TSUKUBA J. MATH.
Vol. 22 No. 3 (1998), 769–781

CAUCHY PROBLEMS RELATED TO DIFFERENTIAL OPERATORS WITH COEFFICIENTS OF GENERALIZED HERMITE OPERATORS

By

Xiaowei XU

1. Introduction

As an example, let us consider the Schrödinger equation:

$$\{D_t + D_x^2 + V(x)\}u(t,x) = 0,$$

where $D_t = (1/i)(\partial/\partial t)$, $D_x = (1/i)(\partial/\partial x)$. In the harmonic oscillator case, where the potential energy function $V(x)$ is equal to x^2, X. Feng ([1]) considered it as follows.

As is well known, the Hermite function

$$\Phi_\alpha(x) = (\alpha!2^\alpha)^{-1/2}(-1)^\alpha \pi^{-1/4} e^{x^2/2}\left(\frac{\partial}{\partial x}\right)^\alpha e^{-x^2}$$

is an eigenfunction of the Hermite operator $H = D_x^2 + x^2$, corresponding to an eigenvalue $2\alpha + 1$, that is,

$$H\Phi_\alpha(x) = (2\alpha+1)\Phi_\alpha(x)$$

for any $\alpha \in I_+ = \{0, 1, \ldots\}$. Moreover, $\{\Phi_\alpha \mid \alpha \in I_+\}$ is a complete orthonormal system of $L^2(R)$, and $\Phi_\alpha(x)$ belongs to $S(R)$, where $S(R)$ is the L. Schwartz space of rapidly decreasing functions in R ([2]).

Suppose $u(t,x) \in S'(R_x)$ for fixed t, where $S'(R)$ is the conjugate space of $S(R)$, and set

$$u_\alpha(t) = \langle u(t,x), \Phi_\alpha(x)\rangle.$$

Then the Cauchy problem

$$(A)\begin{cases} (D_t + H)u(t,x) = 0 & (0 \le t \le T,\ x \in R), \\ u(0,x) = \delta(x) & (x \in R) \end{cases}$$

Received December 8, 1997.
Revised February 4, 1998.

is reduced to the Cauchy problems of ordinary differential equations

$$(a)_\alpha \begin{cases} \{D_t + (2\alpha+1)\}u_\alpha(t) = 0 & (0 \le t \le T), \\ u_\alpha(0) = \Phi_\alpha(0). \end{cases}$$

Therefore, the solution $u(t,x)$ of the problem (A) can be formally represented by making use of solutions $\{u_\alpha(t)\}$ of the problems $(a)_\alpha$. More precisely,

$$u(t,x) = \sum_{\alpha \in I_+} u_\alpha(t)\Phi_\alpha(x) = \sum_{\alpha \in I_+} \Phi_\alpha(0)e^{-i(2\alpha+1)t}\Phi_\alpha(x).$$

X. Feng proved that $u(t,x) \in S'(R_x)$, owing to the Hermite expression theory in $S'(R)$ (B. Simon [3]).

How about anharmonic oscillator cases? Suppose that $\{\phi_\alpha(x)\}$ is a complete orthonormal system in $L^2(R)$, where $\phi_\alpha(x)$ is an eigenfunction of the generalized Hermite operator $L = D_x^2 + x^2 + x^4 + 1$, corresponding to an eigenvalue λ_α. Then the solution of the Cauchy problem of the Schrödinger equation

$$\begin{cases} (D_t + L)u(t,x) = 0 & (0 \le t \le T,\ x \in R), \\ u(0,x) = \delta(x) & (x \in R) \end{cases}$$

can be given by

$$u(t,x) = \sum_{\alpha \in I_+} \phi_\alpha(0)e^{-i\lambda_\alpha t}\phi_\alpha(x).$$

Our aim in this paper is to prove $u(t,x) \in S'(R_x)$. In the following, this problem will be considered in a more general situation.

2. Preparations

Let us define a generalized Hermite operator L by

$$L = (L_1, \dots, L_n),$$

$$L_j = D_{x_j}^2 + V_j(x_j) \quad (j = 1, 2, \dots, n),$$

where $V_j(s)$ is a $C^\infty(R)$-function satisfying the following conditions: there exist $\delta_j > 0$, $c_0 > 0$, and $C_k > 0$ $(k \in I_+)$ such that

$$\begin{cases} V_j(s) \ge c_0(1+|s|)^{2\delta_j} & (\forall s \in R), \\ |D_s^k V_j(s)| \le C_k(1+|s|)^{2\delta_j} & (\forall s \in R). \end{cases}$$

LEMMA 1. *There exist $\{\phi_{jk}(s)\}_{k \in I_+}$ satisfying the following conditions, where $\phi_{jk}(s)$ is an eigenfuction of L_j, corresponding to an eigenvalue λ_{jk}.*

1) $0 < \lambda_{j0} \leq \lambda_{j1} \leq \cdots \leq \lambda_{jk} \leq \cdots$, *and there exists* $p_0 > 0$ *such that*

$$\sum_{k=0}^{\infty} \lambda_{jk}^{-p_0} < +\infty.$$

2) $\phi_{jk}(s)$ *is real valued, and* $\{\phi_{jk}(s)\}_{k \in I_+}$ *is a complete orthonomal system of* $L^2(R)$.

3) $\phi_{jk}(s) \in S(R)$, *and there exist* $C(l) > 0$ *and* $p(l) > 0$ *for any* $l \in I_+$ *such that*

$$\|\phi_{jk}\|_l := \sum_{\alpha+\beta \leq l} \sup_{x \in R} |s^\alpha D_s^\beta \phi_{jk}(s)| \leq C(l) \lambda_{jk}^{p(l)} \quad (\forall k \in I_+)$$

holds.

Lemma 1 is proved in [4] under assumptions slightly different to ours, but it is proved similarly.

Now, for any $\alpha = (\alpha_1, \ldots, \alpha_n)$, $\beta = (\beta_1, \ldots, \beta_n) \in I_+^n$, we put

$$\phi_\alpha(x) = \Pi_{j=1}^n \phi_{j,\alpha_j}(x_j), \quad \lambda_\alpha = (\lambda_{1,\alpha_1}, \ldots, \lambda_{n,\alpha_n}),$$

$$L^\beta = \Pi_{j=1}^n L_j^{\beta_j} \quad \lambda_\alpha^\beta = \Pi_{j=1}^n \lambda_{j,\alpha_j}^{\beta_j}$$

and denote

$$\begin{aligned}\Lambda &= \{\lambda_\alpha \mid \alpha \in I_+^n\} \\ &= \{(\lambda_{10}, \lambda_{20}, \ldots, \lambda_{n0}), (\lambda_{11}, \lambda_{20}, \ldots, \lambda_{n0}), (\lambda_{10}, \lambda_{21}, \ldots, \lambda_{n0}), \ldots\}.\end{aligned}$$

Using Lemma 1, it is easy to prove

LEMMA 2. *$\phi_\alpha(x)$ is an eigenfunction of L^β corresponding to an eigenvalue λ_α^β, and they satisfy*

1) *there exists* $p_0 > 0$ *such that*

$$\sum_{\alpha \in I_+^n} |\lambda_\alpha|^{-p_0} < +\infty,$$

2) $\{\phi_\alpha(x)\}_{\alpha \in I_+^n}$ *is a complete orthonomal system of* $L^2(R^n)$,

3) $\phi_\alpha(x) \in S(R^n)$, *and there exist* $C(l) > 0$ *and* $p(l) > 0$ *for any* $l \in I_+$ *such that*

$$\|\phi_\alpha\|_l \leq C(l) |\lambda_\alpha|^{p(l)} \quad (\forall \alpha \in I_+^n).$$

Here we call $\{\phi_\alpha(x)\}_{\alpha \in I_+^n}$ a family of generalized Hermite functions.

Let s be a space, whose element

$$a=(a_\alpha)_{\alpha\in I_+^n}=(a_{0,\dots,0},a_{1,0,\dots,0},a_{0,1,0,\dots,0},\dots)\quad (a_\alpha\in C)$$

satisfies

$$|a|_h:=\sup_{\alpha\in I_+^n}|a_\alpha|\,|\lambda_\alpha|^{h/2}<+\infty\quad (\forall h\in I_+).$$

s is a Fréchet space with a countable set of seminorms $\{|a|_h\}_{h\in I_+}$. Let s' be the conjugate space of s. Namely, s' is a set of all linear continuous mappings from s to C. More precisely, let $b\in s'$, that is, $b:s\ni a\to\langle b,a\rangle\in C$. Then there exist $h>0$ and $C>0$ such that it holds

$$|\langle b,a\rangle|\le C|a|_h\quad (\forall a\in s).$$

LEMMA 3. 1) *Let $f(x)\in S(R^n)$ and set*

$$a(f)=\{a_\alpha(f)\}_{\alpha\in I_+^n},\quad a_\alpha(f)=\langle f,\phi_\alpha\rangle.$$

Then

$$S(R^n)\ni f(x)\to a(f)\in s$$

is linear continuous. More precisely, there exists $C_h>0$ for any h such that

$$|a(f)|_h\le C_h\|f\|_{2n+(\delta+1)h}\quad \left(\delta=\max_j\delta_j\right).$$

2) *Conversely, let $a\in s$ and set*

$$f(x)=\sum_\alpha a_\alpha\phi_\alpha(x).$$

Then

$$s\ni a\to f\in S(R^n)$$

is linear and continuous. More precisely, there exists $C_l>0$ for any l such that

$$\|f\|_l\le C_l|a|_{2p(l)+2p_0}.$$

Moreover, $a(f)=a$ holds.

PROOF. 1) For any $h\in I_+$, it holds

$$\begin{aligned}|a(f)|_{2h}&=\sup_{\alpha\in I_+^n}|a_\alpha(f)||\lambda_\alpha|^h\\&\le C_h\sup_{\alpha\in I_+^n}(\lambda_{1,\alpha_1}^h+\cdots+\lambda_{n,\partial_n}^h)|\langle f,\phi_\alpha\rangle|\end{aligned}$$

and

$$\lambda_{j,\alpha_j}^h |\langle f, \phi_\alpha \rangle| = |\langle f, L_j^h \phi_\alpha \rangle| = |\langle L_j^h f, \phi_\alpha \rangle| \leq \|L_j^h f\|_{L^2}.$$

Hence

$$|a(f)|_{2h} \leq C_h \sum_{j=1}^{n} \|L_j^h f\|_{L^2}.$$

On the other hand, since

$$|D_{x_j}^k V_j(x_j)| \leq C_k (1 + |x_j|)^{2\delta_j},$$

we have

$$\|L_j^h f\|_{L^2} = \|(D_{x_j}^2 + V(x_j))^h f\|_{L^2} \leq C_h \sum_{\substack{r \leq 2\delta h \\ \beta \leq 2h}} \|x_j^r D_{x_j}^\beta f\|_{L^2} \leq C_h' \|f\|_{2n+2(\delta+1)h}.$$

Therefore, we have

$$|a(f)|_{2h} \leq C_h \|f\|_{2n+2(\delta+1)h}.$$

2) Conversely, let $a = \{a_\alpha\}_{\alpha \in I_+^n}$, then we have from 1) and 3) of Lemma 2,

$$\begin{aligned}\sum_{\alpha \in I_+^n} \|a_\alpha \phi_\alpha\|_l &= \sum_{\alpha \in I_+^n} |a_\alpha| \, \|\phi_\alpha\|_l \\ &\leq \sum_{\alpha \in I_+^n} |a_\alpha| C(l) |\lambda_\alpha|^{p(l)} \\ &\leq C(l) \sup_\alpha |a_\alpha| \, |\lambda_\alpha|^{p(l)+p_0} \sum_{\alpha \in I_+^n} |\lambda_\alpha|^{-p_0} \\ &= C'(l) |a|_{2p(l)+2p_0}\end{aligned}$$

for any l. Therefore, $\sum_{\alpha \in I_+^n} a_\alpha \phi_\alpha(x)$ is a convegent sequence in $S(R)$. Hence, set

$$f(x) = \sum_{\alpha \in I_+^n} a_\alpha \phi_\alpha(x).$$

Then it holds that

$$\|f\|_l \leq C'(l) |a|_{2p(l)+2p_0},$$

and

$$a_\beta(f) = \langle f, \phi_\beta \rangle = \sum_{\alpha \in I_+^n} a_\alpha \langle \phi_\alpha, \phi_\beta \rangle = a_\beta.$$

Thus $a(f) = a$ holds. □

LEMMA 4. 1) *Let* $T \in S'(R^n)$, *and put*

$$b = \{b_\alpha\}_{\alpha \in I_+^n}, \quad b_\alpha = \langle T, \phi_\alpha \rangle.$$

Then

i) *there exists* $h > 0$ *such that* $|b|_{-h} := \sup_\alpha |b_\alpha| \, |\lambda_\alpha|^{-h/2} < \infty$,

ii) $b : s \ni \forall a \to \sum_\alpha a_\alpha b_\alpha \in C$ *belongs to* s',

iii) *for any* $f \in S(R^n)$, *it holds*

$$\langle T, f \rangle = \sum_\alpha b_\alpha a_\alpha(f), \quad a_\alpha(f) = \langle f, \phi_\alpha \rangle.$$

2) *Conversely, let* $b \in s'$. *Then* $T : S(R^n) \ni f \to \langle b, a(f) \rangle$ *belongs to* $S'(R^n)$.

PROOF. 1) i) Since $T : S(R^n) \ni \phi \to \langle T, \phi \rangle \in C$ is continuous, there exist $C > 0$ and $l > 0$ such that

$$|b_\alpha| = |\langle T, \phi_\alpha \rangle| \le C \|\phi_\alpha\|_l \le CC(l) |\lambda_\alpha|^{p(l)},$$

using 3) of Lemma 2. Hence we have $|b|_{-2p(l)} < +\infty$.

ii) Let h be the number in i). Then we have

$$|b_\alpha| \, |\lambda_\alpha|^{-h/2} \le C \quad (\forall \alpha \in I_+^n).$$

Therefore, we have

$$\begin{aligned} \sum_\alpha |a_\alpha| \, |b_\alpha| &\le C \sum_\alpha |a_\alpha| |\lambda_\alpha|^{h/2} \\ &\le C \sum_\alpha |\lambda_\alpha|^{-p_0} \sup_\alpha |a_\alpha| \, |\lambda_\alpha|^{h/2+p_0} \\ &= C' |a|_{h+2p_0}, \end{aligned}$$

for any $a = \{a_\alpha\}_{\alpha \in I_+^n} \in s$, where we used 1) of Lemma 2. Hence

$$b : s \ni a \to \langle b, a \rangle = \sum_\alpha a_\alpha b_\alpha \in C$$

is a linear continuous mapping, that is, b belong to s'.

iii) Let $f(x) \in S(R^n)$. Then we have

$$f(x) = \sum_{\alpha} a_\alpha(f)\phi_\alpha(x) \quad \text{in } S(R^n),$$

where $a_\alpha(f) = \langle f, \phi_\alpha \rangle$ $(\alpha \in I_+^n)$ from 2) of Lemma 3. Hence we have

$$\langle T, f \rangle = \left\langle T, \sum_{\alpha} a_\alpha(f)\phi_\alpha \right\rangle = \sum_{\alpha} a_\alpha(f)\langle T, \phi_\alpha \rangle = \sum_{\alpha} a_\alpha(f) b_\alpha.$$

2) Conversely, for any $f(x) \in S(R^n)$, we have $a(f) = \{a_\alpha(f)\} \in s$, and there exists $C_h > 0$ for any $h \in I_+$ such that

$$|a(f)|_h \leq C_h \|f\|_{2n+(\delta+1)h},$$

from 1) of Lemma 3. Let $b \in s'$. Then there exist $C > 0$ and $h > 0$ such that

$$|\langle b, a(f) \rangle| \leq C|a(f)|_h.$$

Therefore, we have

$$|\langle b, a(f) \rangle| \leq C\|f\|_{2n+(\delta+1)h}.$$

Hence

$$T : S(R^n) \ni f \to \langle b, a(f) \rangle \in C$$

is a linear continuous mapping, namely $T \in S'(R^n)$. □

We say that $u(t,x) \in B^h([0,T], S'(R^n))$, iff

$$u : [0,T] \ni t \to u(t,x) \in S'(R^n)$$

is continuously differentiable up to order h in the sense of simple topology of $S'(R^n)$.

LEMMA 5. 1) *Suppose* $u(t,x) \in B^h([0,T], S'(R_x^n))$, *and set*

$$u_\alpha(t) = \langle u(t,x), \phi_\alpha(x) \rangle.$$

Then there exist $C > 0$ *and* $p > 0$ *such that*

$$|D_t^j u_\alpha(t)| \leq C|\lambda_\alpha|^p \quad (\alpha \in I_+^n,\ 0 \leq j \leq h).$$

2) *Conversely, suppose*

$$|D_t^j u_\alpha(t)| \leq C|\lambda_\alpha|^p \quad (\alpha \in I_+^n,\ 0 \leq j \leq h),$$

and set

$$u(t,x) = \sum_{\alpha} u_\alpha(t)\phi_\alpha(x),$$

that is,

$$u : S(R^n) \ni f \to \langle u(t,x), f\rangle = \sum_{\alpha} u_\alpha(t)\langle\phi_\alpha(x), f(x)\rangle = \sum_{\alpha} u_\alpha(t)a_\alpha(f) \in C$$

for $t \in [0,T]$. *Then* $u(t,x) \in B^h([0,T], S'(R^n_x))$.

PROOF. 1) Suppose $u(t,x) \in B^h([0,T], S'(R^n))$, then $H = \{u(t,x) \mid t \in [0,T]\}$ is a bounded set in $S'(R^n)$ in the sense of simple topology. By using the fundamental lemma of Fréchet space ([5]), there exist $C > 0$ and $l_0 > 0$ such that

$$|\langle u(t,x), \phi(x)\rangle| \le C\|\phi\|_{l_0} \quad (\forall t \in [0,T], \forall\phi \in S(R^n)).$$

Therefore, it holds

$$|u_\alpha(t)| = |\langle u(t,x), \phi_\alpha(x)\rangle| \le C\|\phi_\alpha\|_{l_0} \quad (\alpha \in I^n_+).$$

Besides, since

$$\|\phi_\alpha\|_{l_0} \le C(l_0)|\lambda_\alpha|^{p(l_0)}$$

from 3) of Lemma 2, we have

$$|u_\alpha(t)| \le CC(l_0)|\lambda_\alpha|^{p(l_0)}.$$

In the same way, we have

$$|D^j_t u_\alpha(t)| \le C|\lambda_\alpha|^p \quad (\alpha \in I^n_+,\ j = 0,1,2,\dots,h).$$

2) Conversely, suppose

$$|D^j_t u_\alpha(t)| \le C|\lambda_\alpha|^p. \quad (\alpha \in I^n_+,\ j = 0,1,2,\dots,h),$$

and set

$$u : S(R^n) \ni f \to \langle u(t,x), f(x)\rangle = \sum_{\alpha\in I^n_+} u_\alpha(t)a_\alpha(f) \in C, \quad a_\alpha(f) = \langle f, \phi_\alpha\rangle.$$

Then $u(t,x)$ belongs to $S'(R^n)$, from 2) of Lemma 4. Since

$$\sum_{\alpha \in I_+^n} D_t^j u_\alpha(t) a_\alpha(f) \quad (j = 0, 1, \ldots, h)$$

are uniformly convergent sequences in $[0, T]$,

$$D_t^j \langle u(t,x), f(x) \rangle = D_t^j \sum_\alpha u_\alpha(t) a_\alpha(f)$$
$$= \sum_\alpha D_t^j u_\alpha(t) a_\alpha(f).$$

Therefore, we have

$$|D_t^j \langle u(t,x), f(x) \rangle| \leq \sum_\alpha |a_\alpha(f)| |D_t^j u_\alpha(t)|$$
$$\leq C \sum_\alpha |a_\alpha(f)| |\lambda_\alpha|^p.$$

By using 1) of Lemma 2, we have

$$\sum_\alpha |a_\alpha(f)| |\lambda_\alpha|^p = \sup_{\alpha \in I_+^n} |a_\alpha(f)| \, |\lambda_\alpha|^{p+p_0} \sum_{\alpha \in I_+^n} |\lambda_\alpha|^{-p_0}$$
$$\leq |a(f)|_{2p+2p_0} \sum_{\alpha \in I_+^n} |\lambda_\alpha|^{-p_0}$$
$$= C |a(f)|_{2p+2p_0}.$$

On the other hand, by using 1) of Lemma 3, we have

$$|a(f)|_{2p+2p_0} \leq C \|f\|_{2n+2(\delta+1)(p+p_0)}.$$

Hence

$$|D_t^j \langle u(t,x), f(x) \rangle| \leq C \|f\|_{2n+2(\delta+1)(p+p_0)} < +\infty \quad (t \in [0,T], \, j = 0, 1, \ldots, h),$$

that is, $u(t,x) \in B^h([0,T], S'(R_x^n))$. □

3. Cauchy problems

Let us consider Cauchy problems related to differential operators with coefficients of generalized Hermite operators

$$P(D_t, L) = P_m(L)D_t^m + \cdots + P_0(L),$$

$$P_j(L) = \sum_{|\beta| \le m_j} a_{j,\beta} L^\beta = \sum_{\beta_1 + \cdots + \beta_n \le m_j} a_{j,\beta_1,\dots,\beta_n} L_1^{\beta_1} \cdots L_n^{\beta_n},$$

where $a_{j,\alpha}$ are constants and m_j are non-negatve integers. $P(D_t, L)$ is called an evolution differential operator with coefficients of generalized Hermite operators, iff

(I) there exist $C_1 > 0$ and $p_1 > 0$ such that

$$|P_m(\lambda)| \ge C_1 |\lambda|^{-p_1} \quad (\forall \lambda \in \Lambda),$$

(II) there exists $k > 0$ such that

$$I_m \tau_j(\lambda) \ge -k. \quad (\forall \lambda \in \Lambda, \ 1 \le j \le m),$$

where

$$P(\tau, \lambda) = P_m(\lambda)(\tau - \tau_1(\lambda)) \cdots (\tau - \tau_m(\lambda)).$$

THEOREM 1. *Suppose $P(D_t, L)$ is an evolution differential operator with coefficients of generalized Hermite operators. Let*

$$f(t,x) \in B^h([0,T], S'(R_x^n)), \quad g_j(x) \in S'(R_x^n) \quad (0 \le j \le m-1).$$

Then there exists unique solution $u(t,x)$, belonging to $B^{h+m}([0,T], S'(R^n))$, of the Cauchy problem:

$$(A)\begin{cases} P(D_t, L)u(t,x) = f(t,x) & (0 \le t \le T, \ x \in R^n), \\ D_t^j u(t,x)|_{t=0} = g_j(x) & (x \in R^n, \ 0 \le j \le m-1). \end{cases}$$

PROOF. Let

$$u_\alpha(t) = \langle u(t,x), \phi_\alpha(x) \rangle,$$

$$f_\alpha(t) = \langle f(t,x), \phi_\alpha(x) \rangle, \quad g_{j,\alpha} = \langle g_j(x), \phi_\alpha(x) \rangle.$$

Then the problem (A) is reduced to the Cauchy problems of ordinary differential equations:

$$(a)_\alpha\begin{cases} \{P_m(\lambda_\alpha)D_t^m + \cdots + P_0(\lambda_\alpha)\}u_\alpha(t) = f_\alpha(t) & (0 \le t \le T), \\ D_t^j u_\alpha(t)|_{t=0} = g_{j,\alpha} & (j = 0, 1, 2, \dots, m-1). \end{cases}$$

The solutions of $(a)_\alpha$ can be represented as

$$u_\alpha(t) = \sum_{j=1}^{m} b_{j-1,\alpha} D_t^{m-j} W(t, \lambda_\alpha) + iP_m(\lambda_\alpha)^{-1} \int_0^t f_\alpha(s) W(t-s, \lambda_\alpha)\, ds,$$

where

$$W(t, \lambda_\alpha) = \frac{P_m(\lambda_\alpha)}{2\pi i} \oint_\gamma \frac{e^{itz}}{P(z, \lambda_\alpha)} dz,$$

$$b_{0,\alpha} = g_{0,\alpha}, \quad b_{j,\alpha} = g_{j,\alpha} - \sum_{i=1}^{j} b_{i-1,\alpha} D_t^{m+j-i} W(0, \lambda_\alpha) \quad (1 \le j \le m-1),$$

and γ is a closed curve inside of which all zeros of $P(z, \lambda_\alpha)$ with respect to z are containd. By evaluating the above representation, there exist $C > 0$ and $p > 0$ such that

$$|D_t^k u_\alpha(t)| \le C|\lambda_\alpha|^p \left\{ \sum_{j=0}^{m-1} |g_{j,\alpha}| + \sum_{j=0}^{\max(k-m,0)} \sup_{0 \le s \le t} |D_s^j f_\alpha(s)| \right\}$$

$$(0 \le t \le T, \ 0 \le k \le m+h).$$

Since $g_j(x) \in S'(R^n)$ $(0 \le j \le m-1)$, there exists $q_1 > 0$ such that

$$\sup_{\alpha \in I_+^n} |g_{j,\alpha}|\, |\lambda_\alpha|^{-q_1} < +\infty$$

from 1) of Lemma 4, and since $f(t,x) \in B^h([0,T], S'(R^n))$, there exists $q_2 > 0$ such that

$$\sup_{\alpha \in I_+^n} \sup_{0 \le t \le T} |D_t^j f_\alpha(t)|\, |\lambda_\alpha|^{-q_2} < +\infty,$$

from 1) of Lemma 5. Therefore, we have

$$|D_t^k u_\alpha(t)| \le C' |\lambda_\alpha|^{p+q} \quad (t \in [0,T]), \alpha \in I_+^n, 0 \le k \le m+h),$$

where $q = \max(q_1, q_2)$. Finally, set

$$u(t,x) = \sum_\alpha u_\alpha(t) \phi_\alpha(x).$$

Then $u(t,x)$ belongs to $B^{h+m}([0,T], S'(R_x^n))$, from 2) of Lemma 5, and becomes a solution of the problem (A). The uniqueness of the problem (A) follows from the uniqueness of the problems $(a)_\alpha$.

REMARK 1. Let

$$P(D_t, L) = D_t - \sum_{|\beta| \leq N} a_\beta L^\beta.$$

Then P is an evolution operator, iff there exists $k > 0$ such that

$$I_m \sum_{|\beta| \leq N} a_\beta \lambda_\alpha^\beta \geq -k \quad (\forall \lambda_\alpha \in \Lambda).$$

REMARK 2. Let

$$P(D_t, L) = D_t^2 - \sum_{|\beta| \leq N} a_\beta L^\beta.$$

Then P is an evolution operator, iff there exists $k > 0$ such that

$$\left(\sum_{|\beta| \leq N} R_e a_\beta \lambda_\alpha^\beta, \sum_{|\beta| \leq N} I_m a_\beta \lambda_\alpha^\beta \right) \in \Omega_k \quad (\forall \lambda_\alpha \in \Lambda),$$

where

$$\Omega_k = \{(X, Y) \mid Y^2 \leq kX \text{ or } X^2 + Y^2 \leq k\}.$$

For example,

$$D_t^2 - \{L_1^2 + \cdots + L_n^2 + i(L_1 - L_2)\},$$

$$D_t^2 - \{L_1^3 + \cdots + L_n^3 + i(L_1 - L_2)\},$$

$$D_t^2 - \{L_1^2 L_2^2 + i(L_1 - L_2)\}$$

are evolution operators.

The paper has finished under the kind guidance of Prof. Reiko Sakamoto and Prof. Sadao Miyatake. I am deeply greateful to them.

References

[1] X. Feng, Solutions to the Cauchy Problems for a Number of Linear Partial Differential eguations in the Space of Tempered Distributions, Master's thesis of Lanzhou University, 1986, 1–47.
[2] L. Schwartz, Theorie des Distributions, Hermann Paris, 1966.
[3] B. Simon, Distributions and Their Hermite Expressions, J. Math. Phys. **12**, 1971, 140–148.

[4] D. Robert, Proprietes spectrales d' operateurs pseuds-differentiels, Comm. in partial differential equations, **3** (9), 1978, 755–826.
[5] S. Mizohata, The Theory of Partial Differential Equations, Cambridge Univ. Press, 1973.

Graduate School of Human Culture,
Nara Woman's University