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ATOMIC DECOMPOSITION FOR SOBOLEV SPACES AND FOR THE C_p^α SPACES ON GENERAL DOMAINS

Dedicated to Professor Satoru Igari on the occasion of his sixtieth birthday

By

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§1. Introduction

Atomic decomposition for the Hardy spaces H^p, $0 < p \leqq 1$, is well known. In this paper, we shall give a variant of atomic decomposition which applies to the Sobolev spaces and to the C_p^α spaces of DeVore and Sharpley ([DS]) on general domains. In this section, we shall briefly review our results.

We shall first fix several notations which will be used throughout the paper. In this paper we consider functions defined on $\boldsymbol{R}^n$ or on a subset of $\boldsymbol{R}^n$; the letter n always denotes the dimension of the basic space $\boldsymbol{R}^n$. We also use the letters k, α, p, and Ω in the following fixed meaning: k denotes a nonnegative integer; α denotes a positive real number; p denotes a positive real number or ∞; and Ω denotes an open subset of $\boldsymbol{R}^n$. We shall call a Lebesgue measurable function merely a function. For a Lebesgue measurable set $E \subset \boldsymbol{R}^n$, the $L^p(E)$-quasinorm of a function f on E is defined by

$$\|f\|_{p,E} = \|f; \Gamma(1/p; E)\| = \left(\int_E |f(x)|^p dx\right)^{1/p}$$

with the usual modification in the case $p = \infty$, and the set of functions f on E such that $\|f\|_{p,E} < \infty$ is denoted by $L^p(E)$ or by $\Gamma(1/p; E)$. (Thus the two symbols $\|f\|_{p,E}$ and $\|f; \Gamma(1/p; E)\|$ denote exactly the same thing and so do the two symbols $L^p(E)$ and $\Gamma(1/p; E)$; we shall use whichever will be convenient.) We often abbreviate $\|f\|_{p,\boldsymbol{R}^n} = \|f; \Gamma(1/p; \boldsymbol{R}^n)\|$ to $\|f\|_p = \|f; \Gamma(1/p)\|$ and $L^p(\boldsymbol{R}^n) = \Gamma(1/p; \boldsymbol{R}^n)$ to $L^p = \Gamma(1/p)$. The Lebesgue measure of $E \subset \boldsymbol{R}^n$ is denoted by $|E|$. For $x = (x_1, \dots, x_n) \in \boldsymbol{R}^n$, we write

$$|x| = \left(\sum_{j=1}^n x_j^2\right)^{1/2} \quad \text{and} \quad \|x\| = \max\{|x_i|; i = 1, \dots, n\}.$$

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By a cube, we mean a closed cube in $\boldsymbol{R}^n$ with sides parallel to the coordinate axes, i.e., a cube Q is a subset of $\boldsymbol{R}^n$ of the form $Q = \{x \in \boldsymbol{R}^n; \|x - a\| \leqq t\}$, $0 < t < \infty$; we write $x_Q = a$ (the center of Q) and $\ell(Q) = 2t$ (the sidelength of Q). For a cube Q and for $0 < t < \infty$, we define tQ as the cube with the same center as Q and with sidelength t times as large as Q. The symbol $\mathscr{P}_k$ denotes the set of the polynomial functions on $\boldsymbol{R}^n$ of degree less than or equal to k. If a function f has classical derivatives $\partial^\nu f$ for all ν with $|\nu| = k$, then we write $|\nabla^k f| = \sum_{|\nu|=k} |\partial^\nu f|$. Some other notations will be explained at the last paragraph of this section.

Now, let D be a cube or an open subset of $\boldsymbol{R}^n$ and let f a function on D. For cubes $Q \subset D$, we define

$$I_{k,p,Q}(f) = \inf\{|Q|^{-1/p}\|f - P\|_{p,Q}; P \in \mathscr{P}_k\}.$$

We define

$$f^{\#}_{k,\alpha,p}(x) = \sup\{\ell(Q)^{-\alpha} I_{k,p,Q}(f); Q \text{ cube}, x \in Q \subset D\}, \quad x \in D,$$

and

$$|f; C^{\alpha,k}_p(D)| = \|f^{\#}_{k,\alpha,p}\|_{p,D}.$$

We shall often abbreviate $|f; C^{\alpha,k}_p(\boldsymbol{R}^n)|$ as $|f; C^{\alpha,k}_p|$.

Variants of the above $f^{\#}_{k,\alpha,p}$ and $|f; C^{\alpha,k}_p(D)|$ are considered by many people. At least, DeVore and Sharpley [DS], M. Christ [Chr], and Bojarski [B] gave almost the same definitions as above. The idea can be traced back to many older works; cf. Triebel [Tri; Remark 1.7.2/1].

In considering $|f; C^{\alpha,k}_p(D)|$, the cases $k < \alpha \leqq k + 1$ and $k \leqq \alpha < k + 1$ seem to be most important. In fact, it is known that the case $k = \alpha - 1$ with α a positive integer corresponds to the classical Sobolev space (see Remark (1°) to be given at the last part of this section), and that, at least for sufficiently nice D, the case $k > \alpha$ is essentially equivalent to the case $k \leqq \alpha < k + 1$ (see [DS; Lemma 4.4]). We shall, however, give our results without imposing restrictions on k so long as possible.

Let $\Omega \neq \boldsymbol{R}^n$. For $x \in \Omega$ and for t with $0 < t < 1$, we define

$$\rho_\Omega(x) = \min\{\|x - y\|; y \in \Omega^c\}$$

and

$$Q_t(x) = \{y; \|y - x\| \leqq t\rho_\Omega(x)\}.$$

For functions f on Ω and for $0 < t < 1$, we define

$$f^{*,t}(x) = \|f\|_{\infty,Q_t(x)}, \quad x \in \Omega.$$

We shall often abbreviate $f^{*,1/2}$ to f^*.

Now, let $\Omega \neq \boldsymbol{R}^n$, $k+1 \geqq \alpha$, and $p < \infty$. Suppose g is a smooth function on Ω such that

$$\|\rho_\Omega^{k+1-\alpha}|\nabla^{k+1}g|^*\|_{p,\Omega} < \infty. \tag{1.1}$$

Also suppose that $\{\varphi_j\}$ is a sequence of functions on $\boldsymbol{R}^n$, $\{Q_j\}$ is a sequence of cubes, and $\{\lambda_j\}$ is a sequence of nonnegative real numbers, and that they satisfy

$$\operatorname{supp}\varphi_j \subset Q_j, \quad 2Q_j \subset \Omega, \quad |\varphi_j; C_\infty^{\alpha,k}(\boldsymbol{R}^n)| \leqq \lambda_j, \tag{1.2}$$

and

$$\left\|\left(\sum_j \lambda_j^q \chi_{Q_j}\right)^{1/q}\right\|_p < \infty \tag{1.3}$$

for $q = \min\{p, 1\}$. Then it is rather easy to see that the series $\sum_j \varphi_j$ converges in $L_{\mathrm{loc}}^r(\Omega)$ for some $r > 0$ and the function $f = g + \sum_j \varphi_j$ satisfies

$$|f; C_p^{\alpha,k}(\Omega)| < \infty; \tag{1.4}$$

see Theorems 3.1 and 4.1 of the present paper. It is the main purpose of the present paper to prove the converse of this fact. To be precise, we shall prove the following result, which is slightly stronger than the pure converse: Every function f on Ω satisfying (1.4) can be written as $f = g + \sum_j \varphi_j$ with a smooth function g satisfying (1.1) and with $(\varphi_j, Q_j, \lambda_j)$ satisfying (1.2) and (1.3) for *all* $q > 0$. This result will be given in Theorems 4.2 and 5.1. We shall also give similar result for the case $\Omega = \boldsymbol{R}^n$; see Theorems 4.3 and 5.2.

R. G. Durán [Dur] has already obtained a result which is closely related to our result. Durán treats a maximal function which is different from ours, but it is known that Durán's maximal function (for the case where his dilation operator A_t is the usual dilation $x \to tx$) is equivalent to our $f_{k,\alpha,p}^{\#}$ with $k < \alpha \leqq k+1$ (see [DS; Theorem 5.3]). One of Durán's result in [Dur] gives an atomic decomposition of functions f on $\boldsymbol{R}^n$ with $|f; C_p^{\alpha,k}(\boldsymbol{R}^n)| < \infty$ in the case $k < \alpha \leqq k+1$ and $1 + \alpha/n > 1/p \geqq 1$. Compared with this result of Durán, our results contain the following improvements. First, we treat the functions on arbitrary open subset of $\boldsymbol{R}^n$; the function g is peculiar to our situation. Secondly, we treat the full range $0 < \alpha$, $p < \infty$; thus, in particular, our result covers also the case of classical Sobolev spaces (cf. Remark (1°) below). Thirdly, we give several 'mod 0' estimates (for example we give L^q estimates for the function g and for the series $\sum_j \varphi_j$), whereas in [Dur] the convergence of the atomic series is considered mod polynomials.

In a forthcoming paper, [Mi4], the same author will consider the estimates of the pointwise product of functions in terms of $|\cdot; C_p^{\alpha,k}(\Omega)|$, where the results of the present paper will be effectively used.

The contents of the succeeding sections are as follows. In Section 2, we give several preliminary lemmas. In Section 3, we consider the series $\sum_j \varphi_j$ arising from (1.2) and (1.3). In Section 4, we consider functions g satisfying (1.1); in particular, we give a method to associate with each function f satisfying (1.4) a function g satisfying (1.1). In Section 5, we give the main decomposition theorems, Theorems 5.1 and 5.2.

The following remarks will help the reader to understand the meaning of the quasinorm $|\cdot; C_p^{\alpha,k}(\Omega)|$.

REMARK. (1°) If m is a positive integer and $1/p < 1 + m/n$, then for locally integrable functions f on Ω the quasinorm $|f; C_p^{m,m-1}(\Omega)|$ is equivalent to

$$\sum_{|\nu|=m} \|\partial^\nu f\|_{p,\Omega} \quad (\text{if } p > 1) \quad \text{or} \quad \sum_{|\nu|=m} \|\partial^\nu f\|_{H^p(\Omega)} \quad (\text{if } 1 \leqq 1/p < 1 + m/n),$$

where $\partial^\nu f$ denotes the derivative in the sense of distribution and $\|\cdot\|_{H^p(\Omega)}$ denotes the quasinorm of the H^p space on Ω as given in [Mi2]. This result for $p > 1$ is due to A. P. Calderón [Cal; Theorem 4 and Lemma 7]; proof can be found also in [Chr; Lemma 2.2] or [DS; Theorem 6.2]. The result for $p \leqq 1$ is due to Durán [Dur] and Miyachi [Mi3].

(2°) Let $k < \alpha \leqq k + 1$ and let f be a function on Ω. Then $|f; C_\infty^{\alpha,k}(\Omega)| < \infty$ if and only if f can be modified on a set of measure 0 so that the modified function, which shall be denoted by f again, is of class C^k and

$$|f|_{\mathrm{Lip}(\alpha)} = \sum_{|\nu|=k} \sup \frac{|\partial^\nu f(x) - \partial^\nu f(y)|}{|x - y|^{\alpha-k}} < \infty,$$

where the sup ranges over distinct points x and y in Ω for which there exists a cube Q such that $x, y \in Q \subset \Omega$. Moreover, the quasinorm $|f; C_\infty^{\alpha,k}(\Omega)|$ is equivalent to $|f|_{\mathrm{Lip}(\alpha)}$. This result is due to Campanato [Cm1], [Cm2], and N. G. Meyers [Mey].

(3°) Let m be a positive integer and f a function on Ω. Then $|f; C_\infty^{m,m}(\Omega)| < \infty$ if and only if f can be modified on a set of measure 0 so that the modified function, which shall be denoted by f again, is of class C^{m-1} and

$$|f|_{\Lambda(m)} = \sum_{|\nu|=m-1} \sup \frac{|\partial^\nu f(x) - 2\partial^\nu f((x+y)/2) + \partial^\nu f(y)|}{|x-y|} < \infty,$$

where the sup ranges over the same x, y as described in (2°). The quasinorm $|f; C_\infty^{m,m}(\Omega)|$ is equivalent to $|f|_{\Lambda(m)}$. For a proof of this result, see [Gre] or [Mi1; §6.2].

(4°) Let $k \geqq [\alpha]$, $1 + \alpha/n > 1/p$, and let Ω be a bounded C^∞ domain. Then the quasinorm $\|f; C_p^{\alpha,k}(\Omega)\|$ is equivalent to the quasinorm of the Triebel-Lizorkin space $F_{p,\infty}^\alpha(\Omega)$. This result is due to Seeger [Se] and Triebel [T1]; see also [T2; 1.7.2, 1.7.3, and 5.3].

We shall end this section by mentioning several other notations which will be used throughout the paper. We use the letter c to denote various positive constants. The value of c may be different in each occasion. To show explicitly the dependence of a constant on other parameters, we write as $c(\alpha, \beta, \ldots)$; this denotes a positive constant depending only on the parameters $\alpha, \beta, \ldots$. Since $\mathscr{P}_k$ is a finite dimensional linear space, it admits a unique (up to isomorphism) normed linear space structure. The convergence of a sequence or a series of polynomials in $\mathscr{P}_k$ and the boundedness of a subset of $\mathscr{P}_k$ refer to the corresponding notions with respect to the unique normed linear space structure of $\mathscr{P}_k$. If D is a cube or an open subset of $\boldsymbol{R}^n$, then for functions f on D we define the maximal function $M_p^D(f)$ by

$$M_p^D(f)(x) = \sup\{|Q|^{-1/p}\|f\|_{p,Q}; Q \text{ cube}, x \in Q \subset D\}, \quad x \in D.$$

We simply write $M_p(f) = M_p^{\boldsymbol{R}^n}(f)$. A *dyadic cube* is a cube of the form $\{x \in \boldsymbol{R}^n; 2^m k_i \leqq x_i \leqq 2^m(k_i+1), i = 1, \ldots, n\}$ with m and $k_i (i = 1, \ldots, n)$ integers.

§ 2. Preliminaries

The first lemma follows from the fact that a finite dimensional linear space admits unique structure of Hausdorff topological linear space and from the invariance of $\mathscr{P}_k$ under dilation and translation.

LEMMA 2.1. *The following inequalities hold for all cubes Q and all $P \in \mathscr{P}_k$.*
(1) *For each p,*

$$|Q|^{-1/p}\|P\|_{p,Q} \leqq \|P\|_{\infty,Q} \leqq c|Q|^{-1/p}\|P\|_{p,Q},$$

where $c = c(n, k, p)$.

(2) *For each* $a \geqq 1$,

$$\|P\|_{\infty,aQ} \leqq ca^k \|P\|_{\infty,Q},$$

where $c = c(n,k)$.

(3) *For each multi-index* ν,

$$\|\partial^\nu P\|_{\infty,Q} \leqq c\ell(Q)^{-|\nu|}\|P\|_{\infty,Q},$$

where $c = c(n,k)$.

In the rest of this section, we assume D is a cube or an open subset of $\boldsymbol{R}^n$ and f is a function on D.

For each k, p, and each cube $Q \subset D$, we define $\Pi_{k,p,Q}(f)$ as the set of the polynomials π in $\mathscr{P}_k$ such that

$$\|f - \pi\|_{p,Q} = \min\{\|f - P\|_{p,Q}; P \in \mathscr{P}_k\}.$$

Since $\mathscr{P}_k$ is a finite dimensional linear space, this set $\Pi_{k,p,Q}(f)$ is not empty. (If $f \notin L^p(Q)$, then $\Pi_{k,p,Q}(f) = \mathscr{P}_k$.)

LEMMA 2.2 (cf. [DS; §4, pp. 23–25, and §12, pp. 104–105]). *Let Q, R, and S be cubes included in D.*

(1) *If* $\pi \in \Pi_{k,p,Q}(f)$, *then*

$$\|\pi\|_{\infty,Q} \leqq c|Q|^{-1/p}\|f\|_{p,Q},$$

where $c = c(n,k,p)$.

(2) *If* $1 \geqq \delta > 0$, $Q \cup R \subset S$, *and* $\ell(Q)$, $\ell(R) \geqq \delta\ell(S)$, *and if* $\pi_Q \in \Pi_{k,p,Q}(f)$ *and* $\pi_R \in \Pi_{k,p,R}(f)$, *then*

$$\|\pi_Q - \pi_R\|_{\infty,S} \leqq c_\delta \ell(S)^\alpha \inf_S f^\#_{k,\alpha,p},$$

where $c_\delta = c(n,k,p,\delta)$.

(3) *If* $1 \leqq t < \infty$ *and* $Q \subset R \subset tR \subset D$, *and if* $\pi_Q \in \Pi_{k,p,Q}(f)$ *and* $\pi_R \in \Pi_{k,p,R}(f)$, *then*

$$\|\pi_Q - \pi_R\|_{\infty,Q} \leqq c_t \ell(R)^\alpha \inf_{tQ} f^\#_{k,\alpha,p},$$

where $c_t = c(n,k,\alpha,p,t)$.

(4) *If* $\pi \in \Pi_{k,p,Q}(f)$, *then*

$$|f(x) - \pi(x)| \leqq c\ell(Q)^\alpha f^\#_{k,\alpha,p}(x) \quad \textit{for a.e. } x \in Q,$$

where $c = c(n,k,\alpha,p)$.

For b with $1 \leqq b < \infty$, we define

$$f_{k,\alpha,p}^{\#,b}(x) = \sup\{\ell(Q)^{-\alpha} I_{k,p,Q}(f); Q \text{ cube}, Q \ni x, bQ \subset D\}$$

for $x \in D^i$ (=the interior of D). Thus $f_{k,\alpha,p}^{\#,1}(x) = f_{k,\alpha,p}^{\#}(x)$ for $x \in D^i$. Of course, $D^i = D$ if D is an open subset of $\boldsymbol{R}^n$ and that $D \backslash D^i$ has measure 0 if D is a cube; in either case, $f_{k,\alpha,p}^{\#,b}$ is defined a.e. on D.

Lemma 2.3 (cf. [DS; Theorem 4.3]). *If* $0 < q \leqq \infty$, $1/q + \alpha/n > 1/p$, *and* $1 \leqq b < \infty$, *then*

$$c^{-1}|f; C_p^{\alpha,k}(D)| \leqq \|f_{k,\alpha,q}^{\#,b}\|_{p,D} \leqq c|f; C_p^{\alpha,k}(D)|,$$

where $c = c(n,k,\alpha,p,q,b)$.

Proof. If $b = 1$, the claim follows from the theorem of DeVore and Sharpley mentioned in the lemma. We shall consider the general case $b \geqq 1$. We can easily generalize Theorem 4.3 of [DS] to the case of $f_{k,\alpha,q}^{\#,b}$ with $b \geqq 1$ and, using the generalized theorem, we see the following: If $1 \leqq b < \infty$, $0 < q_1$, $q_2 \leqq \infty$, and $1/q_i + \alpha/n > 1/p (i = 1,2)$, then

$$c^{-1}\|f_{k,\alpha,q_1}^{\#,b}\|_{p,D} \leqq \|f_{k,\alpha,q_2}^{\#,b}\|_{p,D} \leqq c\|f_{k,\alpha,q_1}^{\#,b}\|_{p,D}.$$

Hence, in order to prove the lemma, it is sufficient to prove the inequalities

$$\|f_{k,\alpha,q}^{\#,b}\|_{p,D} \leqq \|f_{k,\alpha,q}^{\#}\|_{p,D} \leqq c\|f_{k,\alpha,q}^{\#,b}\|_{p,D} \tag{2.1}$$

for $b > 1$ and for q satisfying $0 < q < p$. Since the left hand inequality of (2.1) is obvious, we shall prove only the right hand inequality. We shall simply write $f^{\#,b} = f_{k,\alpha,q}^{\#,b}$.

Fix $b > 1$ and fix a cube $R \subset D$. For each cube $Q \subset R$, we choose a $\pi_Q \in \Pi_{k,q,Q}(f)$. Let $\mathscr{G}$ be the set of the maximal dyadic cubes Q such that $3bQ \subset R$. The interiors of the cubes in $\mathscr{G}$ are pairwise disjoint and the union of all the cubes in $\mathscr{G}$ is equal to the interior of R. Fix a cube $Q_0 \in \mathscr{G}$ which contains x_R. Let Q be a cube in $\mathscr{G}$. We can find a finite number of cubes, $Q_j (j = 1, \ldots, m)$, in $\mathscr{G}$ such that $Q_j \cap Q_{j-1} \neq \varnothing$ $(j = 1, \ldots, m)$, $Q_m = Q$, $Q \subset c(n,b)Q_j$ for each j, and

$$\sum_{j=1}^{m} \ell(Q_j)^\alpha \leqq c(n,b,\alpha)\ell(R)^\alpha.$$

(For example, Q_j can be chosen from those cubes in $\mathscr{G}$ which intersects the line segment joining x_R and x_Q.) For each j, let $Q_j' = 3Q_j$ if $\ell(Q_j) > \ell(Q_{j-1})$ and let $Q_j' = 3Q_{j-1}$ if otherwise. Then $Q_j \cup Q_{j-1} \subset Q_j'$ and, by Lemma 2.2(2),

$$\|\pi_{Q_j} - \pi_{Q_{j-1}}\|_{\infty,Q_j} \leqq c\ell(Q_j')^\alpha \inf_{Q_j'} (f|Q_j')^\#_{k,\alpha,q}.$$

Since $cQ_j \supset Q$ and since $bQ_j' \subset R \subset D$, the above inequality combined with Lemma 2.1(2) implies that

$$\begin{aligned}\|\pi_{Q_j} - \pi_{Q_{j-1}}\|_{\infty,Q} &\leqq c\ell(Q_j)^\alpha \inf_{Q_j} f^{\#,b} \\ &\leqq c\ell(Q_j)^\alpha |Q_j|^{-1/q} \|f^{\#,b}\|_{q,Q_j} \leqq c\ell(Q_j)^\alpha \inf_{Q} M_q^D(f^{\#,b}).\end{aligned}$$

Taking the sum over $j = 1, 2, \dots, m$, we obtain

$$\|\pi_{Q_0} - \pi_Q\|_{\infty,Q} \leqq c\ell(R)^\alpha \inf_Q M_q^D(f^{\#,b}).$$

From this estimate, we obtain

$$\begin{aligned}\|f - \pi_{Q_0}\|_{q,Q}^q &\leqq c(\|f - \pi_Q\|_{q,Q}^q + \|\pi_Q - \pi_{Q_0}\|_{q,Q}^q) \\ &\leqq c\ell(Q)^{\alpha q}|Q| \left(\inf_Q f^{\#,b}\right)^q + c\ell(R)^{\alpha q}\left(\inf_Q M_q^D(f^{\#,b})\right)^q |Q|.\end{aligned}$$

Taking sum over $Q \in \mathscr{G}$, we obtain

$$\|f - \pi_{Q_0}\|_{q,R}^q \leqq c\ell(R)^{\alpha q} \sum_{Q \in \mathscr{G}} \left(\inf_Q M_q^D(f^{\#,b})\right)^q |Q| \leqq c\ell(R)^{\alpha q} \|M_q^D(f^{\#,b})\|_{q,R}^q,$$

which implies

$$\ell(R)^{-\alpha} I_{k,q,R}(f) \leqq c|R|^{-1/q} \|M_q^D(f^{\#,b})\|_{q,R}.$$

Now taking the supremum over R's, we obtain

$$f^\#_{k,\alpha,q}(x) \leqq cM_q^D(M_q^D(f^{\#,b}))(x) \quad \text{for all } x \in D.$$

The right hand inequality of (2.1) follows from the above pointwise inequality. Lemma 2.3 is proved.

LEMMA 2.4 (cf. [DS; Proof of Theorem 12.5, pp. 111–112]). *If* $\alpha > \beta > 0$, $0 < p_1 < p_2 \leqq \infty$, *and* $1/p_1 - \alpha/n = 1/p_2 - \beta/n$, *then*

$$|f; C_{p_2}^{\beta,k}(D)| \leqq c|f; C_{p_1}^{\alpha,k}(D)|,$$

where $c = c(n, k, \alpha, \beta, p_1, p_2)$.

LEMMA 2.5 (cf. [DS; Theorem 9.1 and Proof of Theorem 4.3]). *Suppose* $D = Q$ *is a cube. Suppose* $1/p > \alpha/n$ *or* $1/p = \alpha/n \geqq 1$. *Also suppose* $\infty > 1/q > 1/p - \alpha/n$ *and* $\pi \in \Pi_{k,q,Q}(f)$. *Then*

$$\|f - \pi; \Gamma(1/p - \alpha/n; Q)\| \leqq c|f; C_p^{\alpha,k}(Q)|,$$

where $c = c(n, k, \alpha, p, q)$.

LEMMA 2.6. *Let* Q *be a cube and* $\{f_j\}$ *a sequence of functions on* Q. *Suppose* $|f_j; C_p^{\alpha,k}(Q)|$ *is bounded. Let* $0 < q \leqq \infty$, $1/q + \alpha/n > 1/p$, *and* $\pi_j \in \Pi_{k,q,Q}(f_j)$. *Then* $\{f_j - \pi_j\}$ *contains a subsequence which converges in* $L^p(Q)$.

PROOF. If is sufficient to show that the set $\{f_j - \pi_j\}$ is totally bounded in $L^p(Q)$. For each positive integer m, let $\mathscr{G}_m$ be the set of the dyadic subcubes of Q with side length $2^{-m}\ell(Q)$. We take $\pi_{j,R} \in \Pi_{k,q,R}(f_j)$ for each $R \in \bigcup_m \mathscr{G}_m$, and define $f_{j,m}$ by

$$f_{j,m} = \sum_{R \in \mathscr{G}_m} \pi_{j,R}\chi_R.$$

We shall prove the following two facts: (i) As $m \to \infty$, the sequence $\{f_{j,m} - \pi_j\}_m$ converges to $f_j - \pi_j$ in $L^p(Q)$ uniformly with respect to j; (ii) For each fixed m, te set $\{f_{j,m} - \pi_j\}_j$ is totally bounded in $L^p(Q)$. The totally boundedness of the set $\{f_j - \pi_j\}$ follows from these two facts. We shall simply write $f_j^{\#} = (f_j)_{k,\alpha,q}^{\#}$.

We shall prove (i) and (ii). First, by Lemma 2.2(4), we have

$$|f_j(x) - f_{j,m}(x)| \leqq \sum_{R \in \mathscr{G}_m} |f_j(x) - \pi_{j,R}(x)|\chi_R(x) \leqq c(2^{-m}\ell(Q))^{\alpha} f_j^{\#}(x).$$

Hence, by Lemma 2.3,

$$\|f_j - f_{j,m}\|_{p,Q} \leqq c(2^{-m}\ell(Q))^{\alpha}\|f_j^{\#}\|_{p,Q} \leqq c(2^{-m}\ell(Q))^{\alpha}|f_j; C_p^{\alpha,k}(Q)|,$$

which obviously implies (i) (since $|f_j; C_p^{\alpha,k}(Q)|$ is bounded). Next, the function

$f_{j,m} - \pi_j$ can be written as

$$f_{j,m} - \pi_j = \sum_{R \in \mathscr{G}_m} (\pi_{j,R} - \pi_j)\chi_R.$$

Hence, in order to prove (ii), it is sufficient to show that for each fixed R the set $\{\pi_{j,R} - \pi_j\}_j$ is bounded in $\mathscr{P}_k$. But this last fact can be immediately proved by the use of Lemma 2.2(2). Lemma 2.6 is proved.

LEMMA 2.7. *Suppose $\{f_j\}$ is a sequence of functions on D such that*

$$|f_j - f_m; C_p^{\alpha,k}(D)| \to 0 \quad \text{as } j, m \to \infty.$$

Then the following (1) *and* (2) *hold.*

(1) *There exists a function g on D such that*

$$|f_j - g; C_p^{\alpha,k}(D)| \to 0 \quad \text{as } j \to \infty \tag{2.2}$$

and

$$|g; C_p^{\alpha,k}(D)| \leqq \liminf_{j\to\infty} |f_j; C_p^{\alpha,k}(D)|. \tag{2.3}$$

(2) *Suppose, in addition, h is a function on D and suppose for each cube $Q \subset D$ there exists an $r = r_Q > 0$ such that*

$$\|f_j - h\|_{r,Q} \to 0 \quad \text{as } j \to \infty. \tag{2.4}$$

Then (2.2) *and* (2.3) *hold for $g = h$.*

PROOF. The inequality (2.3) follows from (2.2) with the aid of the triangular inequality

$$|f + g; C_p^{\alpha,k}(D)|^q \leqq |f; C_p^{\alpha,k}(D)|^q + |g; C_p^{\alpha,k}(D)|^q, \quad q = \min\{p, 1\}.$$

Hence it is sufficient to show only the assertions concerning (2.2). We shall prove (2) first.

PROOF OF (2). We shall first show that (2.4) actually holds with $r = p$. In order to prove this, it is sufficient to show that, for each cube $Q \subset D$,

$$\|f_j - f_m\|_{p,Q} \to 0, \quad \text{as } j, m \to \infty. \tag{2.5}$$

Fix a cube $Q \subset D$ and let r be a positive number for which (2.4) holds. Replacing r by a smaller number, if necessary, we may assume that $1/r + \alpha/n > 1/p$. Take

$\pi_{j,m} \in \Pi_{k,r,Q}(f_j - f_m)$. Then, by Lemma 2.2(4), Lemma 2.3, and by the assumption of the lemma,

$$\|f_j - f_m - \pi_{j,m}\|_{p,Q} \leqq c\ell(Q)^{\alpha} |f_j - f_m; C_p^{\alpha,k}(Q)| \to 0 \quad \text{as } j, m \to \infty. \tag{2.6}$$

On the other hand, by Lemma 2.2(1) and by the assumption (2.4),

$$\|\pi_{j,m}\|_{p,Q} \leqq c|Q|^{1/p-1/r} \|f_j - f_m\|_{r,Q} \to 0 \quad \text{as } j, m \to \infty. \tag{2.7}$$

The assertion (2.5) follows from (2.6) and (2.7).

Now, from the fact that (2.4) holds with $r = p$, it follows that

$$I_{k,p,Q}(f_j - h) = \lim_{m\to\infty} I_{k,p,Q}(f_j - f_m).$$

From this, we obtain

$$(f_j - h)^{\#}_{k,\alpha,p}(x) \leqq \liminf_{m\to\infty} (f_j - f_m)^{\#}_{k,\alpha,p}(x)$$

for all $x \in D$. Taking $L^p(D)$-norm and using Fatou's lemma, we obtain

$$|f_j - h; C_p^{\alpha,k}(D)| \leqq \liminf_{m\to\infty} |f_j - f_m; C_p^{\alpha,k}(D)|,$$

from which follows (2.2) with $g = h$. Thus (2) is proved.

Proof of (1). It is sufficient to show that $\{f_j\}$ contains a subsequence which converges to a function with respect to $|\cdot; C_p^{\alpha,k}(D)|$. We may and shall assume that $|f_j; C_p^{\alpha,k}(D)|$ is bounded (if this is not the case, it is enough to consider $f_j - f_{j_0}$, with a fixed j_0, instead of f_j). For each $Q \subset D$, we take $\pi_{j,Q} \in \Pi_{k,p,Q}(f_j)$.

First we consider the case where $D = Q$ is a cube. In this case, by Lemma 2.6, $\{f_j - \pi_{j,Q}\}$ contains a subsequence $\{f_{j'} - \pi_{j',Q}\}$ which converges in $L^p(Q)$. If we denote the limit by g, then, by (2) proved above, we have

$$|f_{j'} - g; C_p^{\alpha,k}(Q)| = |f_{j'} - \pi_{j',Q} - g; C_p^{\alpha,k}(Q)| \to 0$$

as desired.

Next, suppose D is a connected open subset of $\boldsymbol{R}^n$. We shall first show that if Q and Q' are cubes included in D then $\{\pi_{j,Q} - \pi_{j,Q'}\}_j$ is bounded in $\mathscr{P}_k$. This immediately follows from Lemma 2.2(2) if there exists a cube R such that $Q \cup Q' \subset R \subset D$. In the general case, we can find a finite number of cubes Q_m, $m = 0, 1, \dots, N$, and cubes R_m, $m = 1, 2, \dots, N$, such that $Q_0 = Q$, $Q_N = Q'$, and $Q_{m-1} \cup Q_m \subset R_m \subset D$. Then, for each m, the sequence $\{\pi_{j,Q_{m-1}} - \pi_{j,Q_m}\}_j$ is bounded in $\mathscr{P}_k$ as seen above, and, hence, the sum

$$\sum_{m=1}^{N}(\pi_j, Q_{m-1} - \pi_{j,Q_m}) = \pi_{j,Q} - \pi_{j,Q'}$$

is also bounded. Now fix a cube $Q_0 \subset D$. Let Q be an arbitrary cube included in D. By Lemma 2.6, the sequence $\{f_j - \pi_{j,Q}\}$ contains a subsequence which converges in $L^p(Q)$. On the other hand, the sequence $\{\pi_{j,Q} - \pi_{j,Q_0}\}$ is bounded in $\mathscr{P}_k$ and hence any subsequence of it contains another subsequence which converges in $\mathscr{P}_k$. Hence the sequence $\{f_j - \pi_{j,Q_0}\} = \{f_j - \pi_{j,Q} + \pi_{j,Q} - \pi_{j,Q_0}\}$ contains a subsequence which converges in $L^p(Q)$. Since this is true for every cube $Q \subset D$, we obtain, by the diagonal method, a subsequence $\{f_{j'} - \pi_{j',Q_0}\}$ which converges in $L^p_{\mathrm{loc}}(D)$. If we denote the limit by g, then by (2) proved above, we have

$$|f_{j'} - g; C_p^{\alpha,k}(D)| = |f_{j'} - \pi_{j',Q_0} - g; C_p^{\alpha,k}(D)| \to 0$$

as desired.

Finally, if D is an arbitrary open subset of $\boldsymbol{R}^n$, then we can obtain the same conclusion by considering on each connected component of D. Lemma 2.7 is proved.

LEMMA 2.8. *Let f be a function on $\boldsymbol{R}^n$, Q a cube, and let $0 < \varepsilon$, $A < \infty$. Suppose* $\mathrm{supp} f \subset Q$ *and suppose* $\ell(R)^{-\alpha} I_{k,p,R}(f) \leqq A$ *for cubes R with* $\ell(R) \leqq \varepsilon\ell(Q)$. *Then* $|f; C_\infty^{\alpha,k}(\boldsymbol{R}^n)| \leqq c_\varepsilon A$ *with* $c_\varepsilon = c(n,k,\alpha,p,\varepsilon)$.

PROOF. We take $\pi_R \in \Pi_{k,p,R}(f)$ for each cube R. If R is a cube with $\ell(R) \leqq \varepsilon\ell(Q)$, then, by Lemma 2.2(4), we have

$$|f(x) - \pi_R(x)| \leqq cA\ell(R)^\alpha \quad \text{for a.e. } x \in R.$$

If R and R' are cubes such that $R \cap R' \neq \varnothing$, $\ell(R) = \ell(R') \leqq \varepsilon\ell(Q)/2$, then, using Lemma 2.2(2), we see that

$$\|\pi_R - \pi_{R'}\|_{\infty, R\cup R'} \leqq cA\ell(R)^\alpha.$$

Using these two facts and the assumption $\mathrm{supp}\, f \subset Q$, we can easily prove that

$$|f(x)| \leqq c_\varepsilon A\, \mathrm{dis}(x, Q^c)^\alpha \quad \text{a.e. } x \in \boldsymbol{R}^n.$$

Thus, in particular, $\|f\|_\infty \leqq c_\varepsilon A\ell(Q)^\alpha$. Hence, for cubes R with $\ell(R) > \varepsilon\ell(Q)$, we have

$$\ell(R)^{-\alpha} I_{k,p,R}(f) \leqq \ell(R)^{-\alpha}\|f\|_\infty \leqq c_\varepsilon A(\ell(Q)/\ell(R))^\alpha \leqq c_\varepsilon A.$$

Hence $f^{\#}_{k,\alpha,p}(x) \leqq c_\varepsilon A$ for all $x \in \boldsymbol{R}^n$, and thus $|f; C^{\alpha,k}_{\infty}(\boldsymbol{R}^n)| \leqq c_\varepsilon A$ (by Lemma 2.3). Lemma 2.8 is proved.

The final lemma, below, easily follows from Taylor's formula.

LEMMA 2.9. *If f is a function of class C^{k+1} on a cube Q, then*

$$I_{k,\infty,Q}(f) \leqq c\ell(Q)^{k+1} \||\nabla^{k+1} f|\|_{\infty,Q}$$

with $c = c(n,k)$.

§ 3. Atomic series

Before we state the main theorem of this section, we shall give a remark and a definition. First, let $\sum_j \varphi_j$ be a series of functions on $\boldsymbol{R}^n$ and fix k, α, and p. The function g on $\boldsymbol{R}^n$ satisfying $|\sum_{j=1}^N \varphi_j - g; C^{\alpha,k}_p| \to 0$ as $N \to \infty$ (if there is any such g) is not acutually unique but is unique only mod $\mathscr{P}_k$. Even so we shall write $g = \sum_j \varphi_j$ and call it *the* sum of the series $\sum_j \varphi_j$. Notice that $|g; C^{\alpha,k}_p|$ is uniquely determined in spite of the non-uniqueness of g. Next, we define the function space C_0 as follows: A function f on $\boldsymbol{R}^n$ belongs to C_0 if there exists a continuous function h on $\boldsymbol{R}^n$ such that $f(x) = h(x)$ a.e. and $h(x) \to 0$ as $|x| \to \infty$. We regard C_0 as a Banach space by identifying functions deffering only on sets of measure 0 and by taking $\|\cdot\|_\infty$ as the norm.

Now, the purpose of this section is to prove the following theorem.

THEOREM 3.1. *Let $\{\varphi_j\}$ be a sequence of functions on $\boldsymbol{R}^n$, $\{\lambda_j\}$ a sequence of nonnegative real numbers, and $\{R_j\}$ a sequence of cubes. Suppose*

$$\operatorname{supp}\varphi_j \subset R_j \quad \textit{and} \quad |\varphi_j; C^{\alpha,k}_{\infty}(\boldsymbol{R}^n)| \leqq \lambda_j,$$

and suppose $p < \infty$ and

$$\left\| \left(\sum_j \lambda_j^q \chi_{R_j} \right)^{1/q} \right\|_p = M_p < \infty \quad \textit{with} \quad q = \min\{p, 1\}.$$

We write $A = \sup \ell(R_j) (\leqq \infty)$. Then the following hold with $c = c(n,k,\alpha,p)$ and $c_r = c(n,k,\alpha,p,r)$.

(1) *The series $\sum_j \varphi_j$ unconditionally converges with respect to $|\cdot; C^{\alpha,k}_p|$ and the sum $\sum_j \varphi_j$ satisfies $|\sum_j \varphi_j; C^{\alpha,k}_p| \leqq cM_p$.*

(2) *If $1/p > \alpha/n$, then the series $\sum_j \varphi_j$ unconditionally converges in $\Gamma(1/p - \alpha/n)$ and $\|\sum_j \varphi_j; \Gamma(1/p - \alpha/n)\| \leqq cM_p$.*

(3) *If* $1/p = \alpha/n \geqq 1$, *then* $\sum_j \varphi_j$ *converges absolutely in* C_0 *and* $\|\sum_j \varphi_j\|_\infty \leqq cM_p$.

(4) *If* $0 < 1/p = \alpha/n < 1$ *and* $A < \infty$, *then, for each* r *with* $p \leqq r < \infty$, *the series* $\sum_j \varphi_j$ *unconditionally converges in* L^r *and* $\|\sum_j \varphi_j\|_r \leqq c_r A^{n/r} M_p$.

(5) *If* $0 < 1/p < \alpha/n$ *and* $A < \infty$, *then* $\sum_j \varphi_j$ *unconditionally converges in* C_0 *and* $\|\sum_j \varphi_j\|_\infty \leqq cA^{\alpha-n/p} M_p$.

In order to prove this theorem, we use the lemmas below.

LEMMA 3.1. *Let* f *be a function on* $\mathbf{R}^n$ *and* Q *a cube. Suppose* $\operatorname{supp} f \subset Q$ *and* $|f; C_\infty^{\alpha,k}(\mathbf{R}^n)| = \lambda < \infty$. *Then:*

(1) *After modified on a set of measure zero,* f *is a continuous function and* $\|f\|_\infty \leqq c\lambda \ell(Q)^\alpha$ *with* $c = c(n, \alpha, k)$;

(2) *For each* p,

$$f_{k,\alpha,p}^{\#}(x) \leqq c\lambda(1 + \ell(Q)^{-1}|x - x_Q|)^{-\alpha-n/p}, \quad x \in \mathbf{R}^n,$$

where $c = c(n, k, \alpha, p)$.

PROOF. (1) The continuity of f follows from a stronger result of DeVore and Sharpley [DS; Theorem 9.1]. (DeVore and Sharpley state the result for the case $k = \alpha$ but their proof works for general k.) The inequality for $\|f\|_\infty$ has been proved in the proof of Lemma 2.8.

(2) The inequality is obvious for $x \in 2Q$. Suppose $x \notin 2Q$. Then, for cubes R containing x and intersecting Q, we have $\ell(R) \geqq \|x - x_Q\|/2$ and hence

$$\begin{aligned} \ell(R)^{-\alpha} I_{k,p,R}(f) &\leqq \ell(R)^{-\alpha-n/p} \|f\|_{p,R} \leqq c\lambda(\ell(Q)/\ell(R))^{\alpha+n/p} \\ &\leqq c\lambda(\ell(Q)/|x - x_Q|)^{\alpha+n/p} \end{aligned}$$

(the second inequality follows from the estimate of $\|f\|_\infty$ as given in (1) and from the assumption $\operatorname{supp} f \subset Q$). Taking supremum over R's, we obtain the desired inequality. Lemma 3.1 is proved.

LEMMA 3.2. *Let* $\{\lambda_j\}$ *be a sequence of nonnegative real numbers,* $\{R_j\}$ *a sequence of cubes, and let* $0 \leqq \beta < \infty$ *and* $0 < \mu < \infty$. *We write* $A = \sup \ell(R_j) (\leqq \infty)$, $f = \sum_j \lambda_j \chi_{R_j}$, *and*

$$g_{\beta,\mu}(x) = \sum_j \lambda_j \ell(R_j)^\beta (1 + \ell(R_j)^{-1}|x - x_{R_j}|)^{-\mu}.$$

Then the following (1) $\sim$ (6) *hold with* $c = c(n, \beta, \mu, p)$.

(1) *If* $n/p > n + \beta$ *and* $\mu > n/p - \beta$, *then*

$$\|g_{\beta,\mu}; \Gamma(1/p - \beta/n)\| \leqq c\left(\sum_j \lambda_j^p |R_j|\right)^{1/p}.$$

(2) *If* $\beta < n/p \leqq n + \beta$ *and* $\mu > n$, *then*

$$\|g_{\beta,\mu}; \Gamma(1/p - \beta/n)\| \leqq c\|f\|_p.$$

(3) *If* $p \leqq 1$, *then*

$$\|g_{n/p,\mu}\|_\infty \leqq \sum_j \lambda_j \ell(R_j)^{n/p} \leqq \|f\|_p.$$

(4) *If* $p > 1$ *and* $\mu > n$ *and if* $\lambda_j = 0$ *except for a finite number of* j*'s, then*

$$\|g_{n/p,\mu}\|_{\mathrm{BMO}} \leqq c\|f\|_p.$$

(5) *If* $n/p < \beta$, $p \leqq 1$, *and* $A < \infty$, *then*

$$\|g_{\beta,\mu}\|_\infty \leqq \sum_j \lambda_j \ell(R_j)^\beta \leqq A^{\beta - n/p}\|f\|_p.$$

(6) *If* $n/p < \beta$, $1 < p < \infty$, $\mu > n - n/p$, *and* $A < \infty$, *then*

$$\|g_{\beta,\mu}\|_\infty \leqq cA^{\beta - n/p}\|f\|_p.$$

Proof. (1) Set $1/q = 1/p - \beta/n$. Then $p \leqq q < 1$ and $\mu > n/q$. Hence

$$\begin{aligned}\|g_{\beta,\mu}\|_q^q &\leqq \sum_j \lambda_j^q \ell(R_j)^{\beta q} \|(1 + \ell(R_j)^{-1}|\cdot|)^{-\mu}\|_q^q \\ &\leqq c\sum_j \lambda_j^q \ell(R_j)^{\beta q + n} = c\sum_j \lambda_j^q |R_j|^{q/p} \leqq c\left(\sum_j \lambda_j^p |R_j|\right)^{q/p}.\end{aligned}$$

(3) The left hand inequality is obvious. The right hand inequality is the same as the inequality $\sum_j \|\lambda_j \chi_{R_j}\|_p \leqq \|\sum_j \lambda_j \chi_{R_j}\|_p$ which holds for $p \leqq 1$.

(4) Let σ_j be the Dirac measure on $\boldsymbol{R}^{n+1}$ concentrated at the point $(x_{R_j}, \ell(R_j))$ and let $\sigma = \sum_j \lambda_j |R_j|^{1/p+1}\sigma_j$. Then we have

$$g_{n/p,\mu}(x) = \int t^{-n}(1 + t^{-1}|x - y|)^{-\mu} d\sigma(y, t).$$

In order to prove the inequality of (4), it is sufficient to show that the inequality

$$\sigma(T(Q)) \leqq c\|f\|_p|Q|, \quad T(Q) = Q \times (0, \ell(Q)), \tag{3.1}$$

holds for every cube Q in $\boldsymbol{R}^n$ (cf. e.g. [Gar; Chapt. VI, Th. 1.6]). If $p \geqq 1$, then

$$\sigma(T(Q)) \leqq \sum_{R_j \subset 2Q} \lambda_j |R_j|^{1/p+1} \leqq c|Q|^{1/p} \sum_{R_j \subset 2Q} \lambda_j |R_j|$$
$$\leqq c|Q|^{1/p} \int_{2Q} f(x)\,dx \leqq c\|f\|_p |Q|$$

(the first inequality follows from the fact that $(x_{R_j}, \ell(R_j)) \in T(Q)$ imlies $R_j \subset 2Q$). If $p < 1$, then

$$\sigma(T(Q)) \leqq \sum_{R_j \subset 2Q} \lambda_j |R_j|^{1/p+1} \leqq c|Q| \sum_j \lambda_j |R_j|^{1/p}$$
$$= c|Q| \sum_j \|\lambda_j \chi_{R_j}\|_p \leqq c|Q| \|f\|_p.$$

Thus (3.1) is proved.

(2) In the case $n/p = n + \beta$, the proof is easy:

$$\|g_{\beta,\mu}; \Gamma(1/p - \beta/n)\| = \|g_{\beta,\mu}\|_1 = c \sum_j \lambda_j |R_j|^{1/p} \leqq c\|f\|_p. \tag{3.2}$$

The claim for the case $\beta < n/p < n + \beta$ follows from this result and from (3) and (4) (proved above) by the use of the interpolation. Some comments shall be necessary, however, since $g_{\beta,\mu}$ is not uniquely defined by f but depends on the representation of f as the sum $\sum_j \lambda_j \chi_{R_j}$. First notice that it is sufficient to prove the inequality in the case where all the cubes R_j are dyadic cubes; this can be seen from the fact that for each cube R_j there exists a dyadic cube Q_j such that $Q_j \subset R_j$ and $\ell(Q_j) > \ell(R_j)/4$. Thus we assume R_j are dyadic cubes. We may also assume that $\lambda_j = 0$ except for a finite number of j's. Now, a key to the real interpolation method is to decompose f as $f = f^t + f_t$, $0 < t < \infty$, where $f^t(x) = \min\{f(x), t\}$ and $f_t(x) = f(x) - f^t(x)$. In the present case, i.e., in the case where $f = \sum_j \lambda_j \chi_{R_j}$ with $0 \leqq \lambda_j < \infty$ and R_j dyadic cubes and with $\lambda_j = 0$ except for a finite number of j's, the following holds: For each $t > 0$, there exist nonnegative real numbers λ_j' and λ_j'' such that $\lambda_j = \lambda_j' + \lambda_j''$, $f^t = \sum_j \lambda_j' \chi_{R_j}$ a.e., and $f_t = \sum_j \lambda_j'' \chi_{R_j}$ a.e. (proof of this fact is left to the reader; everyone will convince this fact once he draws a picture of the graph of f and will see that it comes from the fact that two dyadic cubes have disjoint interiors unless one is included in the other). We define $g_{\beta,\mu}'$ and $g_{\beta,\mu}''$ in the same way as $g_{\beta,\mu}$ using λ_j' and λ_j'', respectively, in place of λ_j. Then we have $g_{\beta,\mu} = g_{\beta,\mu}' + g_{\beta,\mu}''$. Now we can use the usual techniques of the real interpolation method to deduce the inequality of (2) for $\beta < n/p < n + \beta$ from (3), (4), and (3.2).

(5) This immediately follows from (3) since $\ell(R_j)^\beta \leqq A^{\beta-n/p}\ell(R_j)^{n/p}$.

(6) Take a function $\theta \in C_0^\infty R^n)$ such that $0 \leqq \theta(x) \leqq 1$ for all $x \in \boldsymbol{R}^n$, $\theta(x) = 1$ for $x \in Q_0 = [-1/2, 1/2]^n$, and supp $\theta \subset 2Q_0$. For a cube Q, we define θ_Q by $\theta_Q(x) = \theta((x - x_Q)/\ell(Q))$. For $t \geqq 1$, we set

$$h_t = \sum_j \lambda_j \ell(R_j)^\beta \theta_{tR_j}.$$

We shall prove that the following inequality holds for every $t \geqq 1$ and every $\varepsilon > 0$:

$$\|h_t\|_\infty \leqq c_\varepsilon A^{\beta-n/p} t^{n-n/p+\varepsilon} \|f\|_p, \tag{3.3}$$

where $c_\varepsilon = c(n, \beta, p, \varepsilon)$. Once this is proved, the desired inequality can be derived as follows. We have

$$(1 + \ell(R_j)^{-1}|x - x_{R_j}|)^{-\mu} \leqq c \sum_{m=0}^\infty 2^{-m\mu} \theta_{2^m R_j}(x)$$

and hence

$$g_{\beta,\mu}(x) \leqq c \sum_{m=0}^\infty 2^{-m\mu} h_{2^m}(x).$$

Thus, using (3.3) with ε satisfying $n - n/p + \varepsilon < \mu$, we obtain

$$\|g_{\beta,\mu}\|_\infty \leqq c \sum_{m=0}^\infty 2^{-m\mu} A^{\beta-n/p} 2^{m(n-n/p+\varepsilon)} \|f\|_p = cA^{\beta-n/p}\|f\|_p.$$

We shall prove (3.3). Fix an arbitrary $x_0 \in \boldsymbol{R}^n$ and set

$$h'_t = \sum_j{}' \lambda_j \ell(R_j)^\beta \theta_{tR_j},$$

where the sum $\sum_j'$ is taken over the j's such that $2tR_j \ni x_0$. Since $\ell(R_j) \leqq A < \infty$,

$$\operatorname{supp} h'_t \subset Q = \{x; \|x - x_0\| \leqq 2tA\}. \tag{3.4}$$

Take k such that $k + 1 \geqq \beta$. By lemma 3.1(2) and a change of variables, we have

$$(\theta_{tR_j})^\#_{k,\beta,1}(x) \leqq c\ell(tR_j)^{-\beta}(1 + \ell(tR_j)^{-1}|x - x_{R_j}|)^{-\beta-n}$$

and hence

$$(h'_t)^\#_{k,\beta,1}(x) \leqq c \sum_j{}' \lambda_j t^{-\beta}(1 + \ell(tR_j)^{-1}|x - x_{R_j}|)^{-\beta-n}.$$

Thus, using (2) twice, we have

$$\|(h_t')^{\#}_{k,\beta,1}\|_p \leqq ct^{-\beta}\left\|{\sum_j}' \lambda_j \chi_{tR_j}\right\|_p$$

$$\leqq c_\varepsilon t^{-\beta}\left\|{\sum_j}' \lambda_j t^{n+\varepsilon}(1+\ell(R_j)^{-1}|x-x_{R_j}|)^{-n-\varepsilon}\right\|_p \leqq c_\varepsilon t^{-\beta+n+\varepsilon}\|f\|_p.$$

Hence, using Lemma 2.4, we have

$$|h_t'; C_\infty^{\beta-n/p,k}| \leqq c\|(h_t')^{\#}_{k,\beta,1}\|_p \leqq c_\varepsilon t^{-\beta+n+\varepsilon}\|f\|_p. \tag{3.5}$$

From (3.4), (3.5), and Lemma 3.1(1), we obtain

$$h_t'(x_0) \leqq c|h_t'; C_\infty^{\beta-n/p,k}|\ell(Q)^{\beta-n/p} \leqq c_\varepsilon A^{\beta-n/p} t^{n-n/p+\varepsilon}\|f\|_p,$$

which implies (3.3) since $h_t'(x_0) = h_t(x_0)$ and x_0 is arbitrary. Lemma 3.2 is proved.

PROOF OF THEOREM 3.1. PROOF OF (1). By Lemma 3.1(2),

$$(\varphi_j)^{\#}_{k,\alpha,q}(x) \leqq c_q \lambda_j (1+\ell(R_j)^{-1}|x-x_{R_j}|)^{-\alpha-n/q} \tag{3.6}$$

for each $q > 0$. Suppose first $\sum_j \varphi_j$ is a finite sum. If $p \leqq 1$, then, using (3.6) with $q = p$, we have

$$\left|\sum_j \varphi_j; C_p^{\alpha,k}\right|^p \leqq \sum_j \|(\varphi_j)^{\#}_{k,\alpha,p}\|_p^p \leqq c\sum_j \lambda_j^p |R_j| = cM_p^p.$$

If $1 < p < \infty$, then, using Lemma 2.3, (3.6) with $q = 1$, and lemma 3.2(2) with $\beta = 0$, we have

$$\left|\sum_j \varphi_j; C_p^{\alpha,k}\right| \leqq c\left\|\sum_j (\varphi_j)^{\#}_{k,\alpha,1}\right\|_p \leqq c\left\|\sum_j \lambda_j \chi_{R_j}\right\|_p = cM_p.$$

In the general case, where $\sum_j \varphi_j$ is an infinite sum, the unconditional convergence and the estimate as mentioned in (1) can now be proved by a routine argument.

PROOF OF (2) $\sim$ (5). By lemma 3.1,

$$\varphi_j \in C_0 \quad \text{and} \quad |\varphi_j(x)| \leqq c\lambda_j \ell(R_j)^\alpha \chi_{R_j}(x). \tag{3.7}$$

The assertion (2) follows from (3.7), (1) and (2) of Lemma 3.2, and the fact that

$$\left\|\sum_j \lambda_j \chi_{R_j}\right\|_p \leqq \left(\sum_j \lambda_j^p |R_j|\right)^{1/p} \quad \text{for } p \leqq 1. \tag{3.8}$$

The assertion (3) folows from (3.7), Lemma 3.2(3), and (3.8). The assertion (4) follows from (3.7) and Lemma 3.2(2); in fact, since $\ell(R_j) \leqq A < \infty$, (3.7) implies

$$|\varphi_j(x)| \leqq cA^{\varepsilon}\lambda_j \ell(R_j)^{\alpha-\varepsilon}\chi_{R_j}(x), \quad 0 < \varepsilon \leqq \alpha,$$

and we can apply Lemma 3.2(2) with $\beta = \alpha - \varepsilon$. The assertion (5) follows from (3.7), (5) and (6) of Lemma 3.2, and (3.8). Theorem 3.1 is proved.

§4. Smooth function

In Sections 4 and 5, we use the following notations. First, we use the letter b to denote various absolute constants, which may be different in each occasion. When we need to distinguish a constant b from other b's, we write it with a subscript as $b_0, b_1, b_2, \ldots$. Secondly, we assume a is a sufficiently large positive number. A close examination of the arguments of Sections 4 and 5 will show that all the arguments work for each $a \geqq 300$, for example, but the exact value of a is not very important. The reader should check that there exists b_0 such that all the arguments of Sections 4 and 5 work for each $a \geqq b_0$ and such that, in particular, the constants b and b_j $(j \geqq 1)$ can be found independent of a so long as $a \geqq b_0$. Thirdly, we take a constant b_1 and define the relation $\approx$ as follows: For two cubes Q and R, the relation $Q \approx R$ means that $Q \subset b_1 R$ and $R \subset b_1 Q$; for two positive real numbers s and t, the relation $s \approx t$ means that $s \leqq b_1 t$ and $t \leqq b_1 s$. The reader will easily check that we can find a b_1 so that the arguments of Sections 4 and 5 work with this definition of $\approx$.

We begin with the following easy theorem.

THEOREM 4.1. *Let $\Omega \neq \boldsymbol{R}^n$ and suppose $k+1 \geqq \alpha$. Then, for smooth functions g on Ω,*

$$|g; C_p^{\alpha,k}(\Omega)| \leqq c\|\rho_\Omega^{k+1-\alpha}|\nabla^{k+1}g|^*\|_{p,\Omega} \tag{4.1}$$

with $c = c(n,k,\alpha,p)$.

PROOF. We fix a sufficiently large real number s. Then, for $x \in \Omega$ and for cubes Q satisfying $x \in Q$ and $sQ \subset \Omega$, we have $Q \subset Q_{1/2}(x)$ and hence, by Lemma 2.9,

$$\ell(Q)^{-\alpha} I_{k,\infty,Q}(g) \leqq c\ell(Q)^{k+1-\alpha}|||\nabla^{k+1}g|||_{\infty,Q} \leqq c\rho_\Omega^{k+1-\alpha}|\nabla^{k+1}g|^*(x).$$

Hence

$$g^{\#,s}_{k,\alpha,\infty}(x) \leqq c\rho_\Omega^{k+1-\alpha}|\nabla^{k+1}g|^*(x).$$

This inequality combined with Lemma 2.3 implies the desired inequality. The theorem is proved.

In the rest of this section, we shall give a method to associate with each function f on Ω a smooth function g on Ω. One of the property of the function g is that the right side of (4.1) is majorized by $c|f; C_p^{\alpha,k}(\Omega)|$. We shall also give a corresponding result for $\Omega = \boldsymbol{R}^n$. We shall begin with some preliminary results.

For open subset U of Ω, we define $\mathscr{G}_\Omega(U)$ as the set of the maximal dyadic cubes Q such that $a^2Q \subset \Omega$ and $aQ \subset U$. Notice that $\mathscr{G}_\Omega(U) = \emptyset$ if $\Omega = U = \boldsymbol{R}^n$.

In Lemmas 4.1 ~ 4.3 below, U and V denote open subsets of Ω.

LEMMA 4.1. (1) *If $U \neq \boldsymbol{R}^n$, then the cubes in $\mathscr{G}_\Omega(U)$ have pairwise disjoint interiors and the union of all the cubes in $\mathscr{G}_\Omega(U)$ is equal to U.*

(2) *If $Q \in \mathscr{G}_\Omega(U)$ and $x \in 2^{-1}aQ$, then*

$$\ell(Q) \approx \min\{a^{-1}\operatorname{dis}(x, U^c), a^{-2}\operatorname{dis}(x, \Omega^c)\}.$$

(3) *There exists b_2 such that: If $U \subset V$, $Q \in \mathscr{G}_\Omega(U)$, $R \in \mathscr{G}_\Omega(V)$, and $2^{-1}aQ \cap 2^{-1}aR \neq \emptyset$, then $\ell(Q) \leqq b_2\ell(R)$.*

(4) *If $U \subset V$ and $Q \in \mathscr{G}_\Omega(U)$, then the number of the cubes R in $\mathscr{G}_\Omega(V)$ satisfying $2^{-1}aQ \cap 2^{-1}aR \neq \emptyset$ does not exceed $c(n, a)$.*

Proof of the above lemma is left to the reader (cf., e.g., [St; Chapt. VI, § 1]).

Let θ and θ_Q be the functions as defined in the proof of Lemma 3.2(6). For $Q \in \mathscr{G}_\Omega(U)$, we define $\varphi_Q^{\Omega,U}$ as follows:

$$\varphi_Q^{\Omega,U}(x) = \theta_Q(x)\left(\sum_{R \in \mathscr{G}_\Omega(U)} \theta_R(x)\right)^{-1} \quad \text{if } x \in U$$

and $\varphi_Q^{\Omega,U}(x) = 0$ if $x \notin U$. For convenience' sake we also define $\varphi_Q^{\Omega,U} \equiv 0$ for dyadic cubes Q not belonging to $\mathscr{G}_\Omega(U)$.

Proof of the next lemma is easy and is left to the reader.

LEMMA 4.2. (1) $\varphi_Q^{\Omega,U} \in C_0^\infty(\boldsymbol{R}^n)$, $\operatorname{supp}\varphi_Q^{\Omega,U} \subset 2Q$, *and* $0 \leqq \varphi_Q^{\Omega,U}(x) \leqq 1$.

(2) *If $U \neq \boldsymbol{R}^n$, then $\sum_{Q \in \mathscr{G}_\Omega(U)} \varphi_Q^{\Omega,U}(x) = 1$ for all $x \in U$.*

(3) $|\partial^\nu \varphi_Q^{\Omega,U}(x)| \leqq c(n, a, \nu)\ell(Q)^{-|\nu|}$ *for each multi-index ν.*

The next lemma can be immediately proved by the use of Leibniz's formula on differentiation and (1) and (3) of Lemma 4.2.

LEMMA 4.3. *Let* $\{f_Q; Q \in \mathscr{G}_\Omega(U)\}$ *be a family of smooth functions on* U. *Set* $f = \sum_{Q \in \mathscr{G}_\Omega(U)} f_Q \varphi_Q^{\Omega,U}$. *Then, for each nonnegative integer* m,

$$|\nabla^m f(x)| \leqq c \sum_{m'+m''=m} \sum_{Q \in \mathscr{G}_\Omega(U)} |\nabla^{m'} f_Q(x)| \ell(Q)^{-m''} \chi_{2Q}(x),$$

where $c = c(n, a, m)$ *and* m' *and* m'' *denote nonegative integers.*

Now, suppose $\Omega \neq \boldsymbol{R}^n$ and fix k (a nonnegative integer) and q such that $0 < q \leqq \infty$. Let f be a function on Ω. For each cube $Q \in \mathscr{G}_\Omega(\Omega)$, we take a polynomial π_Q in $\Pi_{k,q,Q}(f)$. We define the function $g_{\Omega,k,q}(f)$ on Ω by

$$g_{\Omega,k,q}(f)(x) = \sum_{Q \in \mathscr{G}_\Omega(\Omega)} \pi_Q(x) \varphi_Q^{\Omega,\Omega}(x).$$

Notice that this function depends also on the choice of π_Q.

THEOREM 4.2. *Let* $\Omega \neq \boldsymbol{R}^n$, f *a function on* Ω, *and* $0 < q \leqq \infty$. *Then* $g = g_{\Omega,k,q}(f)$ *satisfies the following.*

(1) g *is a* C^∞ *function on* Ω.

(2) *If* $q < r \leqq \infty$, *then, for each nonnegative integer* m,

$$\|\rho_\Omega^m |\nabla^m g|^*\|_{r,\Omega} \leqq c_m \|f\|_{r,\Omega},$$

where $c_m = c(n, a, k, q, m, r)$.

(3) *If* $1/q + \alpha/n > 1/p$, *then*

$$\|\rho_\Omega^{k+1-\alpha} |\nabla^{k+1} g|^*\|_{p,\Omega} \leqq c|f; C_p^{\alpha,k}(\Omega)|$$

with $c = c(n, a, k, \alpha, p, q)$.

To prove this theorem, we use the following lemma.

LEMMA 4.4. *Let* $\Omega \neq \boldsymbol{R}^n$, f *a function on* Ω, *and let* $a_Q (Q \in \mathscr{G}_\Omega(\Omega))$ *be nonnegative real numbers.*

(1) *For each* t *with* $0 < t < 1$ *and for each real number* β, *there exists* $c = c(n, t, \beta)$ *such that*

$$c^{-1}(\rho_\Omega^\beta f)^{*,t}(x) \leqq \rho_\Omega(x)^\beta f^{*,t}(x) \leqq c(\rho_\Omega^\beta f)^{*,t}(x).$$

(2) *For each* t, t' *with* $0 < t$, $t' < 1$ *and for each* p, *there exists* $c = c(n, p, t, t')$ *such that*

$$c^{-1}\|f^{*,t}\|_{p,\Omega} \leqq \|f^{*,t'}\|_{p,\Omega} \leqq c\|f^{*,t}\|_{p,\Omega}.$$

(3) *If* $0 < t < 1$, $1 \leqq u < a^2$, *and* $v = u + 4a^2t/(1-t)$, *then*

$$\left(\sum_{Q \in \mathscr{G}_\Omega(\Omega)} a_Q \chi_{uQ}\right)^{*,t}(x) \leqq \sum_{Q \in \mathscr{G}_\Omega(\Omega)} a_Q \chi_{vQ}(x) \quad \textit{for all } x \in \Omega.$$

PROOF. (1) Easy.

(2) We shall simply write $\rho = \rho_\Omega$. We may assume $t \leqq t'$. Then the left hand inequality is obvious since $f^{*,t} \leqq f^{*,t'}$ pointwise. We shall prove the right hand inequality. Fix an $x \in \Omega$ and write $Q_{t'}(x) = Q$. Take a positive integer N such that $N^{-1}t' \leqq t(1-t')$ and decompose Q into N^n cubes Q_j ($j = 1, \dots, N^n$) each with sidelength $\ell(Q_j) = N^{-1}\ell(Q) = N^{-1}t'\rho(x)$. For every $y \in Q_j$, we have $t\rho(y) \geqq t(1-t')\rho(x) \geqq N^{-1}t'\rho(x) = \ell(Q_j)$ and hence $Q_j \subset Q_t(y)$. Hence $\|f\|_{\infty,Q_j} \leqq \inf_{Q_j} f^{*,t}$ for every j. Hence, for each $r > 0$,

$$\begin{aligned} f^{*,t'}(x) &= \max\{\|f\|_{\infty,Q_j}\}_j \leqq \max\left\{\inf_{Q_j} f^{*,t}\right\}_j \leqq \max\{|Q_j|^{-1/r}\|f^{*,t}\|_{r,Q_j}\}_j \\ &\leqq N^{n/r}|Q|^{-1/r}\|f^{*,t}\|_{r,Q} \leqq N^{n/r}M_r^\Omega(f^{*,t})(x). \end{aligned}$$

Thus the desired inequality follows from the boundedness of M_r^Ω, $r < p$, in $L^p(\Omega)$.

(3) This can be readily deduced from the following simple geometric fact: If $Q \in \mathscr{G}_\Omega(\Omega)$ and $uQ \cap Q_t(x) \neq \emptyset$, then $x \in vQ$ (with v given in the lemma). Lemma 4.4 is proved.

PROOF OF THEOREM 4.2. In this proof, sums over cubes are taken over $\mathscr{G}_\Omega(\Omega)$. We shall simply write $\rho = \rho_\Omega$ and $f^\# = f^\#_{k,\alpha,q}$.

(1) Obvious.

(2) By Lemma 2.2(1),

$$\|\pi_Q\|_{\infty,Q} \leqq c|Q|^{-1/q}\|f\|_{q,Q} \leqq c \inf_{3Q} M_q^\Omega(f).$$

Hence, using Lemma 4.3 and Lemma 2.1(3), we have

$$|\nabla^m g| \leqq c \sum_Q \ell(Q)^{-m}\left(\inf_{3Q} M_q^\Omega(f)\right)\chi_{2Q}$$

and, thus,

$$\rho^m|\nabla^m g| \leqq c \sum_Q \left(\inf_{3Q} M_q^\Omega(f)\right)\chi_{2Q}. \tag{4.2}$$

We take t such that $0 < t < 1$ and $2 + 4a^2 t/(1-t) \leqq 3$. Then using (4.2) and Lemma 4.4, we have

$$\|\rho^m |\nabla^m g|^*\|_{r,\Omega} \leqq c\|(\rho^m |\nabla^m g|)^*\|_{r,\Omega} \leqq c\left\|\left(\sum_Q \left(\inf_{3Q} M_q^\Omega(f)\right)\chi_{2Q}\right)^{*,t}\right\|_{r,\Omega}$$

$$\leqq c\left\|\sum_Q \left(\inf_{3Q} M_q^\Omega(f)\right)\chi_{3Q}\right\|_{r,\Omega} \leqq c\|M_q^\Omega(f)\|_{r,\Omega} \leqq c\|f\|_{r,\Omega}.$$

(3) Fix a $Q \in \mathscr{G}_\Omega(\Omega)$. For cubes $R \in \mathscr{G}_\Omega(\Omega)$ satisfying $2R \cap Q \neq \varnothing$, we have $R \approx Q$ (Lemma 4.1(3)) and, hence,

$$\|\pi_R - \pi_Q\|_{\infty,Q} \leqq c\ell(Q)^\alpha \inf_{2Q} f^\#$$

(by Lemma 2.2(2)). Using this inequality together with Lemma 2.1(3) and Lemma 4.3, we obtain

$$\|\nabla^{k+1} g\|_{\infty,Q} = \left\|\nabla^{k+1}\sum_R (\pi_R - \pi_Q)\varphi_R^{\Omega,\Omega}\right\|_{\infty,Q} \leqq c\ell(Q)^{\alpha-k-1}\inf_{2Q} f^\#$$

(we used also the fact that $\nabla^{k+1}\sum_R \pi_Q \varphi_R^{\Omega,\Omega} = \nabla^{k+1}\pi_Q = 0$ on Ω). Hence

$$|\nabla^{k+1} g| \leqq c\sum_Q \ell(Q)^{\alpha-k-1}\left(\inf_{2Q} f^\#\right)\chi_Q.$$

Taking t such that $0 < t < 1$ and $1 + 4a^2 t/(1-t) \leqq 2$ and using Lemma 4.4(3), we have

$$\rho^{k+1-\alpha}|\nabla^{k+1} g|^{*,t} \leqq c\sum_Q \left(\inf_{2Q} f^\#\right)\chi_{2Q} \leqq cf^\#,$$

which combined with (1) and (2) of Lemma 4.4 implies the desired inequality. Theorem 4.2 is proved.

Finally we shall consider the case $\Omega = \boldsymbol{R}^n$. We introduce some notations. We define $\mathscr{A}$ as the set of functions f on $\boldsymbol{R}^n$ such that the set $\{x; |f(x)| > t\}$ has finite measure for every $t > 0$. If, for a given function f on $\boldsymbol{R}^n$, there exists a polynmial P such that $f - P \in \mathscr{A}$, then such P is unique (since 0 is the only polynomial contained in $\mathscr{A}$) and we write the unique P as $P_0(D128;)$.

THEOREM 4.3. *Suppose that either* (i) $1/p > \alpha/n$ *or* (ii) $1/p = \alpha/n \geqq 1$ *and* $k+1 \geqq \alpha$. *Let f be a function on $\boldsymbol{R}^n$ satisfying $|f; C_p^{\alpha,k}| < \infty$. Then the following hold.*

(1) *$P_0(f)$ exists and $P_0(f) \in \mathscr{P}_k$.*

(2) *There exists $c = c(n,k,\alpha,p)$ such that $\|f - P_0(f); \Gamma(1/p - \alpha/n)\| \leqq c|f; C_p^{\alpha,k}|$.*

(3) *Let $\{R_j\}$ be a sequence of cubes such that $\ell(R_j) \to \infty$ as $j \to \infty$ and that there exists a positive number u for which the intersection of the cubes $\{uR_j\}_j$ is nonempty. Suppose $\infty > 1/q > 1/p - \alpha/n$ and $\pi_j \in \Pi_{k,q,R_j}(f)$. Then $\pi_j \to P_0(f)$ in $\mathscr{P}_k$ as $j \to \infty$.*

(4) *In the case* (ii), *it holds that $f - P_0(f) \in C_0$.*

PROOF. Here we shall not completely prove the theorem; to be precise, for the case (i), we shall prove all the claims, but for the case (ii), we shall prove only a part of the claims. Proof for the case (ii) shall be completed in Section 5.

First consider the case (i). Suppose $\infty > 1/q > 1/p - \alpha/n$. Let $Q_0 = [-1/2, 1/2]^n$ and $Q_m = 2^m Q_0$ and let $\pi'_m \in \Pi_{k,q,Q_m}(f)$. From Lemma 2.2(2), it follows that

$$\|\pi'_{m+1} - \pi'_m\|_{\infty,Q_m} \leqq c \int_{|Q_m|}^{|Q_{m+1}|} (f^\#)^\sim(s) s^{\alpha/n-1}\, ds, \tag{4.3}$$

where $f^\# = f^\#_{k,\alpha,q}$ and $(f^\#)^\sim$ denotes the nonincreasing rearrangement of $f^\#$. Since $f^\# \in L^p$, we have $(f^\#)^\sim(s) = o(s^{-1/p})$ as $s \to \infty$ and hence

$$\int_1^\infty (f^\#)^\sim(s) s^{\alpha/n-1}\, ds < \infty. \tag{4.4}$$

From (4.3) and (4.4), it follows that $\sum_{m=1}^\infty \|\pi'_{m+1} - \pi'_m\|_{\infty,Q_m} < \infty$ and, a fortiori, that $\lim_{m\to\infty} \pi'_m$ exists in $\mathscr{P}_k$. Set $P = \lim_{m\to\infty} \pi'_m$. From Lemma 2.5 and Fatou's lemma, we see that the inequality of (2) holds if $P_0(f)$ is replaced by P. In particular $f - P \in \Gamma(1/p - \alpha/n)$, which in turn implies $f - P \in \mathscr{A}$ (since $1/p > \alpha/n$) and thus $P = P_0(f)$. Finally let R_j and π_j be as mentioned in (3). For each j, let $m(j)$ be the integer such that $2^{m(j)} < \ell(R_j) \leqq 2^{m(j)+1}$. Then, from Lemma 2.2(2),

$$\|\pi'_{m(j)} - \pi_j\|_{\infty,Q_{m(j)}} \leqq c \int_{|R_j|}^{2|R_j|} (f^\#)^\sim(s) s^{\alpha/n-1}\, ds,$$

which combined with (4.4) implies that $\pi'_{m(j)} - \pi_j \to 0$ in $\mathscr{P}_k$ as $j \to \infty$. Hence $\lim \pi_j = \lim \pi'_m = P = P_0(f)$. Thus the proof is complete in the case (i).

Next consider the case (ii). Since $p \leqq 1$, it holds that $L^p \subset L^{p,1}$ (where $L^{p,1}$ is the Lorentz space; cf. e.g. [SW; Chapt, V. §3]) and, hence, (4.4) holds again. Hence, by the same reasoning as in the case (i), we can prove the following: (a)

If $\{\pi_j\}$ is a sequence of polynomials as mentioned in (3), then $\lim_{j\to\infty} \pi_j$ exists in $\mathscr{P}_k$ and this limit (may depend on q but) does not depend on the choice of $\{R_j\}$ and $\{\pi_j\}$ (we write $\lim \pi_j = P$); (b) the inequality of (2) holds if $P_0(f)$ is replaced by P. (In fact, the assumption $k+1 \geqq \alpha$ is not necessary to prove these results.) In Section 5, Proof of Theorem 5.2, we shall complete the proof by showing that $f - P \in C_0$ (which will imply $P = P_0(f)$).

§5. Atomic decomposition

Fix k, α, and p such that $k+1 \geqq \alpha$ and $p < \infty$. Suppose $\Omega \neq \boldsymbol{R}^n$ and $\{\varphi_j\}$ is a sequence of functions on $\boldsymbol{R}^n$ which satisfies the assumptions of Theorem 3.1 with cubes R_j such that $2R_j \subset \Omega$ and suppose g is a smooth function on Ω satisfying

$$\|\rho_\Omega^{k+1-\alpha}|\nabla^{k+1} g|^*\|_{p,\Omega} = B < \infty.$$

Then, by (2) $\sim$ (5) of Theorem 3.1, the series $\sum_j \varphi_j$. converges unconditionally in $L^r_{\mathrm{loc}}(\Omega)$ for some $r > 0$. We set $f = g + \sum_j \varphi_j$. Combining Theorem 3.1(1) and Theorem 4.1, we have

$$|f; C_p^{\alpha,k}(\Omega)| \leqq c(B + M_p),$$

where M_p is as mentioned in Theorem 3.1 and $c = c(n,k,\alpha,p)$.

The main purpose of the present section is to prove the converse to the above fact. The results are given in the following two theorems. Recall that we are assuming a is a sufficiently large positive number.

THEOREM 5.1. *Suppose* $\Omega \neq \boldsymbol{R}^n$, $k+1 \geqq \alpha$, *and* $p < \infty$. *Also suppose* $0 < q \leqq \infty$ *and* $1/q + \alpha/n > 1/p$. *Let* f *be a function on* Ω *such that* $|f; C_p^{\alpha,k}(\Omega)| < \infty$. *Then there exist* φ_m, Q_m, *and* λ_m $(m = 1, 2, \ldots)$ *which satisfy the following* (0) $\sim$ (v):

(0) φ_m *are functions on* $\boldsymbol{R}^n$, Q_m *are cubes, and* λ_m *are nonnegative numbers;*

(i) $\operatorname{supp} \varphi_m \subset Q_m$;

(ii) $2^{-1}a^2 Q_m \subset \Omega$;

(iii) $|\varphi_m; C_\infty^{\alpha,k}| \leqq \lambda_m$;

(iv) *for each* $r > 0$,

$$\left\| \left(\sum_m \lambda_m^r \chi_{Q_m} \right)^{1/r} \right\|_p \leqq c_r |f; C_p^{\alpha,k}(\Omega)|,$$

where $c_r = c(n,k,\alpha,p,q,a,r)$;

(v) $f = g_{\Omega,k,q}(f) + \sum_m \varphi_m$.

THEOREM 5.2. *Let* $k+1 \geqq \alpha$ *and let* f *be a function on* $\boldsymbol{R}^n$ *such that* $|f; C_p^{\alpha,k}| < \infty$. *Then*:

(I) *If* $1/p > \alpha/n$ *or* $1/p = \alpha/n \geqq 1$, *then there exist* φ_m, Q_m, *and* λ_m $(m = 1, 2, \ldots)$ *which satisfy* (0), (i), (iii), *and* (iv) (*with* $\Omega = \boldsymbol{R}^n$ and with $c_r = c(n, k, \alpha, p, r)$) *of Theorem* 5.1 *and also satisfy*

(v′) $f = P_0(f) + \sum_m \varphi_m$.

(II) *If* $0 < 1/p < \alpha/n$ *or* $0 < 1/p = \alpha/n < 1$, *then there exist* φ_m, Q_m, *and* λ_m $(m = 1, 2, \ldots)$ *which satisfy* (0), (i), (iii), *and* (iv) (*with the same replacement as in* (I)) *of Theorem* 5.1 *and also satisfy*

(v″) $f - \sum_m \varphi_m \in \mathscr{P}_k$.

REMARK. The series $\sum_m \varphi_m$ in Theorem 5.1 converges unconditionally in $L^r_{\mathrm{loc}}(\Omega)$ for some $r > 0$ as we already saw at the beginning of this section. The series $\sum_m \varphi_m$ in the case (I) of Theorem 5.2 converges unconditionally in $\Gamma(1/p - \alpha/n)$ (when $1/p > \alpha/n$) or converges absolutely in C_0 (when $1/p = \alpha/n \geqq 1$) by (2) and (3) of Theorem 3.1. The series $\sum_m \varphi_m$ in the case (II) of Theorem 5.2 converges unconditionally with respect to $|\cdot; C_p^{\alpha,k}|$ by Theorem 3.1(1) (cf. also the remark given in the first paragraph of Section 3).

In the rest of this section, we shall prove the above theorems and, as a corollary to the proof of the case (I) of Theorem 5.2, we shall complete the proof of Theorem 4.3. The main idea of the proof of Theorem 5.1 is the same as in [Mi2; §3].

For the proofs of Theorems 5.1 and 5.2, we use Lemmas 5.1 $\sim$ 5.4 to be given below. In these lemmas, we fix an Ω and assume U, V, and W are open subsets of Ω, and, for open subsets $D \subset \Omega$, we abbreviate $\mathscr{G}_\Omega(D)$ to $\mathscr{G}(D)$.

LEMMA 5.1. (1) $\{Q \in \mathscr{G}(\Omega); 3aQ \subset U\} = \{Q \in \mathscr{G}(U); 3aQ \subset U\}$.

(2) *There exists* b_3 *such that*: *If* Q *is a dyadic cube satisfying* $b_3 aQ \subset U$, *then* $\varphi_Q^{\Omega,U} = \varphi_Q^{\Omega,\Omega}$.

(3) *There exists* b_4 *such that*: *If* $U \subset V \subset W$, $R \in \mathscr{G}(V)$, $b_4 aR \subset U$, *and if* Q *is a dyadic cube satisfying* $2Q \cap 2R \neq \emptyset$, *then* $\varphi_Q^{\Omega,U} = \varphi_Q^{\Omega,W} = \varphi_Q^{\Omega,\Omega}$.

PROOF. For a dyadic cube Q, we denote by $\tilde{Q}$ the unique dyadic cube such that $\tilde{Q} \supset Q$ and $\ell(\tilde{Q}) = 2\ell(Q)$.

(1) Suppose $Q \in \mathscr{G}(U)$ and $3aQ \subset U$. Then the maximality of $Q \in \mathscr{G}(U)$ implies that $a^2\tilde{Q} \not\subset \Omega$ or $a\tilde{Q} \not\subset U$. But the latter relation is impossible since $a\tilde{Q} \subset 3aQ \subset U$. Hence $a^2\tilde{Q} \not\subset \Omega$ and, hence, $Q \in \mathscr{G}(\Omega)$. This shows that the right

hand set in (1) is included in the left hand set. The converse inclusion is obvious from the definition of $\mathscr{G}(\Omega)$ and $\mathscr{G}(U)$.

(2) We may and shall assume $Q \in \mathscr{G}(\Omega) \cup \mathscr{G}(U)$. In order to show $\varphi_Q^{\Omega,\Omega} = \varphi_Q^{\Omega,U}$, it is sufficient to show

$$\{R \in \mathscr{G}(\Omega); 2Q \cap 2R \neq \varnothing\} = \{R \in \mathscr{G}(U); 2Q \cap 2R \neq \varnothing\}. \tag{5.1}$$

First, suppose $3aQ \subset U$. Then $Q \in \mathscr{G}(\Omega) \cap \mathscr{G}(U)$ by (1). Hence, for cubes R belonging to either the right or the left hand set of (5.1), we have $\ell(R) \approx \ell(Q)$ (by Lemma 4.1(3)) and, hence, $3aR \subset b_3aQ$ for some $b_3 > 3$. Now, with this b_3, assume $b_3aQ \subset U$. Then: If R belongs to the left hand set of (5.1), then $3aR \subset b_3aQ \subset U$ and hence, by (1), R belongs to the right hand set of (5.1); and the same holds if we interchange 'left' and 'right'. Thus (5.1) is proved.

(3) First suppose $R \in \mathscr{G}(V)$ and $3aR \subset U$. Then $R \in \mathscr{G}(\Omega) \cap \mathscr{G}(U) \cap \mathscr{G}(W)$ by (1). If $Q \in \mathscr{G}(\Omega) \cup \mathscr{G}(U) \cup \mathscr{G}(W)$ and $2R \cap 2Q \neq \varnothing$, then $\ell(Q) \approx \ell(R)$ (by Lemma 4.1(3)) and hence $b_3aQ \subset b_4aR$ for some $b_4 > 3$. Now assume $R \in \mathscr{G}(V)$ and $b_4aR \subset U$ and assume Q is a dyadic cube such that $2Q \cap 2R \neq \varnothing$. Then: If $Q \in \mathscr{G}(\Omega) \cup \mathscr{G}(U) \cup \mathscr{G}(W)$, then $b_3aQ \subset b_4aR \subset U \subset W$ and hence $\varphi_Q^{\Omega,\Omega} = \varphi_Q^{\Omega,U} = \varphi_Q^{\Omega,W}$ by (2); and if $Q \notin \mathscr{G}(\Omega) \cup \mathscr{G}(U) \cup \mathscr{G}(W)$, then $\varphi_Q^{\Omega,\Omega} = \varphi_Q^{\Omega,U} = \varphi_Q^{\Omega,W} = 0$. Lemma 5.1 is proved.

Lemma 5.2. *If T is a cube such that $a^2T \subset \Omega$ and $aT \not\subset U$, and if $Q \in \mathscr{G}(U)$ and $2Q \cap T \neq \varnothing$, then $\ell(Q) \leqq 2\ell(T)$, $Q \subset 7T$, and $4aQ \not\subset U$.*

Proof. Suppose T and Q are cubes satisfying the assumptions of the lemma. If $\ell(T) < \ell(Q)/2$, then, since $2Q \cap T \neq \varnothing$ and a is large, we have $aT \subset (a/2 + b)Q \subset aQ \subset U$, which contradicts the assumption $aT \not\subset U$. Thus $\ell(T) \geqq \ell(Q)/2$, which combined with the assumption $2Q \cap T \neq \varnothing$ implies that $Q \subset 7T$. If $\ell(Q) < \ell(T)/3$, then, since $2Q \cap T \neq \varnothing$ and a is large, we have $a^2\tilde{Q} \subset (2a^2/3 + b)T \subset a^2T \subset \Omega$ and hence, by the maximality of $Q \in \mathscr{G}(U)$, we have $a\tilde{Q} \not\subset U$ and thus $3aQ \not\subset U$ (since $3aQ \supset a\tilde{Q}$). Finally, if $\ell(Q) \geqq \ell(T)/3$, then, since $2Q \cap T \neq \varnothing$, we have $aT \subset (3a + b)Q \subset 4aQ$ and hence $4aQ \not\subset U$ (since $aT \not\subset U$). Lemma 5.2 is proved.

Lemma 5.3. *If T is a cube such that $a^2T \subset \Omega$ and $aT \subset U$, and if Q, $Q' \in \mathscr{G}(U)$, $2Q \cap T \neq \varnothing$, and $2Q' \cap T \neq \varnothing$, then $\ell(T) \leqq b\ell(Q)$, $T \subset bQ$, and $Q \approx Q'$.*

The above lemma easily follows from (2) and (3) of Lemma 4.1.

LEMMA 5.4. *There exists* b_5 *such that: If* $U \subset V$, $R \in \mathcal{G}(V)$, $b_4 aR \not\subset U$, $Q \in \mathcal{G}(U)$, *and* $2Q \cap 2R \neq \emptyset$, *then* $b_5 aQ \not\subset U$.

PROOF. Take $b_5 > 3$ so that $2Q \cap 2R \neq \emptyset$ and $\ell(R) \leqq b_2 \ell(Q)$ imply $b_4 aR \subset b_5 aQ$. Now suppose U, V, R, and Q satisfy the assumptions of the lemma. We shall prove $b_5 aQ \not\subset U$. This is obvious if $3aQ \not\subset U$. Hence we assume $3aQ \subset U$. Then, by Lemma 5.1(1) and Lemma 4.1(3), we have $Q \in \mathcal{G}(\Omega)$ and $\ell(R) \leqq b_2 \ell(Q)$. Hence $b_4 aR \subset b_5 aQ$, which combined with the assumption $b_4 aR \not\subset U$ implies that $b_5 aQ \not\subset U$. The lemma is proved.

PROOF OF THEOREM 5.1. In this proof, c denotes various positive constants depending only on n, k, α, p, q, a, and other explicitly indicated parameters (if any). We simply write $f^{\#} = f^{\#}_{k,\alpha,q}$ and $\mathcal{G}(U) = \mathcal{G}_{\Omega}(U)$.

For each cube $Q \subset \Omega$, we take a $\pi_Q \in \Pi_{k,q,Q}(f)$. With each pair (Q, R) of dyadic cubes, we associate a dyadic cube $K(Q, R)$ in such a way that $K(Q, R)$ is equal to either Q or R, $\ell(K(Q, R)) = \min\{\ell(Q), \ell(R)\}$, and that $K(Q, R) = K(R, Q)$ for all Q and R. We set $P_{Q,R} = \pi_{K(Q,R)}$ for dyadic cubes Q, $R \subset \Omega$. Notice that $P_{Q,R} = P_{R,Q}$.

If $|f; C_p^{\alpha,k}(\Omega)| = 0$, then $f = g_{\Omega,k,q}(f)$ and the conclusion of the theorem is obvious. Thus we assume $|f; C_p^{\alpha,k}(\Omega)| > 0$. We take a continuous function h on Ω such that $h(x) > 0$ for all $x \in \Omega$ and $\|h\|_{p,\Omega} \leqq |f; C_p^{\alpha,k}(\Omega)|$. For each integer j, we set

$$U_j = \{x \in \Omega; f^{\#}(x) + h(x) > 2^j\}.$$

Since $f^{\#} + h$ is a lower semicontinuous function, U_j is an open subset of Ω. We shall simply write $\varphi_Q^j = \varphi_Q^{\Omega, U_j}$ and $\varphi_Q^{\Omega} = \varphi_Q^{\Omega,\Omega}$. In the sequel, sums over cubes shall be taken over all the dyadic cubes unless the contrary is explicitly stated. Now we define

$$g^j = f\chi_{\Omega \setminus U_j} + \sum_{R,S} P_{R,S} \varphi_R^j \varphi_S^{j-1}.$$

It is easy to see that $g^{j+1} - g^j$ can be written as

$$g^{j+1} - g^j = \sum_R h_R^j$$

with

$$h_R^j = \varphi_R^j\left(f - f\chi_{U_{j+1}} + \sum_Q P_{Q,R}\varphi_Q^{j+1} - \sum_S P_{R,S}\varphi_S^{j-1}\right).$$

It is obvious that

(5.2) $$h_R^j \neq 0 \quad \text{only if } R \in \mathcal{G}(U_j)$$

and

(5.3) $$\operatorname{supp} h_R^j \subset 2R.$$

The following estimate is the core of the proof:

(5.4) $$|h_R^j; C_\infty^{\alpha,k}| \leqq c2^j.$$

For the moment we assume this estimate, (5.4), and show that this implies the desired decomposition of f.

It is obvious that $g^j(x) \to f(x)$ a.e. as $j \to \infty$ (since $U_j \supset U_{j+1}$ and $|U_j| \to 0$ as $j \to \infty$). On the other hand, since $U_j \to \Omega$ as $j \to -\infty$ (this is because $h(x) > 0$ for all $x \in \Omega$), it holds that

$$g^j(x) \to \sum_{R,S} P_{R,S}(x)\varphi_R^\Omega(x)\varphi_S^\Omega(x) \quad \text{as } j \to -\infty$$

for every $x \in \Omega$ (cf. [Mi2; Proof of (3.12), pp. 219–220]). Hence

$$f - \sum_{R,S} P_{R,S}\varphi_R^\Omega\varphi_S^\Omega = \sum_{j=-\infty}^{\infty}\left(\sum_R h_R^j\right) \quad \text{a.e.}$$

and, thus,

(5.5) $$f - g_{\Omega,k,q}(f) = \sum_{R,S}(P_{R,S} - \pi_R)\varphi_R^\Omega\varphi_S^\Omega + \sum_{j=-\infty}^{\infty}\left(\sum_R h_R^j\right) \quad \text{a.e..}$$

We shall prove that the right side of this equality forms a series which satisfies the estimates as mentioned in the theorem.

First, for each $r > 0$,

$$\left(\sum_{j=-\infty}^{\infty}\sum_{R\in\mathcal{G}(U_j)} 2^{jr}\chi_{2R}(x)\right)^{1/r} \leqq \left(c\sum_j 2^{jr}\chi_{U_j}(s)\right)^{1/r} \leqq c_r(f^{\#}(x) + h(x)),$$

and hence

$$\left\| \left(\sum_j \sum_{R \in \mathscr{G}(U_j)} 2^{jr} \chi_{2R} \right)^{1/r} \right\|_p \leqq c_r \| f^\# + h \|_{p,\Omega} \leqq c_r |f; C_p^{\alpha,k}(\Omega)|,$$

which together with (5.2) $\sim$ (5.4) implies that the families $\{h_R^j\}$, $\{2R\}$, and $\{c2^j\}$ each indexed by the set $\{(j,R); j \in \boldsymbol{Z}, R \in \mathscr{G}(U_j)\}$ form a triplet of sequences satisfying the conditions as mentioned in (0) $\sim$ (iv) of the theorem.

Next, the first sum on the right side of (5.5) can be written as $\sum_R h_R$ with

$$h_R = \sum_S (P_{R,S} - \pi_R) \varphi_R^\Omega \varphi_S^\Omega.$$

If $\varphi_R^\Omega \varphi_S^\Omega \not\equiv 0$, then $2R \cap 2S \neq \emptyset$, $\ell(S) \approx \ell(R)$, and $S \approx R$ (by Lemma 4.1(3)), and hence $K(R,S) \approx R$ and

$$\|\nabla^m (P_{R,S} - \pi_R)\|_{\infty,2R} \leqq c\ell(R)^{\alpha-m} \inf_{2R} f^\#$$

for every nonnegative integer m (by Lemma 2.2(2) and Lemma 2.1(3)) Using these estimates and Lemma 4.2(3), we obtain

$$\|\nabla^{k+1} h_R\|_\infty \leqq c\ell(R)^{\alpha-k-1} \inf_{2R} f^\#.$$

From this, using Lemmas 2.9 and 2.8, we obtain

$$|h_R; C_\infty^{\alpha,k}| \leqq c \inf_{2R} f^\# \tag{5.6}$$

(here we also used the assumption $k+1 \geqq \alpha$). For each $r > 0$, we have

$$\left\| \left(\sum_{R \in \mathscr{G}(\Omega)} \left(\inf_{2R} f^\# \right)^r \chi_{2R} \right)^{1/r} \right\|_p \leqq \|c_r f^\#\|_{p,\Omega} \leqq c_r |f; C_p^{\alpha,k}(\Omega)|. \tag{5.7}$$

The estimates (5.6) and (5.7) together with the obvious fact $\operatorname{supp} h_R \subset 2R$ imply that the families $\{h_R\}$, $\{2R\}$, and $\{c \inf_{2R} f^\#\}$ each indexed by $\mathscr{G}(\Omega)$ also form a triplet of sequences satisfying the conditions as mentioned in (0) $\sim$ (iv) of the theorem.

Now the rest of the proof is to show (5.4). In order to do this, we first observe that

$$h_R^j = 0 \quad \text{if} \quad b_4 a R \subset U_{j+1}. \tag{5.8}$$

In fact this can be seen as follows. If $R \notin \mathscr{G}(U_j)$, then obviously $h_R^j = 0$. Suppose $R \in \mathscr{G}(U_j)$ and $b_4 aR \subset U_{j+1}$. By Lemma 5.1(3), we have $\varphi_Q^{j+1} = \varphi_Q^{j-1} = \varphi_Q^{\Omega}$ for all dyadic cubes Q satisfying $2R \cap 2Q \neq \varnothing$. Hence, on $2R$,

$$\sum_Q P_{Q,R}\varphi_Q^{j+1} - \sum_S P_{R,S}\varphi_S^{j-1} = \sum_Q P_{Q,R}\varphi_Q^{\Omega} - \sum_S P_{R,S}\varphi_S^{\Omega} = 0,$$

where the last equality follows from the symmetry property $P_{Q,R} = P_{R,Q}$. On the other hand, $f - f\chi_{U_{j+1}} = 0$ on $2R$ since $2R \subset b_4 aR \subset U_{j+1}$. Hence $h_R^j = 0$. Thus (5.8) holds.

We shall now prove (5.4). By (5.2) and (5.8), we may and shall assume $R \in \mathscr{G}(U_j)$ and $b_4 aR \not\subset U_{j+1}$. We shall prove the estimate

$$\ell(T)^{-\alpha} I_{k,\infty,T}(h_R^j) \leqq c2^j \tag{5.9}$$

for cubes T such that

$$T \cap 2R \neq \varnothing \quad \text{and} \quad \ell(T) \leqq \ell(R)/2. \tag{5.10}$$

This estimate together with (5.3) will imply (5.4) (by Lemma 2.8). In the sequel, we assume T is a cube satisfying (5.10). Notice that $T \subset 3R$ and $a^2 T \subset (a^2/2 + b)R \subset a^2 R \subset \Omega$.

Case 1: $aT \not\subset U_{j+1}$.

We decompose h_R^j as follows:

$$\begin{aligned} h_R^j &= h_1 + h_2 + h_3, \\ h_1 &= \varphi_R^j (f - \pi_T)\chi_{\Omega \setminus U_{j+1}}, \\ h_2 &= \varphi_R^j \sum_Q (P_{Q,R} - \pi_T)\varphi_Q^{j+1}, \\ h_3 &= \varphi_R^j \sum_S (\pi_T - P_{R,S})\varphi_S^{j-1}. \end{aligned}$$

We shall prove the estimate

$$\ell(T)^{-\alpha} I_{k,\infty,T}(h_i) \leqq c2^j \tag{5.11}$$

for $i = 1, 2$, and 3, which will of course imply (5.9).

We first estimate h_1. By Lemma 2.2(4), the following holds for a.e. $x \in T \cap (\Omega \setminus U_{j+1})$:

$$|f(x) - \pi_T(x)| \leqq c\ell(T)^{\alpha} f^{\#}(x) \leqq c\ell(T)^{\alpha} 2^j.$$

Hence $\|h_1\|_{\infty,T} \leqq c\ell(T)^{\alpha} 2^j$, which implies (5.11) for $i = 1$.

Next, we estimate h_2. Suppose Q is a dyadic cube such that

$$Q \in \mathscr{G}(U_{j+1}) \quad \text{and} \quad 2Q \cap 2R \cap T \neq \emptyset. \tag{5.12}$$

By Lemma 5.2, we have $\ell(Q) \leqq 2\ell(T)$, $Q \subset 7T$, and $4aQ \not\subset U_{j+1}$. By Lemma 4.1(3), we have $\ell(Q) \leqq b\ell(R)$ and thus $K(Q,R) \approx Q$. Thus $K(Q,R) \subset b_6 T$ for some $b_6 > 1$. Since $4aQ \subset b_7 a K(Q,R)$ for some b_7 and since $b_7 a b_6 T \subset a^2 T \subset \Omega$, we can use Lemma 2.2(3) to obtain

$$\|\pi_{K(Q,R)} - \pi_{b_6 T}\|_{\infty, K(Q,R)} \leqq c\ell(T)^{\alpha} \inf_{4aQ} f^{\#} \leqq c\ell(T)^{\alpha} 2^{j}$$

(the last inequality follows from the relation $4aQ \not\subset U_{j+1}$). Similarly, using Lemma 2.2(2), we have

$$\|\pi_{b_6 T} - \pi_T\|_{\infty, T} \leqq c\ell(T)^{\alpha} \inf_{aT} f^{\#} \leqq c\ell(T)^{\alpha} 2^{j}$$

(the last inequality follows from the assumption $aT \not\subset U_{j+1}$). Using these inequalities and using Lemma 2.1(2), we obtain

$$\|P_{Q,R} - \pi_T\|_{\infty, 2Q} \leqq c\|\pi_{K(Q,R)} - \pi_{b_6 T}\|_{\infty, K(Q,R)} + c\|\pi_{b_6 T} - \pi_T\|_{\infty, T} \leqq c\ell(T)^{\alpha} 2^{j}$$

and thus

$$\|h_2\|_{\infty, T} = \left\| \varphi_R^{j} \sum_{Q:(5.12)} (P_{Q,R} - \pi_T)\varphi_Q^{j+1} \right\|_{\infty, T} \leqq c\ell(T)^{\alpha} 2^{j},$$

which implies (5.11) for $i = 2$.

Finally, we shall estimate h_3. Suppose S is a dyadic cube such that

$$S \in \mathscr{G}(U_{j-1}) \quad \text{and} \quad 2R \cap 2S \neq \emptyset. \tag{5.13}$$

By Lemma 4.1(3), we have $\ell(R) \leqq b\ell(S)$ and hence $K(R,S) \approx R$. We take $b_8 > 3$ such that $T \subset b_8 R$ and $K(R,S) \subset b_8 R$ and take cubes T_i $(i = 0, 1, \ldots, N)$ such that

$$T = T_0 \subset T_1 \subset \cdots \subset T_N = b_8 R,$$

$$\ell(T_i) = 2\ell(T_{i-1}) \quad \text{for} \quad 1 \leqq i < N, \quad \text{and} \quad \ell(T_{N-1}) < \ell(T_N) \leqq 2\ell(T_{N-1}). \tag{5.14}$$

From the assumption $aT \not\subset U_{j+1}$, it follows that $\inf_{aT_i} f^{\#} \leqq 2^{j+1}$ for each i (notice that $aT_i \subset ab_8 R \subset a^2 R \subset \Omega$). Hence, by lemma 2.2(2) and Lemma 2.1(3),

$$\|\nabla^m(\pi_{T_i} - \pi_{T_{i-1}})\|_{\infty, T_i} \leqq c\ell(T_i)^{\alpha - m} \inf_{aT_i} f^{\#} \leqq c\ell(T_i)^{\alpha - m} 2^{j}$$

and

$$\|\nabla^m(P_{R,S} - \pi_{T_N})\|_{\infty,T_N} \leqq c\ell(T_N)^{\alpha-m} \inf_{aT_N} f^\# \leqq c\ell(T_N)^{\alpha-m}2^j,$$

and thus

$$\begin{aligned}\|\nabla^m(P_{R,S} - \pi_T)\|_{\infty,T} &\leqq \|\nabla^m(P_{R,S} - \pi_{T_N})\|_{\infty,T_N} + \sum_{i=1}^{N} \|\nabla^m(\pi_{T_i} - \pi_{T_{i-1}})\|_{\infty,T_i} \\ &\leqq c2^j \sum_{i=1}^{N} \ell(T_i)^{\alpha-m},\end{aligned}$$

where m denotes an arbitrary nonnegative integer. Using this inequality, Leibniz' formula on differentiation, and Lemma 4.2(3), we obtain

$$(5.15) \qquad \|\nabla^{k+1}h_3\|_{\infty,T} \leqq c \sum_{m,m',m''} \sum_{S:(5.13)} \ell(R)^{-m} \left(2^j \sum_{i=1}^{N} \ell(T_i)^{\alpha-m'}\right) \ell(S)^{-m''},$$

where the sum with respect to m, m', and m'' is taken over the nonnegative integers m, m', and m'' satisfying $m + m' + m'' = k + 1$ and $m' \leqq k$ (the restriction $m' \leqq k$ comes from the fact that $\nabla^{m'}(P_{R,S} - \pi_T) = 0$ for $m' > k$). Using (3) and (4) of Lemma 4.1, we see that the right side of (5.15) is majorized by

$$c2^j \sum_{m=0}^{k} \ell(R)^{-k-1+m} \sum_{i=1}^{N} (2^i \ell(T))^{\alpha-m}.$$

Evaluating this series and using Lemma 2.9, we finally obtain the following estimates: If $k < \alpha$, then

$$\ell(T)^{-\alpha} I_{k,\infty,T}(h_3) \leqq c2^j(\ell(T)/\ell(R))^{k+1-\alpha};$$

if $k \geqq \alpha$, then

$$\ell(T)^{-\alpha} I_{k,\infty,T}(h_3) \leqq c2^j \left(\frac{\ell(T)}{\ell(R)}\right)^{k+1-\alpha} \left(1 + \log \frac{b\ell(R)}{\ell(T)}\right) + c2^j \frac{\ell(T)}{\ell(R)}$$

(the log term is necessary only when α is a positive integer). Since $k + 1 \geqq \alpha$ and since $\ell(T) \leqq \ell(R)/2$, the above estimates imply (5.11) for $i = 3$. Thus (5.9) is proved in Case 1.

Case 2: $aT \subset U_{j+1}$.

From the cubes Q satisfying (5.12), choose a cube Q_0. On U_{j+1} (and hence on T in particular), the function h_R^j can be written as follows:

$$h_R^j = h_4 + h_5,$$
$$h_4 = \varphi_R^j \sum_Q (P_{Q,R} - \pi_{Q_0})\varphi_Q^{j+1},$$
$$h_5 = \varphi_R^j \sum_S (\pi_{Q_0} - P_{R,S})\varphi_S^{j-1}.$$

We shall prove that (5.11) holds for $i = 4$ and 5.

Suppose Q is a dyadic cube satisfying (5.12). In the present case, we have $\ell(T) \leqq b\ell(Q)$, $T \subset b_9 Q$ with some b_9, and $Q \approx Q_0$ (by Lemma 5.3). By Lemma 4.1(3), we have $\ell(Q) \leqq b\ell(R)$ and, hence, $K(Q,R) \approx Q$ and $K(Q,R) \approx Q_0$. By Lemma 5.4, we have $b_5 aQ \not\subset U_{j+1}$ and hence $\inf_{b_5 aQ} f^{\#} \leqq 2^{j+1}$.

Now we shall estimate h_4. For Q satisfying (5.12), we have

$$\|P_{Q,R} - \pi_{Q_0}\|_{\infty,Q_0} \leqq c\ell(Q_0)^\alpha \inf_{b_5 aQ_0} f^{\#} \leqq c2^j \ell(Q_0)^\alpha$$

and hence

$$\|\nabla^m (P_{Q,R} - \pi_{Q_0})\|_{\infty,T} \leqq c\|\nabla^m (P_{Q,R} - \pi_{Q_0})\|_{\infty,Q_0} \leqq c2^j \ell(Q_0)^{\alpha-m}$$

for every nonnegative integer m. Using this estimate and Lemma 4.3, we obtain

$$\|\nabla^{k+1} h_4\|_{\infty,T} \leqq c2^j \ell(Q_0)^{\alpha-k-1}.$$

Hence, by Lemma 2.9,

$$\ell(T)^{-\alpha} I_{k,\infty,T}(h_4) \leqq c2^j (\ell(T)/\ell(Q_0))^{k+1-\alpha} \leqq c2^j.$$

This proves (5.11) for $i = 4$.

Next we shall estimate h_5. The argument is similar to that given in the estimate of h_3 in Case 1. In the present case, we take b_{10} such that $Q_0 \subsetneqq b_{10} R$ and that $K(R,S) \subset b_{10} R$ for every S satisfying (5.13). We take cubes T_i $(i = 0, 1, \ldots, N)$ such that $Q_0 = T_0 \subset T_1 \subset \cdots \subset T_N = b_{10} R$ and that (5.14) holds. Since $b_5 aQ_0 \not\subset U_{j+1}$, we have $\inf_{b_5 aT_i} f^{\#} \leqq 2^{j+1}$ for every i. Hence, arguing in the same way as in the estimate of h_3 in Case 1, we obtain

$$\|\nabla^{k+1} h_5\|_{\infty,b_9 Q_0} \leqq c2^j \sum_{m=0}^{k} \ell(R)^{-k-1+m} \sum_{i=1}^{N} (2^i \ell(Q_0))^{\alpha-m}.$$

From this and from the fact that $T \subset b_9 Q_0$, we obtain, using Lemma 2.9, the following estimates: If $k < \alpha$, then

$$\ell(T)^{-\alpha} I_{k,\infty,T}(h_5) \leqq c2^j (\ell(T)/\ell(R))^{k+1-\alpha};$$

if $k \geqq \alpha$, then

$$\ell(T)^{-\alpha} I_{k,\infty,T}(h_5) \leqq c2^j \left(\frac{\ell(T)}{\ell(R)}\right)^{k+1-\alpha} \left(1 + \log \frac{b\ell(R)}{\ell(Q_0)}\right) + c2^j \frac{\ell(T)}{\ell(R)} \left(\frac{\ell(T)}{\ell(Q_0)}\right)^{k-\alpha}$$

(the log term appears only when α is a positive integer). Since $b^{-1}\ell(T) \leqq \ell(Q_0) \leqq b\ell(R)$, the above estimates imply (5.11) for $i = 5$. Thus (5.9) is proved in Case 2 as well. Theorem 5.1 is proved.

Proof of Theorem 5.2. We use the same notations as in the proof of Theorem 5.1 (we only replace Ω by $\boldsymbol{R}^n$). We take q such that $0 < q < p$ and define g^j and h_R^j in the same way as in the proof of Theorem 5.1 (notice that $U_j \neq \boldsymbol{R}^n$). By exactly the same argument as in the proof of Theorem 5.1, we see the following: (1°) The families $\{h_R^j\}$, $\{2R\}$, and $\{c2^j\}$ each indexed by the set $\{(j,R); j \in \boldsymbol{Z}, R \in \mathscr{G}(U_j)\}$ form a triplet which satisfies the conditions as mentioned in (0), (i), (iii), and (iv) of Theorem 5.1; (2°) for each integer N,

$$f(x) - g^N(x) = \sum_{j=N}^{\infty} \left(\sum_R h_R^j(x) \right) \quad \text{a.e..} \tag{5.16}$$

In the rest of the proof, we shall treat the cases (I) and (II) separately.

Case (I) $1/p > \alpha/n$ or $1/p = \alpha/n \geqq 1$.

It is sufficient to show that

$$g^N(x) \to P_0(f)(x) \quad \text{as} \quad N \to -\infty \quad \text{for all} \quad x \in \boldsymbol{R}^n. \tag{5.17}$$

If $1/p > \alpha/n$, then (5.17) is easily seen from Theorem 4.3. (Recall that Theorem 4.3 for the case $1/p > \alpha/n$ is completely proved.) Suppose $1/p = \alpha/n \geqq 1$. Let P be the polynomial as given in the claim (a) in the proof of Theorem 4.3. Then, by that claim (a), we have $g^N(x) \to P(x) (N \to \infty)$ for every $x \in \boldsymbol{R}^n$. By the above (1°) and by Theorem 3.1(3), the series $\sum_{j,R} h_R^j$ converges absolutely in C_0. Conbining these facts with the above (2°), we see that $f - P \in C_0$ and hence $P = P_0(f)$. Thus (5.17) is proved in the case $1/p = \alpha/n \geqq 1$ as well. (Notice that this argument also completes the proof of Theorem 4.3.)

Case (II) $0 < 1/p < \alpha/n$ or $0 < 1/p = \alpha/n < 1$.

The series $\sum_{j,R} h_R^j$ unconditionally converges with respect to $|\cdot; C_p^{\alpha,k}|$ (this follows from (1°) and Theorem 3.1(1)). For each integer N, the series $\sum_{j,R} h_R^j$

unconditionally converges in C_0 (when $0 < 1/p < \alpha/n$) or in L^p (when $0 < 1/p = \alpha/n < 1$); this follows from the assertion (1°) and the fact that $\ell(R)$ for $R \in \mathscr{G}(U_j)$, $j \geqq N$, is bounded, with the aid of (4) and (5) of Theorem 3.1. Hence the function on the right side of (5.16) coincides mod $\mathscr{P}_k$ with the $|\cdot; C_p^{\alpha,k}|$-unconditional sum $\sum_{j \geqq N,R} h_R^j$ (cf. Lemma 2.7). Thus it is sufficient to show that

$$|g^N; C_p^{\alpha,k}| \to 0 \quad \text{as} \quad N \to -\infty. \tag{5.18}$$

To show this, we shall prove that the following two estimates hold for all integers j and all $x \in \boldsymbol{R}^n$:

$$(g^j)^{\#}_{k,\alpha,q}(x) \leqq c2^j, \tag{5.19}$$

$$(g^j)^{\#}_{k,\alpha,q}(x) \leqq cM_q(f^{\#})(x). \tag{5.20}$$

The claim (5.18) will follow from these estimates with the aid of Lebesgue's convergence theorem.

The estimates (5.19) and (5.20) can be proved in a way similar to the proof of (5.4) given in the proof of Theorem 5.1; here we shall only indicate the key steps. (A similar argument can also be found in [DS; Lemma 8.1].) To make the reference to the proof of Theorem 5.1 easy, we shall treat g^{j+1} instead of g^j. We fix a cube T. We shall prove

$$\ell(T)^{-\alpha} I_{k,q,T}(g^{j+1}) \leqq c \min\{2^j, |T|^{-1/q} \|f^{\#}\|_{q,T}\}, \tag{5.21}$$

which will imply (5.19) and (5.20), We write

$$(*) = (\text{the right side of } (5.21)).$$

First, suppose $aT \not\subset U_{j+1}$. We consider $g^{j+1} - \pi_T$, which can be written as $g^{j+1} - \pi_T = h_1 + h_2$ with

$$h_1 = (f - \pi_T)\chi_{\Omega \setminus U_{j+1}} \quad \text{and} \quad h_2 = \sum_{Q,R} (P_{Q,R} - \pi_T)\varphi_Q^{j+1}\varphi_R^j.$$

For h_1, we have

$$|T|^{-\alpha/n-1/q} \|h_1\|_{q,T} \leqq |T|^{-\alpha/n-1/q} \|f - \pi_T\|_{q,T} \leqq c \inf_{aT} f^{\#} \leqq (*).$$

If Q and R are dyadic cubes for which $\varphi_Q^{j+1}\varphi_R^j \not\equiv 0$ on T, the $Q \subset 7T$ and

$$\|P_{Q,R} - \pi_T\|_{\infty,Q} \leqq c\ell(T)^{\alpha} \min\left\{2^j, \inf_{2Q} f^{\#}\right\}.$$

Hence

$$|h_2(x)| \leqq c\ell(T)^{\alpha}\min\{2^j, f^{\#}(x)\} \quad \text{for all } x \in T$$

and thus

$$|T|^{-\alpha/n-1/q}\|h_2\|_{q,T} \leqq (*).$$

The inequality (5.21) follows from the above estimates of h_1 and h_2.

Next, suppose $aT \subset U_{j+1}$. We fix a cube $Q_0 \in \mathscr{G}(U_{j+1})$ such that $2Q_0 \cap T \neq \varnothing$. If Q and R are dyadic cubes for which $\varphi_Q^{j+1}\varphi_R^j \not\equiv 0$ on T, then $T \subset bQ_0$ and

$$\|\nabla^m(P_{Q,R} - \pi_{Q_0})\|_{\infty,T} \leqq c\ell(Q_0)^{\alpha-m}\inf_{baQ_0} f^{\#} \leqq \ell(Q_0)^{\alpha-m}(*)$$

for every nonnegative integer m. Hence

$$\|\nabla^{k+1}g^{j+1}\|_{\infty,T} = \left\|\nabla^{k+1}\sum_{Q,R}(P_{Q,R} - \pi_{Q_0})\varphi_Q^{j+1}\varphi_R^j\right\|_{\infty,T} \leqq \ell(Q_0)^{\alpha-k-1}(*),$$

from which follows (5.21). Theorem 5.2 is proved.

As we mentioned in the above proof, Theorem 4.3 is also completely proved.

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