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# THE CENTER CONSTRUCTION FOR WEAK HOPF ALGEBRAS

By

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## 1. Introduction

Weak Hopf algebras have been proposed recently in [2], [3], [11], as a new generalization of Hopf algebras. In contrast to other Hopf algebraic constructions such as the quasi-Hopf algebras or weak quasi-Hopf algebras and rational Hopf algebras, weak Hopf algebras are coassociative but they have "weaker" axioms related to the unit and counit. Nevertheless the category of left (or right) modules carries a monoidal structure.

The *center construction* was introduced by Joyal and Street [7], Majid [8] and also by Drinfel'd (unpublished). It associates to a tensor category $\mathscr{C}$ a braided tensor category $\mathscr{Z}(\mathscr{C})$, called the *center* of $\mathscr{C}$. The goal of this paper is to prove that the center of the category of finite dimensional left modules over a weak Hopf algebra $H$ is braided equivalent to the category of finite dimensional left modules over $D(H)$, the Drinfel'd double associated to $H$. This generalize the case of Hopf algebras (see [6]). Similar results have been obtained in [4], [5], [9] for quasi-Hopf algebras and in [14] for $\times_R$-Hopf algebras.

## 2. Preliminaries

We work over a field $k$, all maps are $k$-linear, unadorned tensor products $\otimes$ means $\otimes_k$ and all $k$-vector spaces are finite dimensional. We shall freely use the $\Sigma$-notation from [10] and [15]

Definition 2.1 ([3]). *A weak bialgebra* (*WBA for short*) *is a quintuple* $(H, \mu, 1, \Delta, \varepsilon)$ *such that*:

(i) $(H, \mu, 1)$ *is a* (*finite dimensional*) *associative algebra*

(ii) $(H, \Delta, \varepsilon)$ *is a coassociative coalgebra*

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(iii) a) $\Delta(xy) = \Delta(x)\Delta(y)$

b) $(\Delta \otimes id)(\Delta(1)) = (\Delta(1) \otimes 1)(1 \otimes \Delta(1)) = (1 \otimes \Delta(1))(\Delta(1) \otimes 1)$

c) $\varepsilon(xyz) = \sum \varepsilon(xy_{(1)})\varepsilon(y_{(2)}z) = \sum \varepsilon(xy_{(2)})\varepsilon(y_{(1)}z)$

Definition 2.2 ([3]). *A weak bialgebra $H$ is called a weak Hopf algebra* (*WHA for short*) *if there exists a k-linear map $S : H \to H$ satisfying the following axioms*:

(i) $\sum x_{(1)}S(x_{(2)}) = \sum \varepsilon(1_{(1)}x)1_{(2)}$

(ii) $\sum S(x_{(1)})x_{(2)} = \sum 1_{(1)}\varepsilon(x1_{(2)})$

(iii) $\sum S(x_{(1)})x_{(2)}S(x_{(3)}) = S(x)$

*The map $S$ is called the antipode of $H$.*

Remark 2.3. 1) *The antipode of a weak Hopf algebra is bijective* (*see* [3]).

2) *For a weak Hopf algebra $(H, \mu, 1, \Delta, \varepsilon, S)$ the following conditions are equivalent* (*see* [3]):

- *$H$ is a Hopf algebra*
- $\Delta(1) = 1 \otimes 1$
- $\varepsilon(xy) = \varepsilon(x)\varepsilon(y)$
- $\sum S(x_{(1)})x_{(2)} = \varepsilon(x)1$
- $\sum x_{(1)}S(x_{(2)}) = \varepsilon(x)1$

A morphism of WHA's is a map between them which is both an algebra and a coalgebra morphism preserving unit and counit and commuting with the antipode.

We define the maps

$$\sqcap^L, \sqcap^R : H \to H$$

$$\sqcap^L(h) := \sum \varepsilon(1_{(1)}h)1_{(2)} = \sum h_{(1)}S(h_{(2)})$$

$$\sqcap^R(h) := \sum 1_{(1)}\varepsilon(h1_{(2)}) = \sum S(h_{(1)})h_{(2)}$$

and we introduce the notation $H_L := \sqcap^L(H)$, $H_R := \sqcap^R(H)$. Then $H_L$ and $H_R$ are subalgebras of $H$ containing 1 which commute with each other and the restriction of $S$ defines an algebra anti-isomorphism between them (see [3]).

If $H$ is a WHA then the dual space denoted by $\hat{H}$ becomes a WHA as in the classical case of finite dimensional Hopf algebras. Between the left (right) subalgebra of $H$ and the right (left) subalgebra of $\hat{H}$ there exist the following correspondences:

Lemma 2.4 ([3], Lemma 2.6). *The maps $k_H^L : H_L \to \hat{H}_R$, $k_H^L(x^L) = x^L \rightharpoonup \varepsilon$*

*and* $k_H^R : H_R \to \hat{H}_L$, $k_H^R(x^R) = \varepsilon \leftharpoonup x^R$ *are algebra isomorphisms, where* $(h \rightharpoonup \phi)(h') = \phi(h'h)$ *and* $(\phi \leftharpoonup h)(h') = \phi(hh')$ *for all* $h \in H$, $\phi \in \hat{H}$.

The notions of left or right $H$-module and left or right $H$-comodule over a WHA $H$ are similar to classical ones. If $M$ is a right $H$-comodule with structure $\rho : M \to M \otimes H$, $\rho(m) = \sum m_{\langle 0\rangle} \otimes m_{\langle 1\rangle}$, then $M$ becomes a left $\hat{H}$-module with the action $\phi \cdot m = \sum \phi(m_{\langle 1\rangle})m_{\langle 0\rangle}$.

We give now the definition of a Yetter Drinfel'd module over a WHA which is slightly different from the classical one, see [13].

Definition 2.5. *A left-right Yetter-Drinfel'd module over the WHA H is a k-linear space M such that*:

(i) *M is a left H-module with the action* $H \otimes M \to M$, $h \otimes m \mapsto h \cdot m$

(ii) *M is a right H-comodule with the coaction* $\rho : M \to M \otimes H$, $\rho(m) = \sum m_{\langle 0\rangle} \otimes m_{\langle 1\rangle}$

(iii) $\sum h_{(1)} \cdot m_{\langle 0\rangle} \otimes h_{(2)}m_{\langle 1\rangle} = \sum (h_{(2)} \cdot m)_{\langle 0\rangle} \otimes (h_{(2)} \cdot m)_{\langle 1\rangle}h_{(1)}$
$\sum 1_{(1)} \cdot m_{\langle 0\rangle} \otimes 1_{(2)}m_{\langle 1\rangle} = \sum m_{\langle 0\rangle} \otimes m_{\langle 1\rangle}$

*for all* $h \in H$, $m \in M$.

Remark 2.6. 1) *The axiom* (iii) *is equivalent to*

$$\sum (h \cdot m)_{\langle 0\rangle} \otimes (h \cdot m)_{\langle 1\rangle} = \sum h_{(2)} \cdot m_{\langle 0\rangle} \otimes h_{(3)}m_{\langle 1\rangle}S^{-1}(h_{(1)}) \tag{1}$$

2) *If* $x^L \in H_L$, $x^R \in H_R$ *and M is a left-right Yetter-Drinfel'd module then for all* $m \in M$ *we have*:

$$\begin{aligned}
(x^L \rightharpoonup \varepsilon) \cdot m &= \sum (x^L \rightharpoonup \varepsilon)(m_{\langle 1\rangle})m_{\langle 0\rangle} \\
&= \sum \varepsilon(m_{\langle 1\rangle}x^L)m_{\langle 0\rangle} \\
&= \sum \varepsilon(1_{(2)}m_{\langle 1\rangle}x^L)1_{(1)}m_{\langle 0\rangle} \\
&= \sum \varepsilon((1_{(2)} \cdot m)_{\langle 1\rangle}1_{(1)}x^L)(1_{(2)} \cdot m)_{\langle 0\rangle} \\
&= \sum \varepsilon((x^L_{(2)} \cdot m)_{\langle 1\rangle}x^L_{(1)})(x^L_{(2)} \cdot m)_{\langle 0\rangle} \\
&= \sum \varepsilon(x^L_{(2)}m_{\langle 1\rangle})x^L_{(1)} \cdot m_{\langle 0\rangle} \\
&= \sum \varepsilon(1_{(2)}m_{\langle 1\rangle})(x^L 1_{(1)}) \cdot m_{\langle 0\rangle} \\
&= \sum \varepsilon(1_{(2)}m_{\langle 1\rangle})x^L \cdot (1_{(1)} \cdot m_{\langle 0\rangle}) \\
&= \sum x^L\varepsilon(m_{\langle 1\rangle})m_{\langle 0\rangle} \\
&= x^L \cdot m
\end{aligned} \tag{2}$$

*where the first and the second equality is the definition of left action of* $\widehat{H_R}$ *and* $\rightharpoonup$ *respectively, the third, fourth, sixth and ninth equality is condition* (iii) *from Definition* 2.5 (*in the fourth equality* $h = 1$ *and in the sixth* $h = x^L$), *the fifth and the seventh equality is* $\Delta(x^L) = 1_{(1)}x^L \otimes 1_{(2)} = x^L 1_{(1)} \otimes 1_{(2)}$ (*see* [3], (2.6a), (2.7a) *and* (2.10)); *the last equality is the counit property.*

$$
\begin{aligned}
(\varepsilon \leftharpoonup x^R) \cdot m &= \sum (\varepsilon \leftharpoonup x^R)(m_{\langle 1\rangle})m_{\langle 0\rangle} \\
&= \sum \varepsilon(x^R m_{\langle 1\rangle})m_{\langle 0\rangle} \\
&= \sum \varepsilon(x^R(1_{(2)}m_{\langle 1\rangle}))1_{(1)} \cdot m_{\langle 0\rangle} \\
&= \sum \varepsilon((x^R 1_{(2)})m_{\langle 1\rangle})1_{(1)} \cdot m_{\langle 0\rangle} \\
&= \sum \varepsilon(x^R_{(2)} m_{\langle 1\rangle})x^R_{(1)} \cdot m_{\langle 0\rangle} \\
&= \sum \varepsilon((x^R_{(2)} \cdot m)_{\langle 1\rangle} x^R_{(1)})(x^R_{(2)} \cdot m)_{\langle 0\rangle} \\
&= \sum \varepsilon(((1_{(2)}x^R) \cdot m)_{\langle 1\rangle} 1_{(1)})((1_{(2)}x^R) \cdot m)_{\langle 0\rangle} \\
&= \sum \varepsilon(1_{(2)}(x^R \cdot m)_{\langle 1\rangle})1_{(1)}(x^R \cdot m)_{\langle 0\rangle} \\
&= \sum \varepsilon((x^R \cdot m)_{\langle 1\rangle})(x^R \cdot m)_{\langle 0\rangle} \\
&= x^R \cdot m
\end{aligned} \tag{3}
$$

*where we used the definition of left action of* $\widehat{H_R}$ *for the first equality and the definition of* $\leftharpoonup$ *for the second equality*; *for the third, sixth, eighth and ninth equality we used Definition* 2.5 (iii) *for* $h = x^R$, $h = 1$ *and* $m = x^R \cdot m$ (*in the sixth, eighth and ninth equality respectively*); *the fifth and the seventh equalities follow from the fact that* $\Delta(x^R) = 1_{(1)} \otimes x^R 1_{(2)} = 1_{(1)} \otimes 1_{(2)}x^R$ (*see* [3] (2.6b), (2.7b) *and* (2.10)); *the last equality is the counit property.*

Similarly we can define the notions of a left-left, right-left and right-right Yetter-Drinfel'd module over a WHA $H$.

We denote by ${}_H\mathscr{YD}^H$ the category of left-right Yetter-Drinfel'd modules whose objects are finite dimensional left-right Yetter-Drinfel'd modules and morphisms are maps which are $H$-linear and $H$-colinear. Moreover we have:

Proposition 2.7. *Let $H$ be a WHA over the field $k$. Then the category ${}_H\mathscr{Y}\mathscr{D}^H$ is braided.*

Proof. For two objects $V, W$ we define the tensor product to be:

$$V \times W := \{x \in V \otimes W \mid x = \Delta(1) \cdot x\} \subset V \otimes W$$

as $k$-space. Then $V \times W \in {}_H\mathscr{Y}\mathscr{D}^H$ with the following action and coaction:

$$h \cdot (v \times w) := \sum h_{(1)} \cdot v \times h_{(2)} \cdot w$$

$$\rho(v \times w) := \sum v_{\langle 0\rangle} \times w_{\langle 0\rangle} \otimes w_{\langle 1\rangle} v_{\langle 1\rangle}$$

for all $v \in V$, $w \in W$. It is easy to see that the action and the coaction are well defined and that $V \times W$ becomes an Yetter-Drinfel'd module. If $f, g$ are morphisms then

$$f \times g := (f \otimes g) \circ \Delta(1)$$

The associativity constraints are trivial. The unit object is $H_R$ with the structures:

$$h \cdot z = \sum 1_{(1)} \varepsilon(1_{(2)} hz)$$

$$\rho(z) = \sum z_{(1)} \otimes z_{(2)}$$

for all $h \in H$, $z \in H_R$ and the unit constraints are:

$$l_V : H_R \times V \to V, \quad l_V(z \times v) = S(z) \cdot v$$

$$r_V : V \times H_R \to V, \quad r_V(v \times z) = z \cdot v$$

for all $v \in V$, $z \in H_R$. It is easy to see that $l_V$ and $r_V$ are natural isomorphisms in the category ${}_H\mathscr{Y}\mathscr{D}^H$ with inverses:

$$l_V^{-1} : V \to H_R \times V$$

$$l_V^{-1}(v) = 1 \times v$$

$$r_V^{-1} : V \to V \times H_R$$

$$r_V^{-1}(v) = \sum \varepsilon(1_{(3)}) \sqcap^R (1_{(2)}) \cdot v \times 1_{(1)}$$

for all $v \in V$, $z \in H_R$. We verify the "Triangle Axiom":

$$\begin{aligned}(r_V \times id_W)(v \times z \times w) &= z \cdot v \times w \\ &= \sum 1_{(1)} z \cdot v \otimes 1_{(2)} \cdot w \\ &= \sum 1_{(1)} \cdot v \otimes 1_{(2)} S(z) \cdot w \quad \text{(see [3], 2.31a)} \\ &= v \times S(z) \cdot w \\ &= (id_V \times l_W)(v \times z \times w)\end{aligned}$$

for all $v \in V,\ w \in W,\ z \in H_R$. So ${}_H\mathscr{Y}\mathscr{D}^H$ is a monoidal category.

For two objects $V, W \in {}_H\mathscr{Y}\mathscr{D}^H$ we define:

$$c_{V,W} : V \times W \to W \times V$$

$$c_{V,W}(v \times w) = \sum w_{\langle 0 \rangle} \times w_{\langle 1 \rangle} \cdot v$$

This map is $H$-linear because we have:

$$\begin{aligned}c_{V,W}(h \cdot (v \times w)) &= c_{V,W}\Big(\sum h_{(1)} \cdot v \times h_{(2)} \cdot w\Big) \\ &= \sum (h_{(2)} \cdot w)_{\langle 0 \rangle} \times (h_{(2)} \cdot w)_{\langle 1 \rangle} h_{(1)} \cdot v \\ &= \sum h_{(1)} \cdot w_{\langle 0 \rangle} \times (h_{(2)} w_{\langle 1 \rangle}) \cdot v \\ &= \sum h_{(1)} \cdot w_{\langle 0 \rangle} \times h_{(2)} \cdot (w_{\langle 1 \rangle} \cdot v) \\ &= \sum h \cdot (w_{\langle 0 \rangle} \times w_{\langle 1 \rangle} \cdot v) \\ &= h \cdot c_{V,W}(v \times w)\end{aligned}$$

Similarly we can prove that $c_{V,W}$ is $H$-colinear. If $f : V \to V'$ and $g : W \to W'$ are morphisms in the category ${}_H\mathscr{Y}\mathscr{D}^H$ then:

$$\begin{aligned}((g \times f) \circ c_{V,W})(v \times w) &= \sum g(w_{\langle 0 \rangle}) \times f(w_{\langle 1 \rangle} \cdot v) \\ &= \sum 1_{(1)} \cdot g(w_{\langle 0 \rangle}) \otimes 1_{(2)} \cdot f(w_{\langle 1 \rangle} \cdot v) \\ &= \sum 1_{(1)} g(w_{\langle 0 \rangle}) \otimes 1_{(2)} (w_{\langle 1 \rangle} \cdot f(v)) \\ &= \sum 1_{(1)} g(w)_{\langle 0 \rangle} \otimes 1_{(2)} (g(w)_{\langle 1 \rangle} \cdot f(v)) \\ &= \sum g(w)_{\langle 0 \rangle} \times g(w)_{\langle 1 \rangle} \cdot f(v) \\ &= (c_{V',W'} \circ (f \times g))(v \times w)\end{aligned}$$

for all $v \in V$, $w \in W$, so $c_{V,W}$ is natural. Moreover it satisfies the "Hexagon Axioms". We have:

$$\begin{aligned} c_{U,V\times W}(u \times (v \times w)) &= \sum (v \times w)_{\langle 0\rangle} \times (v \times w)_{\langle 1\rangle} \cdot u \\ &= \sum v_{\langle 0\rangle} \times w_{\langle 0\rangle} \times (w_{\langle 1\rangle} v_{\langle 1\rangle}) \cdot u \\ &= (id_V \times c_{U,W})\Big(\sum v_{\langle 0\rangle} \times v_{\langle 1\rangle} \cdot u \times w\Big) \\ &= (id_V \times c_{U,W}) \circ (c_{U,V} \times id_W) \end{aligned}$$

$$\begin{aligned} c_{U\times V,W}((u \times v) \times w) &= \sum w_{\langle 0\rangle} \times w_{\langle 1\rangle} \cdot (u \times v) \\ &= \sum w_{\langle 0\rangle} \times w_{\langle 1\rangle_{(1)}} \cdot u \times w_{\langle 1\rangle_{(2)}} \cdot v \\ &= \sum w_{\langle 0\rangle_{\langle 0\rangle}} \times w_{\langle 0\rangle_{\langle 1\rangle}} \cdot u \times w_{\langle 1\rangle} \cdot v \\ &= (c_{U,W} \times id_V)\Big(\sum u \times w_{\langle 0\rangle} \times w_{\langle 1\rangle} \cdot v\Big) \\ &= (c_{U,W} \times id_V) \circ (id_U \times c_{V,W})(u \times v \times w) \end{aligned}$$

The map $c_{V,W}$ is bijective, the inverse is given by:

$$c_{V,W}^{-1} : W \times V \to V \times W$$
$$c_{V,W}^{-1}(w \times v) = \sum S(w_{\langle 1\rangle}) \cdot v \times w_{\langle 0\rangle}$$

We compute:

$$\begin{aligned} (c_{V,W} \circ c_{V,W}^{-1})(w \times v) &= c_{V,W}\Big(\sum S(w_{\langle 1\rangle}) \cdot v \times w_{\langle 0\rangle}\Big) \\ &= \sum w_{\langle 0\rangle_{\langle 0\rangle}} \times (w_{\langle 0\rangle_{\langle 1\rangle}} \cdot (S(w_{\langle 1\rangle}) \cdot v) \\ &= \sum w_{\langle 0\rangle} \times (w_{\langle 1\rangle_{(1)}} S(w_{\langle 1\rangle_{(2)}})) \cdot v \\ &= \sum w_{\langle 0\rangle} \times \varepsilon(1_{(1)} w_{\langle 1\rangle}) 1_{(2)} \cdot v \\ &= \sum \varepsilon(1_{(1)} w_{\langle 1\rangle}) w_{\langle 0\rangle} \times 1_{(2)} \cdot v \\ &= \sum 1_{(1)} \cdot w \times 1_{(2)} \cdot v \quad \text{(using (3) for } 1_{(1)} \in H_R) \\ &= w \times v \end{aligned}$$

Similarly $c_{V,W}^{-1} \circ c_{V,W} = id_{V\times W}$. ■

For a WHA $H$ we denote by $Rep(H)$ the category of representations of $H$, whose objects are finite dimensional left $H$-modules and whose morphisms are $H$-linear maps.

PROPOSITION 2.8 (see [12]). *The category $Rep(H)$ is a monoidal category with unit object $H_L$.*

PROOF. The tensor product is defined in the same way as the tensor product of the category ${}_H\mathscr{Y}\mathscr{D}^H$. The unit object is $H_L$ with the left $H$-module structure given by:

$$h \cdot z = \sqcap^L(hz)$$

for all $h \in H$, $z \in H_L$. The unit constraints are the following:

$$l_V : H_L \otimes V \to V, \quad l_V\Big(\sum 1_{(1)} \cdot z \otimes 1_{(2)} \cdot v\Big) = z \cdot v$$

$$r_V : V \otimes H_L \to V, \quad r_V\Big(\sum 1_{(1)} \cdot v \otimes 1_{(2)} \cdot z\Big) = S(z) \cdot v$$

for all $v \in V$, $z \in H_L$. The inverses are given by:

$$l_V^{-1} : V \to H_L \otimes V, \quad l_V^{-1}(v) = \sum 1_{(1)} \otimes 1_{(2)} \cdot v$$

$$r_V^{-1} : V \to V \otimes H_L, \quad r_V^{-1}(v) = \sum 1_{(1)} \cdot v \otimes 1_{(2)} \qquad \blacksquare$$

DEFINITION 2.9 (see [12]). *A quasitriangular weak Hopf algebra is a pair $(H, \mathscr{R})$ where $H$ is a WHA and $\mathscr{R} \in \Delta^{op}(1)(H \otimes H)\Delta(1)$ such that the following conditions are fulfilled*:

(qt1) $(id \otimes \Delta)(\mathscr{R}) = \mathscr{R}_{13}\mathscr{R}_{12}$

(qt2) $(\Delta \otimes id)(\mathscr{R}) = \mathscr{R}_{13}\mathscr{R}_{23}$

(qt3) $\Delta^{cop}(h)\mathscr{R} = \mathscr{R}\Delta(h)$

(qt4) *there exists* $\tilde{\mathscr{R}} \in \Delta(1)(H \otimes H)\Delta^{op}(1)$ *with*

$$\mathscr{R}\tilde{\mathscr{R}} = \Delta^{op}(1) \quad and \quad \tilde{\mathscr{R}}\mathscr{R} = \Delta(1)$$

*where* $\mathscr{R}_{12} = \mathscr{R} \otimes 1$, $\mathscr{R}_{23} = 1 \otimes \mathscr{R}$ *etc., as usual.*

Note that $\tilde{\mathscr{R}}$ is uniquely determined by $\mathscr{R}$. As in the Hopf algebra case we shall denote $\mathscr{R} = \sum \mathscr{R}^1 \otimes \mathscr{R}^2$. For any two objects $V, W \in Rep(H)$ we define:

$$c_{V,W} : V \times W \to W \times V$$

$$c_{V,W}(v \times w) = \sum \mathscr{R}^2 \cdot w \times \mathscr{R}^1 \cdot v$$

then we have the following result from [12], Proposition 5.2:

Proposition 2.10. *The family of homomorphisms* $\{c_{V,W}\}_{V,W}$ *defines a braiding in* $Rep(H)$. *Conversely, if* $H$ *is a WHA such that* $Rep(H)$ *is braided, then there exists* $\mathscr{R} \in \Delta^{op}(1)(H \otimes H)\Delta(1)$ *satisfying the properties in Definition* 2.9 *and inducing the given braiding.*

For more details about WHA's and quasitriangular weak Hopf algebras see [3], [11], [12].

## 3. The Drinfel'd Double for WHA's

In this section we give the generalization to WHA's of the double construction due to Drinfel'd for Hopf algebras. This also appears in [1] and [12].

Let $H$ be a WHA. By $\hat{H}$ we denoted the dual space which is again a WHA (see [3]). Consider on the vector space $H \otimes \hat{H}$ a multiplication given by

$$(g \otimes \phi)(h \otimes \psi) := \sum gh_{(2)} \otimes \phi_{(2)} \psi \phi_{(1)}(h_{(3)}) \phi_{(3)}(S^{-1}(h_{(1)}))$$

where $g, h \in H$ and $\phi, \psi \in \hat{H}$. With this muliplication $H \otimes \hat{H}$ becomes an associative algebra with unit $1 \otimes \varepsilon$. We denote by $J$ the two-sided ideal generated by:

$$z \otimes \varepsilon - 1 \otimes \varepsilon \leftharpoonup z, \quad z \in H_R$$

$$y \otimes \varepsilon - 1 \otimes y \rightharpoonup \varepsilon, \quad y \in H_L$$

where $y \in H_L$, $z \in H_R$. We define the Drinfel'd double, denoted by $D(H)$, to be the factor algebra $(H \otimes \hat{H})/J$ and let $[h \otimes \phi]$ denote the class of $h \otimes \phi$ in $D(H)$.

Proposition 3.1 (see [1]). *$D(H)$ is a WHA with the following structures*:
$[g \otimes \phi][h \otimes \psi] = \sum [gh_{(2)} \otimes \phi_{(2)} \psi \phi_{(1)}(h_{(3)}) \psi_{(3)}(S^{-1}(h_{(1)}))]$
$1_{D(H)} = [1 \otimes \varepsilon]$
$\Delta([g \otimes \phi]) = \sum [g_{(1)} \otimes \phi_{(2)}] \otimes [g_{(2)} \otimes \phi_{(1)}]$
$\varepsilon([g \otimes \phi]) = \sum \varepsilon(g 1_{(1)}) \phi(1_{(2)})$
$S([g \otimes \phi]) = \sum [1 \otimes \phi \circ S^{-1}][S(g) \otimes \varepsilon]$

As in the classical case we have:

PROPOSITION 3.2. *The Drinfel'd double $D(H)$ of a weak Hopf algebra $H$ has a quasitriangular structure given by*:

$$\mathscr{R} = \Sigma_i \, [f_i \otimes \varepsilon] \otimes [1 \otimes \xi^i]$$

$$\tilde{\mathscr{R}} = \Sigma_i \, [f_i \otimes \varepsilon] \otimes [1 \otimes \hat{S}(\xi^i)]$$

*where $\{f_i\}_i$ and $\{\xi^i\}_i$ are dual bases in $H$ and $\hat{H}$ respectively.*

PROOF. The identities $(id \otimes \Delta)(\mathscr{R}) = \mathscr{R}_{13}\mathscr{R}_{12}$ and $(\Delta \otimes id)(\mathscr{R}) = \mathscr{R}_{13}\mathscr{R}_{23}$ can be written (identifying $[H \otimes \varepsilon]$ with $H$ and $[1 \otimes \hat{H}]$ with $\hat{H}$) as:

$$\Sigma_i \, f_i \otimes \xi^i_{(2)} \otimes \xi^i_{(1)} = \Sigma_{ij} \, f_i f_j \otimes \xi^j \otimes \xi^i$$

$$\Sigma_i \, f_{i_{(1)}} \otimes f_{i_{(2)}} \otimes \xi^i = \Sigma_{ij} \, f_i \otimes f_j \otimes \xi^i \xi^j$$

These equalities can be verified by evaluating both sides on an element $g \otimes h \in H \otimes H$ in the first two factors (respectively on $\phi \otimes \psi \in \hat{H} \otimes \hat{H}$).

To check (qt3) we compute:

$$\begin{aligned}
\mathscr{R}\Delta([\phi \otimes h]) &= \sum [f_i h_{(1)} \otimes \phi_{(2)}] \otimes [h_{(3)} \xi^i_{(2)} \phi_{(1)} \xi^i_{(1)}(h_{(4)}) \xi^i(S^{-1}(h_{(1)}))] \\
&= \sum [h_{(4)} f_i S^{-1}(h_{(2)}) h_{(1)} \otimes \phi_{(2)}] \otimes [h_{(3)} \otimes \xi^i \phi_{(1)}] \\
&= \sum [h_{(3)} f_i \varepsilon(h_{(1)} 1_{(1)}) 1_{(2)} \otimes \phi_{(2)}] \otimes [h_{(2)} \otimes \xi^i \phi_{(1)}] \\
&= \sum [h_{(3)} f_i \otimes \varepsilon_{(1)} \phi_{(2)} \varepsilon_{(2)}(1_{(2)})] \otimes [\varepsilon'(h_{(1)} 1_{(1)}) h_{(2)} \otimes \xi^i \phi_{(1)}] \\
&= \sum [h_{(2)} f_i \otimes \varepsilon'(1'_{(1)} 1_{(1)}) \varepsilon_{(2)}(1_{(2)}) \varepsilon_{(1)} \phi_{(2)}] \otimes [h_{(2)} 1'_{(2)} \otimes \xi^i \phi_{(1)}] \\
&= \sum [h_{(2)} f_i \otimes \phi_{(2)}] \otimes [\varepsilon_{(1)}(1'_{(1)}) \varepsilon_{(2)}(1_{(1)}) \phi_{(3)}(1_{(2)}) h_{(1)} 1'_{(2)} \otimes \xi^i \phi_{(1)}] \\
&= \sum [h_{(2)} f_i \otimes \phi_{(2)}] \otimes [h_{(1)} \otimes \varepsilon_{(1)} \hat{\varepsilon}(\varepsilon_{(2)} \phi_{(3)}) \xi^i \phi_{(1)}] \\
&= \sum [h_{(2)} f_i \otimes \phi_{(2)}] \otimes [h_{(1)} \otimes \phi_{(4)} S^{-1}(\phi_{(3)}) \xi^i \phi_{(1)}] \\
&= \sum [h_{(2)} f_{i_{(2)}} \otimes \phi_{(2)} \phi_{(1)}(f_{i_{(3)}}) \phi_{(3)}(S^{-1}(f_{i_{(1)}}))] \otimes [h_{(1)} \otimes \phi_{(4)} \xi^i] \\
&= \Delta^{cop}([h \otimes \phi])\mathscr{R}
\end{aligned}$$

where we used the relations from the definition of the double and

$$\Sigma_i \xi^i(a) f_i = a, \quad \Sigma_i \phi(f_i)\xi^i = \phi$$

Finally let us check that $\tilde{\mathscr{R}}$ satisfies $\tilde{\mathscr{R}}\mathscr{R} = \Delta(1)$, $\mathscr{R}\tilde{\mathscr{R}} = \Delta^{op}(1)$. The first property is equivalent to:

$$\sum [f_i f_j \otimes \varepsilon] \otimes [1 \otimes \hat{S}(\xi^i)\xi^j] = \sum [1'_{(1)} 1_{(1)} \varepsilon_{(2)}(1'_{(2)}) \otimes \varepsilon] \otimes [1 \otimes \varepsilon_{(1)} \varepsilon'_{(1)} \varepsilon'_{(2)}(1_{(2)})]$$

which can be regarded as an equality in $H \otimes \hat{H}$:

$$\Sigma_{ij}\, f_i f_j \otimes \hat{S}(\xi^i)\xi^j = \sum 1'_{(1)} 1_{(1)} \varepsilon_{(2)}(1'_{(2)}) \varepsilon'_{(2)}(1_{(2)}) \varepsilon_{(1)} \varepsilon'_{(1)}$$

Evaluating both sides on an arbitrary $\phi \in \hat{H}$ in the first factor we get:

$$\begin{aligned} S(\phi_{(2)})\phi_{(1)} &= \widehat{\sqcap^R}(\phi_{(1)})\, \widehat{\sqcap^R}(\phi_{(2)}) \\ &= \widehat{\sqcap^R}(\phi) \end{aligned}$$

The second property can be proved in a similar way. ■

Remark 3.3. *In* [12] *the authors introduce the definition of the Drinfel'd double using $\hat{H}^{op} \otimes H$ instead of $H \otimes \hat{H}$ with a different multiplication. They prove that their double is a quasitriangular weak Hopf algebra.*

Theorem 3.4. *Let $H$ be a WHA. Then the category ${}_H\mathscr{Y}\mathscr{D}^H$ of left-right Yetter-Drinfel'd modules can be identified with the category $Rep(D(H))$ of left modules over the Drinfel'd double $D(H)$.*

Proof. Let $V$ be a left $D(H)$-module. Then $V$ becomes a left $H$-module and a left $\hat{H}$-module in an obvious way. The left action of $\hat{H}$ on $V$ transposes into a right coaction of $H$ as follows:

$$\rho : V \to V \otimes H, \quad \rho(v) = \Sigma_i \xi^i(v) \otimes f_i$$

where $\{f_i\}_i$ and $\{\xi^i\}_i$ are dual bases in $H$ and $\hat{H}$ respectively. We check now the condition (1). We have:

$$\phi \cdot (h \cdot v) = \sum \phi((h \cdot v)_{\langle 1 \rangle})(h \cdot v)_{\langle 0 \rangle}$$

$$\begin{aligned} \phi \cdot (h \cdot v) &= ([1 \otimes \phi][h \otimes \varepsilon]) \cdot v = \sum [h_{(2)} \otimes \phi_{(2)} \phi_{(1)}(h_{(3)}) \phi_{(3)}(S^{-1}(h_{(1)}))] \cdot v \\ &= \sum \phi(h_{(3)} v_{\langle 1 \rangle} S^{-1}(h_{(1)})) h_{(2)} \cdot v_{\langle 0 \rangle} \end{aligned}$$

for all $h \in H$, $\phi \in \hat{H}$. Thus

$$\sum (h \cdot v)_{\langle 0 \rangle} \otimes (h \cdot v)_{\langle 1 \rangle} = \sum h_{(2)} \cdot v_{\langle 0 \rangle} \otimes h_{(3)} v_{\langle 1 \rangle} S^{-1}(h_{(1)})$$

Conversely if $V$ is a left-right Yetter-Drinfel'd module it is easy to see that $V$ becomes a left $D(H)$-module with the action:

$$[h \otimes \phi] \cdot v = \sum \phi(v_{\langle 1 \rangle}) h \cdot v_{\langle 0 \rangle}$$

for all $[h \otimes \phi] \in D(H)$. ■

## 4. The Center Construction

Let $(\mathscr{C}, \otimes_{\mathscr{C}}, I, a, l, r)$ be a tensor category. To this category we shall associate a braided tensor category $\mathscr{Z}(\mathscr{C})$, called the center of $\mathscr{C}$. When $\mathscr{C}$ is $Rep(H)$ for a WHA $H$ then we shall prove that $\mathscr{Z}(\mathscr{C})$ is braided equivalent to $Rep(D(H))$.

DEFINITION 4.1. *An object of $\mathscr{Z}(\mathscr{C})$ is a pair $(V, c_{-,V})$ where $V$ is an object of $\mathscr{C}$ and $c_{-,V}$ is a family of natural isomorphisms*

$$c_{X,V} : X \otimes_{\mathscr{C}} V \to V \otimes_{\mathscr{C}} X$$

*for all $X \in \mathscr{O}b(\mathscr{C})$ such that for all $X, Y \in \mathscr{O}b(\mathscr{C})$ we have:*

$$c_{X \otimes_{\mathscr{C}} Y, V} = a_{V,X,Y} \circ (c_{X,V} \otimes_{\mathscr{C}} id_Y) \circ a^{-1}_{X,V,Y} \circ (id_X \otimes_{\mathscr{C}} c_{Y,V}) \circ a_{X,Y,V} \tag{2}$$

$$c_{I,V} = r_V^{-1} \circ l_V \tag{3}$$

*A morphism $f : (V, c_{-,V}) \to (W, c_{-,W})$ is a morphism $f : V \to W$ in $\mathscr{C}$ such that for each object $X$ of $\mathscr{C}$ we have:*

$$(f \otimes_{\mathscr{C}} id_X) \circ c_{X,V} = c_{X,W} \circ (id_X \otimes_{\mathscr{C}} f) \tag{4}$$

*The composition of two morphisms in $\mathscr{Z}(\mathscr{C})$ is the same as in $\mathscr{C}$ and $id_{(V, c_{-,V})} = id_V$.*

THEOREM 4.2. *Let $(\mathscr{C}, \otimes, I, a, l, r)$ be a tensor category. Then $\mathscr{Z}(\mathscr{C})$ is a braided tensor category.*

PROOF. We follow the same steps as in [6]. The tensor product of two objects $(V, c_{-,V})$ and $(W, c_{-,W})$ is given by:

$$(V, c_{-,V}) \otimes_{\mathscr{Z}(\mathscr{C})} (W, c_{-,W}) = (V \otimes_{\mathscr{C}} W, c_{-, V \otimes_{\mathscr{C}} W})$$

where

$$c_{X,V\otimes_{\mathscr{C}}W} : X \otimes_{\mathscr{C}} (V \otimes_{\mathscr{C}} W) \to (V \otimes_{\mathscr{C}} W) \otimes_{\mathscr{C}} X$$

$$c_{X,V\otimes_{\mathscr{C}}W} = a^{-1}_{V,W,X} \circ (id_V \otimes_{\mathscr{C}} c_{X,W}) \circ a_{V,X,W} \circ (c_{X,V} \otimes_{\mathscr{C}} id_W) \circ a^{-1}_{X,V,W}$$

The unit is $(I, c_{-,I})$ and the braiding is:

$$c_{V,W} : (V, c_{-,V}) \otimes_{\mathscr{Z}(\mathscr{C})} (W, c_{-,W}) \to (W, c_{-,W}) \otimes_{\mathscr{Z}(\mathscr{C})} (V, c_{-,V})$$

In order to prove our main result we fix a WHA $H$ and we consider $\mathscr{C} = Rep(H)$. This is a tensor category with unit $H_L$ and trivial associativity constraints. We need some preliminary results.

Lemma 4.3. *Any object of $\mathscr{Z}(Rep(H))$ becomes a left-right Yetter-Drinfel'd module.*

Proof. Let $(V, c_{-,V}) \in \mathscr{Z}(Rep(H))$. We define $\rho_V : V \to V \otimes H$, $\rho_V(v) := c_{H,V}(1 \times v)$. We claim that $\rho_V$ is a right $H$-comodule structure. As in [6] Lemma XIII. 5.2 we have that for all $X \in \mathscr{Z}(Rep(H))$:

$$c_{X,V}(x \times v) = \sum v_{\langle 0\rangle} \otimes v_{\langle 1\rangle} \cdot x$$

To prove that $\rho_V$ is coassociative we consider $X, Y \in \mathscr{Z}(Rep(H))$ and using condition (2) we obtain:

$$\sum v_{\langle 0\rangle} \times v_{\langle 1\rangle_{(1)}} \cdot x \times v_{\langle 2\rangle_{(2)}} \cdot y = \sum v_{\langle 0\rangle_{\langle 0\rangle}} \times v_{\langle 0\rangle_{\langle 1\rangle}} \cdot x \times v_{\langle 1\rangle} \cdot y$$

for all $x \in X$, $y \in Y$, $v \in V$. Taking $X = Y = H$ and $x = y = 1$ we get that $\rho_V$ is coassociative. Since $H_L$ is the unit of $Rep(H)$ we have:

$$c_{H_L,V}(1 \times v) = \sum v_{\langle 0\rangle} \times v_{\langle 1\rangle} \cdot 1 = \sum v_{\langle 0\rangle} \times \varepsilon(1_{(1)} v_{\langle 1\rangle}) 1_{(2)}$$

But $c_{H_L,V} = r_V^{-1} \circ l_V$, hence

$$\sum v_{\langle 0\rangle} \times \varepsilon(1_{(1)} v_{\langle 1\rangle}) 1_{(2)} = v \times 1$$

Applying $I \times \varepsilon$ to both sides of this equality we obtain $v = \sum v_{\langle 0\rangle} \varepsilon(v_{\langle 1\rangle})$, so $V$ is a right $H$-comodule.

Let us now express the fact that $c_{H,V}$ is $H$-linear. For $h \in H$, $v \in V$ we have:

$$c_{H,V}(h \cdot (1 \times v)) = h \cdot c_{H,V}(1 \times v)$$

Replacing $c_{H,V}$ by its expression in $\rho_V$ we get:

$$\sum (h_{(2)} \cdot v)_{\langle 0\rangle} \otimes (h_{(2)} \cdot v)_{\langle 1\rangle} h_{(1)} = \sum h_{(1)} \cdot v_{\langle 0\rangle} \otimes h_{(2)} v_{\langle 1\rangle}$$

which is exactly condition (iii) from Definition 2.5. ■

So if $(V, c_{-,V}) \in \mathscr{Z}(Rep(H))$ then $V$ becomes a right comodule, hence a left $\hat{H}$-module. Under the same hypotheses we have:

LEMMA 4.4. *Let $(V, c_{-,V})$ be an object of $\mathscr{Z}(Rep(H))$ and $X$ a left $H$-module. Then the isomorphism $c_{X,V}$ is determined by*:

$$c_{X,V}(x \times v) = \Sigma_i \, \xi^i \cdot v \times f_i \cdot x$$

*for all $x \in X$, $v \in V$, where $\{f_i\}_i$ and $\{\xi^i\}_i$ are dual bases in $H$ and $\hat{H}$ respectively.*

PROOF. We have

$$\begin{aligned} c_{X,V}(x \times v) &= \sum v_{\langle 0\rangle} \times v_{\langle 1\rangle} \cdot x \\ &= \sum v_{\langle 0\rangle} \times \xi^i (v_{\langle 1\rangle}) f_i \cdot x \\ &= \sum \xi^i \cdot v \times f_i \cdot x \end{aligned}$$ ■

THEOREM 4.5. *For any weak Hopf algebra $H$ the tensor braided categories $\mathscr{Z}(Rep(H))$ and $Rep(D(H))$ are braided equivalent.*

PROOF. We define

$$F : \mathscr{Z}(Rep(H)) \to Rep(D(H))$$

$$F((V, c_{-,V})) = V$$

$$F(f) = f$$

for all $(V, c_{-,V}) \in \mathscr{Z}(Rep(H))$ and $f$ a map in $\mathscr{Z}(Rep(H))$.

From Lemma 4.3 $V$ is a left-right Yetter-Drinfel'd module so $V$ is a left $D(H)$-module (see Theorem 3.4) with the action:

$$[h \otimes \phi] \cdot v = \sum \phi(v_{\langle 1\rangle}) h \cdot v_{\langle 0\rangle}$$

where $h \in H$, $\phi \in \hat{H}$, $v \in V$.

Because $f$ is a map in $\mathscr{Z}(Rep(H))$ it follows that $f$ is $H$-linear and relation (4) for $X = H$ implies that $f$ is $H$-colinear, hence $\hat{H}$-linear. Consequently $f$ is $D(H)$-linear. This proves that $F$ is well defined.

Let us show that $F$ is a braided functor. The unit object is $H_L$ in both categories. The tensor product of $(V, c_{-,V})$ and $(W, c_{-,W})$ is $(V \times W, c_{-,V\times W})$. We define $F((V, c_{-,V}) \times (W, c_{-,W})) = V \times W$ with the left $D(H)$ action given by $[h \otimes \phi] \triangleright (v \times w) = \sum \phi((v \times w)_{\langle 1\rangle})h \cdot (v \times w)_{\langle 0\rangle}$. In the same time we can consider the tensor product in $Rep(D(H))$ of $V = F((V, c_{-,V}))$ and $W = F((W, c_{-,W}))$ with the action of $D(H)$ given via $\Delta$. We shall prove that these two actions of $D(H)$ coincide. We compute:

$$
\begin{aligned}
[h \otimes \phi] \triangleright (v \times w) &= \sum \phi((v \times w)_{\langle 1\rangle})h \cdot (v \times w)_{\langle 0\rangle} \\
&= \sum \phi(w_{\langle 1\rangle} v_{\langle 1\rangle})h_{(1)} \cdot v_{\langle 0\rangle} \times h_{(2)} \cdot w_{\langle 0\rangle} \\
&= \sum \phi_{(2)}(v_{\langle 1\rangle})h_{(1)} \cdot v_{\langle 0\rangle} \times \phi_{(1)}(w_{\langle 1\rangle})h_{(2)} \cdot w_{\langle 0\rangle} \\
&= \sum [h_{(1)} \otimes \phi_{(2)}] \cdot v \times [h_{(2)} \otimes \phi_{(1)}] \cdot w \\
&= [h \otimes \phi] \cdot (v \times w)
\end{aligned}
$$

We prove now that $F$ is braided functor, that is, if $c_{V,W}$ is the braiding of $\mathscr{Z}(Rep(H))$ and $\widetilde{c_{V,W}}$ the braiding of $Rep(D(H))$, then $F(c_{V,W}) = \widetilde{c_{V,W}}$, which is equivalent to

$$c_{V,W}(v \times w) = \sum \mathscr{R}^2 \cdot w \times \mathscr{R}^1 \cdot v$$

Using the definition of $\mathscr{R}$ the later is equivalent to

$$c_{V,W}(v \times w) = \Sigma_i \xi^i \cdot w \times f_i \cdot v$$

which is exactly the result from Lemma 4.4.

We shall construct an inverse for the functor $F$. We define:

$$G : Rep(D(H)) \to \mathscr{Z}(Rep(H))$$

$$G(V) = (V, c_{-,V})$$

where $c_{X,V}(x \times v) = \sum \mathscr{R}^2 \cdot v \times \mathscr{R}^1 \cdot x$ for all $X \in Rep(H)$ and $x \in X$, $v \in V$. The map $c_{X,V}$ is equal to $\widetilde{c_{X,V}}$, the braiding of the category $Rep(D(H))$ (see Proposition 2.10), and it is easy to see that it satisfies all the properties from Definition 4.1.

If $f : V \to W$ is a $D(H)$-linear map we define $G(f) = f$. We check that $f$ is a morphism in $\mathscr{Z}(Rep(H))$. First $f$ is $H$-linear since it is $D(H)$-linear. Next we have:

$$\begin{aligned}((f \times id_X) \circ c_{X,V})(x \times v) &= \sum f(\mathscr{R}^2 \cdot v) \times \mathscr{R}^1 \cdot x \\ &= \sum \mathscr{R}^2 \cdot f(v) \times \mathscr{R}^1 \cdot x \\ &= (c_{X,V} \circ (id_X \times f))(x \times v)\end{aligned}$$

This proves that $G$ is well defined. Clearly $F \circ G = id$ and the equality $G \circ F = id$ follows from Lemma 4.4, so the functor $F$ is an equivalence. ■

REMARK 4.6. *The natural embedding $H \subset D(H)$ of WHA's induces a tensor functor $U : Rep(D(H)) \to Rep(H)$. It is easy to check that $U$ corresponds to the universal functor $\Pi : \mathscr{Z}(Rep(H)) \to Rep(H)$ under the equivalence of Theorem* 4.5.

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