(\boldsymbol{Z},N\tau\boldsymbol{Z},\psi\rho_\tau)\,\middle|\,a'(n)=0\ \text{if}\ \tau\nmid n\right\}.$$

Moreover,

$$\langle f,f\rangle=\tau^{-(2k+1)/2}\tau\langle h,h\rangle,$$

where for given two cusp forms f, g *of weight* l *with respect to* Γ, *their inner product* $\langle f, g\rangle$ *means*

$$\langle f, g\rangle = \mathrm{vol}(\Gamma\backslash\mathfrak{H})^{-1}\int_{\Gamma\backslash\mathfrak{H}} \overline{f(z)}g(z)\mathfrak{I}(z)^l d_{\mathfrak{H}}z \quad \textit{with}$$

$$d_{\mathfrak{H}}z = \frac{dxdy}{y^2} \quad (x = \mathfrak{R}(z), y = \mathfrak{I}(z)).$$

Throughout the rest of the paper we assume that N is an odd integer. Let χ be a Dirichlet character modulo N such that $\chi(-1) = \varepsilon$. Put $\chi_1 = (\frac{4\varepsilon}{*})\chi$. We introduce a subspace $\mathscr{S}_{(2k+1)/2,\lambda'}(N,\chi)$ of $\mathscr{S}_{(2k+1)/2,\lambda'}(4N,\chi_1)$ defined by

(1-15) $\mathscr{S}_{(2k+1)/2,\lambda'}(N,\chi) =$

$$\left\{ f \in \mathscr{S}_{(2k+1)/2,\lambda'}(4N,\chi_1) \,\middle|\, f(z) = \sum_{\varepsilon(-1)^k n \equiv 0, 1(4), n\neq 0} a(n) W_{\alpha',\beta'}(ny)e[nx] \right\}.$$

We call $\mathscr{S}_{(2k+1)/2,\lambda'}(N,\chi)$ the Kohnen's space of Maass wave forms of weight $(2k+1)/2$ and of level N with character χ. For a positive integer m, we define a function $f|U(m)$ on $\mathfrak{H}$ by

$$f|U(m)(z) = \sum_{n\in \boldsymbol{Z}-\{0\}} a(mn) W_{\alpha',\beta'}(ny)e[nx]$$

for every function $f(z) = \sum_{n\in\boldsymbol{Z}-\{0\}} a(n) W_{\alpha',\beta'}(ny)e[nx]$ on $\mathfrak{H}$. By the same manner as that of [3, p. 42 and p. 46], we define Hecke operators $T^N_{(2k+1)/2,\lambda',\chi}(p)$ $(p \nmid N)$ on $\mathscr{S}_{(2k+1)/2,\lambda'}(N,\chi)$. We can show the following lemma (cf. [10]).

Lemma 1.3. *The notation being as above, for* $f(z) = \sum_{\varepsilon(-1)^k n\equiv 0,1(4), n\neq 0} a(n) W_{\alpha',\beta'}(ny)e[nx] \in \mathscr{S}_{(2k+1)/2,\lambda'}(N,\chi)$, $f|T^N_{(2k+1)/2,\lambda',\chi}(p)(z)$ $(p\nmid N)$ *and* $f|U(p^2)(z)$ $(p|N)$ *belong to* $\mathscr{S}_{(2k+1)/2,\lambda'}(N,\chi)$. *Moreover, Fourier expansions of* $f|T^N_{(2k+1)/2,\lambda',\chi}(p)(z)$ $(p\nmid N)$ *and* $f|U(p^2)(z)$ $(p|N)$ *are given as follows*:

(1-16) $f|T^N_{(2k+1)/2,\lambda',\chi}(p)(z)$

$$= \sum_{n\in\boldsymbol{Z}-\{0\}, \varepsilon(-1)^k n\equiv 0,1(4)}$$

$$\times \left(a(p^2 n) + \chi(p)\left(\frac{\varepsilon(-1)^k n}{p}\right) p^{k-1} a(n) + \chi^2(p) p^{2k-1} a(n/p^2)\right)$$

$W_{\alpha',\beta'}(ny)e[nx](p\nmid N)$ *and*

$$f|U(p^2)(z)=\sum_{n\in Z-\{0\},\varepsilon(-1)^k n\equiv 0,1(4)} a(p^2n)W_{\alpha',\beta'}(ny)e[nx]\ (p|N),$$

where $a(n/p^2)$ *means* 0 *if* $p^2\nmid n$.

Observe that $T^N_{(2k+1)/2,\lambda',\chi}(p)$ $(p\nmid N)$ coincides with the restriction of $T^{4N}_{(2k+1)/2,\lambda',\chi_1}(p^2)$ to $\mathscr{S}_{(2k+1)/2,\lambda'}(N,\chi)$.

§2. Shimura Correspondence and Some Formulas of Theta Integrals

This section is devoted to confirming an integral expression of Shimura correspondence and key proposition concerning theta integrals which are essential and useful for our results. Put

$$V=\left\{\xi=\begin{pmatrix} a & b\\ c & d\end{pmatrix}\in M_2(\boldsymbol{Q})\,\middle|\,\mathrm{tr}\,\xi=0\right\}\quad\text{and}$$

$\mathscr{S}(V)=\{\eta: V\to \boldsymbol{C}\,|\,\eta$ is a locally constant function in the sence of [14]$\}$.

Let η be an element of $\mathscr{S}(V)$. Define a theta function $\Theta(z,w;\eta)$ on $\mathfrak{H}\times\mathfrak{H}$ by

$$\Theta(z,w;\eta)=y^{1/2}\mathfrak{I}(w)^{-2k}\sum_{\xi\in V}\eta(\xi)[\xi,\bar{w}]^k e[2^{-1}R[\xi,z,w]] \tag{2-1}$$

for every $(z,w)\in\mathfrak{H}\times\mathfrak{H}$, where

$$[\xi,w]=[\xi,w,w],[\xi,w,w']=(cww'+dw-aw'-b)\quad\text{and}$$

$$R[\xi,z,w]=(\det\xi)z+\frac{iy}{2}\mathfrak{I}(w)^{-2}|[\xi,w]|^2\quad\left(\xi=\begin{pmatrix} a & b\\ c & d\end{pmatrix}\in V, y=\mathfrak{I}(z)\right).$$

Let ψ be the Hecke character given in (1-11). We denote by τ a positive square free integer such that

$$\tau\equiv 2,3\ (\mathrm{mod}\ 4)\quad\text{and}\quad(\tau,N)=1. \tag{2-2}$$

Put $\varphi=\psi\rho_\tau$ with the Hecke character ρ_τ associated with the quadratic field $\boldsymbol{Q}(\sqrt{\tau})$.

We put $e=2N$ and

$$\mathfrak{o}[e^{-1},e]=\left\{x=\begin{pmatrix} a_x & b_x\\ c_x & d_x\end{pmatrix}\in M_2(\boldsymbol{Q})\,\middle|\,a_x\in\boldsymbol{Z},b_x\in e^{-1}\boldsymbol{Z},c_x\in e\boldsymbol{Z}\text{ and }d_x\in\boldsymbol{Z}\right\}.$$

Define an element $\eta \in \mathscr{S}(V)$ by

(2-3) $$\eta(x) = \begin{cases} 0 & \text{if } x \notin \mathfrak{o}[e^{-1}, e], \\ \sum_{t \in (1/2\tau)\boldsymbol{Z}/2N\boldsymbol{Z}} \varphi_a(t)\varphi^*((2t\tau))e[-b_x t] & \text{otherwise.} \end{cases}$$

We may deduce the following proposition by the same method as that of [12] (cf. Niwa [9], Makdisi [1] and Kojima [6]).

PROPOSITION 2.1. *For* $f(z) = \sum_{n=-\infty}^{\infty} a(n) W_{\alpha',\beta'}(ny/2)e[nx/2] \in \mathscr{S}_{(2k+1)/2,\lambda'}(\boldsymbol{Z}, N\boldsymbol{Z}; \psi)$, *put* $h(z) = L_\tau(f)(z)$. *Suppose that* ψ_0, τ *and* ψ *satisfy the conditions* (2-2) *and* η *is a function on* V *determined by* (2-3). *Then there exists the even form*

(2-4)
$$g_\tau(w) = \sum_{m \in \boldsymbol{Z}-\{0\}} \left(\sum_{d|m, d>0} \psi_0(d) \left(\frac{-1}{d}\right)^k \left(\frac{\tau}{d}\right) d^{k-1} a(\tau(m/d)^2) \right) W_{2\alpha', 2\beta'}(mv)e[mu]$$

belonging to $\mathscr{S}_{2k,4\lambda'}(2N, \psi_0^2)$ *such that*

$$C' g_\tau(w) = \int_{\Gamma[2,2\tau N]\backslash\mathfrak{H}} h(z)\Theta(z, w; \eta) y^{(2k+1)/2} d_{\mathfrak{H}} z$$

with $w = u + iv$ *and* $C' = 2^{2+k} i^k \tau 4N$.

Define a mapping $\Psi^{4N,\psi_0}_{(2k+1)/2,\lambda',\tau} : \mathscr{S}_{(2k+1)/2,\lambda'}(\boldsymbol{Z}, N\boldsymbol{Z}; \psi) \to \mathscr{S}_{2k,4\lambda'}(2N, \psi_0^2)$ by

(2-5) $\quad \Psi^{4N,\psi_0}_{(2k+1)/2,\lambda',\tau}(f)(w) = g_\tau(w)$ for every $f(z) \in \mathscr{S}_{(2k+1)/2,\lambda'}(\boldsymbol{Z}, N\boldsymbol{Z}; \psi)$.

We call it the Shimura correspondence. Since $\mathscr{S}_{(2k+1)/2,\lambda'}(N, \chi)$ is contained in $\mathscr{S}_{(2k+1)/2,\lambda'}(4N, (\frac{4\varepsilon}{*})\chi)$, we have the following diagram.

(2-6) $\quad \mathscr{S}_{(2k+1)/2,\lambda'}(N, \chi) \subset \mathscr{S}_{(2k+1)/2,\lambda'}(4N, \psi_0) \xrightarrow{L^{-1}} \mathscr{S}_{(2k+1)/2,\lambda'}(\boldsymbol{Z}, N\boldsymbol{Z}; \psi)$ with

$$\psi_0 = \left(\frac{4\varepsilon}{*}\right)\chi.$$

By this relation, we may identify elements $f(z)$ of $\mathscr{S}_{(2k+1)/2,\lambda'}(N, \chi)$ with those of $\mathscr{S}_{(2k+1)/2,\lambda'}(\boldsymbol{Z}, N\boldsymbol{Z}; \psi)$.

Now we impose the following assumption.

(2-7) $\varepsilon = \chi(-1)$ satisfies $(-1)^k\varepsilon > 0$, χ is a primitive Dirichlet character modulo N, $f \in \mathscr{S}_{(2k+1)/2,\lambda'}(N, \chi)$ is an eigenfunction of all Hecke operators $T^N_{(2k+1)/2,\lambda',\chi}(p)$ $(p \nmid N)$ and $U(p^2)$ $(p|N)$, i.e., $f|T^N_{(2k+1)/2,\lambda',\chi}(p)(z) = \omega(p)f(z)$ $(p \nmid N)$ and $f|U(p^2)(z) = \omega(p)f(z)$ $(p|N)$

and $a(4\tau_0) \neq 0$ for some square free positive integer τ_0 such that $\tau_0 \equiv 2, 3 \pmod 4$.

We may derive the following lemma in a manner similar to that of Kojima [8].

LEMMA 2.2. *Suppose that the notation is the same as that of proposition* 2.1 *and the assumptions in proposition* 2.1 *and* (2-7) *are satisfied. Then*

$$\text{(2-8)} \quad \Psi^{4N,\chi(\frac{4\varepsilon}{*})}_{(2k+1)/2,\lambda',\tau}(f)(w)$$

$$= \sum_{m \in \boldsymbol{Z}-\{0\}} \left(\sum_{d|m, d>0} \left(\frac{4\tau}{d}\right) \chi(d) d^{k-1} a(\tau(m/d)^2) \right) W_{2\alpha', 2\beta'}(mv) e[mu]$$

coincides with an element $a(4\tau)g(2w)$ *of* $\mathscr{S}_{2k,4\lambda'}(2N, \chi^2)$, *where* $w = u + iv$ *and*

$$g(w) = \sum_{n \in \boldsymbol{Z}-\{0\}} b(n) W_{2\alpha', 2\beta'}(nv) e[nu]$$

is an element of $\mathscr{S}_{2k,4\lambda'}(N, \chi^2)$ *such that* $b_1 = 1$ *and* $g | T^N_{2k,4\lambda',\chi^2}(p)(w) = \omega(p) g(w)$ *for every prime* p.

We can define a function $f|U(4)$ on $\mathfrak{H}$ by

$$\text{(2-9)} \qquad f|U(4)(z) = \sum_{n \in \boldsymbol{Z}-\{0\}} a(4n) W_{\alpha', \beta'}(ny) e[nx] \in \mathscr{S}_{(2k+1)/2, \lambda'}(4N, \psi_0)$$

for every $f(z) = \sum_{n \in \boldsymbol{Z}-\{0\}} a(n) W_{\alpha', \beta'}(ny) e[nx] \in \mathscr{S}_{(2k+1)/2, \lambda'}(4N, \psi_0)$ (*cf. Shimura* [10]). Here we impose the further assumption on f in (2-7).

(2-10) The Dirichlet character χ is primitive and if $f' \in \mathscr{S}_{(2k+1)/2,\lambda'}\left(4N, \chi\left(\frac{4\varepsilon}{*}\right)\right)$ satifies $f' | T^{4N}_{(2k+1)/2,\lambda',\chi(\frac{4\varepsilon}{*})}(p^2) = \omega(p) f'$

for every prime $p (p \nmid 4\tau N)$, then

$$f'(z) = \tilde{c} f(z) + \tilde{c}' f|U(4)(z) \quad \text{for some constants } \tilde{c} \text{ and } \tilde{c}'.$$

Moreover, we impose the condition that

(2-11) if $g' \in \mathscr{S}_{2k,4\lambda'}(N, \chi^2)$ satifies $g' | T^N_{2k,4\lambda',\chi^2}(p)(w) = \omega(p) g'(w)$ for almost all prime p,

then g' is a constant multiple of g and $f|U(4)(z)$ is not a constant times $f(z)$, where $f(z)$, $g(w)$ and $\omega(p)$ are the same elements given in Lemma 2.2. We consider the assumption that

(2-12) if $2|\tau$, then the conductor of φ is $4N\tau$ and $\varphi_2(1+4x) = \varphi_2(1+4x^2)$ for every $x \in \boldsymbol{Z}_2$,

where φ_2 is the restriction of φ to $\boldsymbol{Q}_2^\times$ and $\boldsymbol{Z}_2$ (resp. $\boldsymbol{Q}_2$) means the ring of all 2-adic integers (numbers).

Let ζ^Z and ζ_Z denote two elements in $\mathscr{S}(V)$ determined by

$$\zeta^Z(x) = \begin{cases} \bar{\varphi}_a(b_x)\bar{\varphi}^*((b_x e)) & \text{if } x \in \mathfrak{o}[e^{-1}, e], \\ 0 & \text{otherwise} \end{cases} \tag{2-13}$$

and

$$\zeta_Z(x) = \begin{cases} \bar{\varphi}_a(b_x)\bar{\varphi}^*((b_x e)) & \text{if } x \in \mathfrak{o}[e^{-1}, e] \text{ and } (b_x e, 4N\tau) = 1, \\ 0 & \text{otherwise,} \end{cases}$$

where $e = 2N$. By the same method as that of Shimura [13], we may derive the following lemma.

LEMMA 2.3. *Suppose that N is odd, the conductor of $\psi\rho_\tau$ is $4N$ with ψ in* (2-6) *and the conditions* (2-2), (2-7), (2-10), (2-11) *and* (2-12) *are satisfied. Then there are constants M and M' such that*

$$\langle \Theta(z, w; \zeta^Z), g(2w) \rangle = Mh(z) + M'h|U(4)(z) \quad \text{with } h(z) = L_\tau(f)(z),$$

where $g(w)$ is the same function given in Lemma 2.2.

By virtue of Proposition 2.1, Lemma 2.2 and Lemma 2.3 and the arguments in Kojima [8], we may deduce the following proposition.

PROPOSITION 2.4. *Suppose that the assumption in Lemma* 2.3 *is satisfied. Then we have*

$$Ah(z) + Bh|U(4)(z) = \langle \Theta(z, w; \eta), g(2w) \rangle$$

and

$$Ch(z) + Dh|U(4)(z) = \langle \Theta(z, w; \zeta_Z), g(2w) \rangle$$

with

$$A=\overline{a(4\tau)}\,\bar{C}'\,\mathrm{vol}(\Gamma[2,2N\tau]\backslash\mathfrak{H})^{-1}\frac{\langle g(2w),g(2w)\rangle\langle h',h'\rangle-\langle g(w),g(2w)\rangle\langle h,h'\rangle}{\langle h,h\rangle\langle h',h'\rangle-\langle h,h'\rangle\langle h',h\rangle},$$

$$B=\overline{a(4\tau)}\,\bar{C}'\,\mathrm{vol}(\Gamma[2,2N\tau]\backslash\mathfrak{H})^{-1}\frac{\langle g(w),g(2w)\rangle\langle h,h\rangle-\langle g(2w),g(2w)\rangle\langle h',h\rangle}{\langle h,h\rangle\langle h',h'\rangle-\langle h,h'\rangle\langle h',h\rangle},$$

$$A=\varphi_a(-1)\overline{\gamma(\varphi)}C,\quad B=\varphi_a(-1)\overline{\gamma(\varphi)}D,$$

where η (*resp.* C') *is the same element given in* (2-3) (*resp.* (2-4)), $h'(z)=h|U(4)(z)$ *and* $\gamma(\varphi)$ *means the Gauss sum of* φ.

§3. Rankin's Convolution of Theta Series, Eisenstein Series and Final Calculation

Put

$$\vartheta(z)=\sum_{n=-\infty}^{\infty}e[n^2z/2]\quad\text{and}\quad L_M(s,\omega)=\sum_{n=1}^{\infty}\omega^*(n\boldsymbol{Z})n^{-s}$$

for each Hecke character ω of $\boldsymbol{Q}$ and for each positive integer M, where n runs over all positive integers such that $(n,M)=1$ and ω^* is the ideal character associated with ω (cf. [14, p. 505]). Let $g(w)$ and $h(z)$ be the same functions in Lemma 2.3. For a subgroup Γ of $SL(2,\boldsymbol{Z})$, we put $\Gamma_\infty=\Gamma\cap\left\{\pm\begin{pmatrix}1&n\\0&1\end{pmatrix}\middle|\,n\in\boldsymbol{Z}\right\}$.

We consider an integral

$$\text{(3-1)}\qquad \int_{\Gamma\backslash\mathfrak{H}}h(z)\overline{\vartheta(z)C(z,\bar{s}+1/2:k,\bar{\varphi},\Gamma)}\,y^{(2k+1)/2}\,d_{\mathfrak{H}}z\quad(z=x+iy),$$

where $\Gamma=\Gamma[2,2\tau N]$, $C(z,s:k,\bar{\varphi},\Gamma)=L_{4N\tau}(2s,\bar{\varphi})E(z,s:k,\bar{\varphi},\Gamma)$, $C(z,s:k,\bar{\varphi},\Gamma)$ and $E(z,s:k,\bar{\varphi},\Gamma)$ means functions given in [14, (4-6) and (4-11)]. By the same method as that of Shimura [14, p. 542], we may reduced (3-1) to the form

$$\text{(3-2)}\qquad L_{4N\tau}(2s+1,\varphi)\int_{\Gamma\backslash\mathfrak{H}}h(z)\overline{\vartheta(z)E(z,\bar{s}+1/2:k,\bar{\varphi},\Gamma)}\,y^{(2k+1)/2}\,d_{\mathfrak{H}}z$$
$$=L_{4N\tau}(2s+1,\varphi)\int_{\Psi}h(z)\overline{\vartheta(z)}\,y^{s+1+k/2}\,d_{\mathfrak{H}}z$$

with $\Psi=\Gamma[2,2\tau N]_\infty\backslash\mathfrak{H}$, which implies that the integral (3-1) equals

$$L_{4N\tau}(2s+1,\varphi)\tau^k \int_0^2\int_0^\infty \sum_{m\in \boldsymbol{Z}-\{0\}} a(m)W_{\alpha',\beta'}(my/2)e[mx/2] \cdot \sum_{n=-\infty}^{\infty} \overline{\exp(2\pi i n^2(x+iy)/2)}\, y^{s+1+k/2}\, d_{\mathfrak{H}}z. \tag{3-3}$$

Since

$$\int_0^\infty V(y;\alpha;\beta)\exp(-y/2)y^{(s/2)-1}\,dy = \frac{\Gamma((s/2)+\alpha)\Gamma((s/2)+\beta)}{\Gamma((s/2)+\alpha+\beta)},$$

the integral (3-1) is equal to

$$a(4\tau)\frac{4\tau^k}{(2\pi\tau^2)^{s+k/2}}\frac{\Gamma(s+(k/2)+\alpha')\Gamma(s+(k/2)+\beta')}{\Gamma(s+(k/2)+\alpha'+\beta')}2^{-(2s+k)}L(2s+k,g) \tag{3-4}$$

with

$$L(s,g)=\sum_{n=1}^{\infty} b(n)n^{-s} \quad\text{and}\quad g(w)=\sum_{n\in\boldsymbol{Z}-\{0\}} b(n)W_{2\alpha',2\beta'}(nv)e[nu].$$

A calculation similar to the above one shows that

$$\int_{\Gamma[2,2\tau N]\backslash\mathfrak{H}} h'(z)\overline{\vartheta(z)C(z,\bar{s}+1/2:k,\bar{\varphi},\Gamma)}\,y^{(2k+1)/2}\,d_{\mathfrak{H}}z \tag{3-5}$$

$$= a(4\tau)\frac{4\tau^k}{(2\pi\tau^2)^{s+k/2}}\frac{\Gamma(s+(k/2)+\alpha')\Gamma(s+(k/2)+\beta')}{\Gamma(s+(k/2)+\alpha'+\beta')}L(2s+k,g).$$

Next we calculate an integral

$$\int_{\Gamma[2\tau,4N]\backslash\mathfrak{H}} g(2w)\overline{C(w,\bar{s}+1/2:\bar{\varphi})E(w,\bar{t}+1/2:\bar{\varphi})}\,\mathfrak{I}(w)^{2k}\,d_{\mathfrak{H}}w, \tag{3-6}$$

where $C(w,s:\bar{\varphi})=C(w,s:k,\bar{\varphi},\Gamma[2\tau,4N])$ and $E(w,t:\bar{\varphi})=E(w,t:k,\bar{\varphi},\Gamma[2\tau,4N])$ are given in [14, (4-6) and (4-11)]. By a method similar to that of [14, p. 550], the integral (3-6) may be reduced to the form:

$$\int_{\Gamma[2\tau,4N]_\infty\backslash\mathfrak{H}} g(2w)\overline{C(w,\bar{s}+1/2:\bar{\varphi})}\,\mathfrak{I}(w)^{t+(1+3k)/2}\,d_{\mathfrak{H}}w \tag{3-7}$$

$$= \int_0^{2\tau}\int_0^\infty g(2w)\overline{C(w,\bar{s}+1/2:\bar{\varphi})}\,\mathfrak{I}(w)^{t+(1+3k)/2}\,d_{\mathfrak{H}}w.$$

We recall the following formula (cf. [14, p. 531]).

(3-8) $$v^{-s+k/2}C(w,s:\bar{\varphi})$$
$$= L_{8N\tau}(2s,\bar{\varphi}) + \gamma(\bar{\varphi})(4N\tau)^{-1} \sum_{h'\in \boldsymbol{Z}, c'\in \boldsymbol{Z}(c'>0)} (4Nc')^{1-2s}\varphi^*((h'))$$
$$\cdot\, \varphi_a(h'/4N\tau)e[(c'h'/\tau)u]\xi(v,h'c'/\tau;s+k/2,s-k/2),$$

where $w = u + iv$ and

$$\xi(y,h;l,m)$$
$$= \int_{-\infty}^{\infty} e[-hx](x+iy)^{-l}(x-iy)^{-m}\,dx \quad (y>0,\ h\in \boldsymbol{R}, l,m\in \boldsymbol{C},\ l-m\in \boldsymbol{Z}).$$

The function $\xi(y,h;l,m)$ has the following integral expression

$$\xi(y,h;l,m)$$
$$= i^{m-l}(2\pi)^{l+m}\Gamma(l)^{-1}\Gamma(m)^{-1}h^{l+m-1}e^{-2\pi hy}\int_0^{\infty} e^{-4\pi hyt}(t+1)^{l-1}t^{m-1}\,dt$$

$(y>0, h>0, (l,m)\in \boldsymbol{C}^2, \Re(m)>0)$ and

$$\xi(y,h;l,m) = |h|^{l+m-1}\xi(|h|y,\mathrm{sgn}(h);l,m).$$

Hence, the above integral (3-7) can be rewritten as follows:

(3-9) $$2\tau\overline{\gamma(\bar{\varphi})}(4N\tau)^{-1} \sum_{h'\in \boldsymbol{Z}, c'\in \boldsymbol{Z}(c'>0)} (4Nc')^{-2s}\overline{\varphi_a(h')\varphi^*((h'))}b(c'h'/2\tau)$$
$$\cdot \int_0^{\infty} W_{2\alpha',2\beta'}(|c'h'/\tau|\,\mathrm{sgn}(c'h'/\tau)v)|h'c'/\tau|^{2s}$$
$$\times \overline{\xi(|h'c'/\tau|v,\mathrm{sgn}(h'c'/\tau);\bar{s}+(1+k)/2,\bar{s}+(1-k)/2)}$$
$$\times v^{t+(1+3k)/2}v^{s+(1/2)-(k/2)-2}\,dv$$

with

$$g(w) = \sum_{n\in \boldsymbol{Z}-\{0\}} b(n)W_{2\alpha',2\beta'}(nv)e[nu].$$

Since $g(w)$ is even,

$$\varphi_a(h')b(c'h'/2\tau) = (-1)^k(-1)^k(2\alpha')_k(2\beta')_k b(|c'h'/2\tau|) \quad (h'<0).$$

Hence (3-9) is equal to

$$(3\text{-}10)\qquad 2\tau\overline{\gamma(\bar{\varphi})}(4N\tau)^{-1}\sum_{h'\in\mathbf{Z},c'\in\mathbf{Z}(h'>0,c'>0)}(4Nc')^{-2s}\overline{\varphi^*((h'))}(h'c'/\tau)^{-(s+t+k-1)}$$

$$\times\, b(c'h'/2\tau)(h'c'/\tau)^{2s-1}\sum_{\sigma\in\{\pm 1\}}M(s,t,\sigma),$$

where

$$M(s,t,\sigma)=\alpha(\sigma)\int_0^\infty W_{2\alpha',2\beta'}(v\sigma)\overline{\xi\left(v,\sigma;\bar{s}+\frac{k+1}{2},\bar{s}+\frac{1-k}{2}\right)}v^{s+t+k-1}\,dv$$

with $\alpha(\sigma)=\begin{cases}1 & \text{if } \sigma=1,\\ (2\alpha')_k(2\beta')_k & \text{if } \sigma=-1.\end{cases}$ We can check that

$$(3\text{-}11)\qquad \sum_{h'=1}^{\infty}\sum_{c'=1,2\tau|c'h'}^{\infty}(c')^{-2s}\overline{\varphi_a(h')\varphi^*((h'))}b(c'h'/2\tau)(h'c'/\tau)^{s-t-k}$$

$$=(\tau)^{-2s}2^{-s-t-k}\sum_{h'=1}^{\infty}\sum_{c''=1}^{\infty}\bar{\varphi}^*((h'))b(h'c'')(c'')^{-s-t-k}(h')^{-t+s-k}.$$

Putting $s=t$, the sum (3-11) equals

$$(3\text{-}12)\qquad (2\tau)^{-2s}2^{-k}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\bar{\varphi}^*((m))b(mn)m^{-k}n^{-2s-k}$$

$$=(2\tau)^{-2s}2^{-k}L_{4N}(2s+1,\varphi)^{-1}L(2s+k,g)L(k,g,\bar{\varphi}),$$

where

$$L(s,g,\bar{\varphi})=\sum_{n=1}^{\infty}\bar{\varphi}^*((n))c(n)n^{-s}\quad\text{and}\quad g(w)=\sum_{n\in\mathbf{Z}-\{0\}}c(n)W_{2\alpha',2\beta'}(nv)e[nu].$$

The following formula was verified by K. K-Makdisi [1]

$$(3\text{-}13)\qquad \sum_{\sigma\in\{\pm 1\}}M(s,s,\sigma)$$

$$=i^k(2\pi)^{-k}\frac{\Gamma(s+(k/2)+\alpha')\Gamma(s+(k/2)+\beta')\Gamma(\alpha'+k)\Gamma(\beta'+k)}{\Gamma(s+(1-k)/2)\Gamma(s+(1+k)/2)}.$$

Employing (3-9), (3-12) and (3-13), we find that

(3-14)
$$\int_{\Gamma[2\tau,4N]\backslash\mathfrak{H}} g(2w)\overline{C(w,\bar{s}+1/2:\bar{\varphi})E(w,\bar{s}+1/2,\bar{\varphi})}\Im(w)^{2k}\,d_{\mathfrak{H}}w$$
$$= A(s)L_{4N}(2s+1,\varphi)^{-1}L(2s+k,g)L(k,g,\bar{\varphi}),$$

where

$$A(s) = 2\tau\overline{\gamma(\bar{\varphi})}(4N\tau)^{-1}(4N)^{-2s}(2\tau)^{-2s}2^{-k}i^k(2\pi)^{-k}$$
$$\times\frac{\Gamma(s+(k/2)+\alpha')\Gamma(s+(k/2)+\beta')\Gamma(\alpha'+k)\Gamma(\beta'+k)}{\Gamma(s+(1-k)/2)\Gamma(s+(1+k)/2)}.$$

Exchanging the order of integration, we have

(3-15)
$$\int_{\Gamma[2,2\tau N]\backslash\mathfrak{H}}\langle\Theta(z,w;\zeta_Z),g(2w)\rangle\,\overline{\vartheta(z)C(z,\bar{s}+1/2:k,\bar{\varphi},\Gamma[2,2\tau N])}\,y^{(2k+1)/2}\,d_{\mathfrak{H}}z$$
$$=\int_{\Gamma[2,2\tau N]\backslash\mathfrak{H}}\operatorname{vol}(\Gamma[2\tau,4N]\backslash\mathfrak{H})^{-1}\left\{\int_{\Gamma[2\tau,4N]\backslash\mathfrak{H}}\overline{\Theta(z,w;\zeta_Z)}g(2w)\right.$$
$$\left.\times\,\Im(w)^{2k}\,d_{\mathfrak{H}}w\right\}\overline{\vartheta(z)C(z,\bar{s}+1/2:k,\bar{\varphi},\Gamma[2,2\tau N])}\,y^{(2k+1)/2}\,d_{\mathfrak{H}}z$$
$$=\operatorname{vol}(\Gamma[2\tau,4N]\backslash\mathfrak{H})^{-1}\int_{\Gamma[2\tau,4N]\backslash\mathfrak{H}}\left\{\overline{\int_{\Gamma[2,2\tau N]\backslash\mathfrak{H}}\vartheta(z)\Theta(z,w;\zeta_Z)}\right.$$
$$\left.\times\,\overline{C(z,\bar{s}+1/2:k,\bar{\varphi})\,y^{(2k+1)/2}\,d_{\mathfrak{H}}z}\right\}g(2w)\Im(w)^{2k}\,d_{\mathfrak{H}}w$$
$$=\langle M'(w,\bar{s}),g(2w)\rangle,$$

where

$$M'(w,s)=\int_{\Gamma[2,2\tau N]\backslash\mathfrak{H}}\vartheta(z)\Theta(z,w;\zeta_Z)C(z,s+1/2:k,\bar{\varphi},\Gamma[2,2\tau N])\,y^{(2k+1)/2}\,d_{\mathfrak{H}}z.$$

For a function $p(z)$ on $\mathfrak{H}$ and a $\sigma=\begin{pmatrix}a & b\\ c & d\end{pmatrix}\in SL(2,\boldsymbol{R})$, $p\|_l\sigma(z)$ means $p(\sigma(z))\cdot(cz+d)^{-l}$. Observing that

(3-16)
$$C(z,s:k,\bar{\varphi},\Gamma[2,2\tau N])$$
$$=L_{4\tau N}(2s,\bar{\varphi})\sum_{\sigma\in\Gamma[2,2\tau N]_\infty\backslash\Gamma[2,2\tau N]}\bar{\varphi}_a(d)\bar{\varphi}^*((d))\Im(z)^{s-(k/2)}\|_k\sigma(z),$$

we obtain

$$\text{(3-17)} \qquad M'(w,s) = L_{4N\tau}(2s+1,\bar{\varphi}) \int_{\sigma \in \Gamma[2,2\tau N]_\infty \backslash \mathfrak{H}} \vartheta(z)\Theta(z,w;\zeta_Z) y^{s+1+(k/2)} \, d_{\mathfrak{H}} z$$

$$= B(s) L_{4N\tau}(2s+1,\bar{\varphi}) S'(w,s),$$

where

$$B(s) = 2\frac{2^{s+(k+1)/2}}{\pi^{s+(k+1)/2}} \Gamma(s+(k+1)/2),$$

$$S'(w,s) = \sum_{(\sigma,b)\in X} \zeta_Z(\sigma)\mu(b)[\sigma,w]^{-k} \left| \frac{[\sigma,w]}{\Im(w)} \right|^{2k-2t'},$$

$$X = \{(\sigma,b) \in V \times \boldsymbol{Q} \mid \sigma \neq 0, -\det\sigma = b^2\}, \quad \mu(b) = \begin{cases} 1 & \text{if } b \in \boldsymbol{Z}, \\ 0 & \text{if } b \in \boldsymbol{Q} - \boldsymbol{Z} \end{cases}$$

and $t' = s + (1+k)/2$.

Combining Proposition 2.4 with (3-4), (3-5), (3-15) and (3-17), we may deduce that

$$\text{(3-18)} \qquad \int_{\Gamma[2,2\tau N]\backslash\mathfrak{H}} Ch(z)\overline{\vartheta(z)C(z,\bar{s}+1/2:k,\bar{\varphi},\Gamma[2,2\tau N])} y^{(2k+1)/2} \, d_{\mathfrak{H}} z$$

$$+ \int_{\Gamma[2,2\tau N]\backslash\mathfrak{H}} Dh'(z)\overline{\vartheta(z)C(z,\bar{s}+1/2:k,\bar{\varphi},\Gamma[2,2\tau N])} y^{(2k+1)/2} \, d_{\mathfrak{H}} z$$

$$= a(4\tau)\frac{4\tau^k}{(2\pi\tau^2)^{s+k/2}} \frac{\Gamma(s+(k/2)+\alpha')\Gamma(s+(k/2)+\beta')}{\Gamma(s+(k/2)+\alpha'+\beta')}$$

$$(C2^{-(2s+k)} + D)L(2s+k,g)$$

$$= \int_{\Gamma[2,2\tau N]\backslash\mathfrak{H}} \langle \Theta(z,w;\zeta_Z), g(2w) \rangle$$

$$\times \overline{\vartheta(z)C(z,\bar{s}+1/2:k,\bar{\varphi},\Gamma[2,2\tau N])} y^{(2k+1)/2} \, d_{\mathfrak{H}} z$$

$$= \overline{B(\bar{s})} L_{4N\tau}(2s+1,\varphi) \langle S'(w,\bar{s}), g(2w) \rangle.$$

By [14, (7.9a) and (7.13)], we have

$$\text{(3-19)} \qquad S'(w,s) = (-1)^k \sum_{q \in 2N\boldsymbol{Z}/4N\boldsymbol{Z}} T'(w,s)\|_{2k} W(q) \quad \text{and}$$

$$T'\left(w, s-\frac{1}{2}\right) = (2N)^{2s} C(w,s;\bar{\varphi}) E(w,s;\bar{\varphi}) \quad \text{with } W(q) = \begin{pmatrix} 1 & 0 \\ q & 1 \end{pmatrix}.$$

Employing (3-14) and (3-19), we derive

(3-20) $\langle M'(w,\bar{s}), g(2w)\rangle$

$$= \overline{B(\bar{s})}L_{4N\tau}(2s+1,\varphi)(-1)^k(2N)^{2s+1}$$

$$\times \left\langle \sum_{q\in 2N\boldsymbol{Z}/4N\boldsymbol{Z}} C(w,\bar{s}+1/2;\bar{\varphi})E(w,\bar{s}+1/2;\bar{\varphi})\|_{2k}W(q), g(2w)\right\rangle$$

$$= 2(-1)^k\overline{B(\bar{s})}L_{4N\tau}(2s+1,\varphi)(2N)^{2s+1}$$

$$\times \langle C(w,\bar{s}+1/2;\bar{\varphi})E(w,\bar{s}+1/2;\bar{\varphi}), g(2w)\rangle$$

$$= 2(-1)^k\overline{B(\bar{s})}(2N)^{2s+1}A(s)\,\mathrm{vol}(\Gamma[2\tau,4N]\backslash\mathfrak{H})^{-1}$$

$$\times L(2s+k,g)L(k,g,\bar{\varphi}).$$

From this equality and (3-18), we conclude that

(3-21) $$a(4\tau)\frac{4\tau^k}{(2\pi\tau^2)^{s+k/2}}(C2^{-(2s+k)}+D)$$

$$= (-1)^k 8\overline{\left(\frac{2^{\bar{s}+(k+1)/2}}{\pi^{\bar{s}+(k+1)/2}}\right)}(2N)^{2s+1}\tau\overline{\gamma(\bar{\varphi})}$$

$$\times (4N\tau)^{-1}(4N)^{-2s}(2\tau)^{-2s}2^{-k}i^k(2\pi)^{-k}\Gamma(\alpha'+k)$$

$$\times \Gamma(\beta'+k)\,\mathrm{vol}(\Gamma[2\tau,4N]\backslash\mathfrak{H})^{-1}L(k,g,\bar{\varphi}).$$

Putting $2s+k=0$, we may deduce that

(3-22)

$$a(4\tau)(C+D)$$

$$= (-1)^k\overline{\gamma(\bar{\varphi})}2^{1/2}i^k\pi^{-k-(1/2)}\Gamma(\alpha'+k)\Gamma(\beta'+k)\,\mathrm{vol}(\Gamma[2\tau,4N]\backslash\mathfrak{H})^{-1}L(k,g,\bar{\varphi}).$$

By Proposition 2.4, we see that

(3-23) $$C+D=\overline{a(4\tau)}\frac{\mathrm{vol}(\Gamma[2,2N\tau]\backslash\mathfrak{H})^{-1}}{(\varphi_a(-1)\overline{\gamma(\varphi)})}\bar{C}'E\tau^{-(2k+1)/2+1} \text{ with}$$

$$E=\frac{\langle g(2w),g(2w)\rangle(\langle f',f'\rangle-\langle f',f\rangle)+\langle g(w),g(2w)\rangle(\langle f,f\rangle-\langle f,f'\rangle)}{\langle f,f\rangle\langle f',f'\rangle-\langle f,f'\rangle\langle f',f\rangle},$$

where $f'(z)=f|U(4)(z)$.

Consequently, by (3-22) and (3-23), we conclude the following theorem.

THEOREM. *Let the notation be as above. Suppose the assumption in Lemma* 2.3. *Then we have*

$$|a(4\tau)|^2 E = (-1)^k \overline{\gamma(\bar{\varphi})} 2^{-(1/2)} i^k \pi^{-k-(1/2)} \bar{C}'^{-1} \tau^{(2k+1)/2-1}$$
$$\times \Gamma(\alpha' + k)\Gamma(\beta' + k)(\varphi_a(-1)\overline{\gamma(\varphi)}) L(k, g, \bar{\varphi}).$$

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