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INDUCED MO-MAPPINGS

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Abstract. A mapping $f: X \to Y$ between continua X and Y is called an MO-mapping provided that it can be represented as the composition of two mappings, $f_1 : X \to Z$ and $f_2 : Z \to Y$, such that f_1 is open and f_2 is monotone. Induced MO-mappings, 2^f and $C(f)$, between hyperspaces are studied. In particular an example is constructed of an open mapping $f: [0,1] \to [0,1]$ for which $C(f)$ is not an MO-mapping. This answers two questions asked by H. Hosokawa.

All spaces considered in this paper are assumed to be metric. A *mapping* means a continuous function. To exclude some trivial statements we assume that all considered mappings are not constant. A *continuum* means a compact connected space. Given a continuum X with a metric d, we let 2^X denote the hyperspace of all nonempty closed subsets of X equipped with the *Hausdorff metric* H defined by

$$H(A,B) = \max\{\sup\{d(a,B) : a \in A\}, \sup\{d(b,A) : b \in B\}\}$$

(see e.g. [9, (0.1), p. 1 and (0.12), p. 10]). Further, we denote by $C(X)$ the hyperspace of all subcontinua of X, i.e., of all connected elements of 2^X. The reader is referred to Nadler's book [9] for needed information on the structure of hyperspaces.

Given a mapping $f: X \to Y$ between continua X and Y, we consider mappings (called the *induced* ones)

$$2^f : 2^X \to 2^Y \quad \text{and} \quad C(f) : C(X) \to C(Y)$$

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defined by

$$2^f(A) = f(A) \text{ for every } A \in 2^X \quad \text{and} \quad C(f)(A) = f(A) \text{ for every } A \in C(X).$$

A mapping $f : X \to Y$ between spaces X and Y is said to be:
— *open*, provided that the image of an open subset of the domain is open in the range;
— *monotone*, provided that it has connected point-inverses;
— *OM-mapping*, provided that it can be represented as the composition of two mappings, $f = f_2 \circ f_1$, such that f_1 is monotone and f_2 is open;
— *MO-mapping*, provided that it can be represented as the composition of two mappings, $f = f_2 \circ f_1$, such that f_1 is open and f_2 is monotone;
— *confluent*, provided that for each subcontinuum Q of Y each component of $f^{-1}(Q)$ is mapped onto Q under f.

Monotone, as well as open mappings of compact spaces are known to be confluent, [12, Theorem 7.5, p. 148]. OM- and MO-mappings were introduced in [7, Section 3, p. 104] and studied in [8]. It is known that OM-mappings coincide with quasi-interior ones, as introduced in [13, p. 9], see [7, Corollary 3.1, p. 104], and that all MO-mappings are OM-mappings, [7, Corollary 3.2, p. 104].

Let $\mathfrak{M}_i$, where $i \in \{1, 2, 3\}$ be some three classes of mappings between continua. A general problem which is related to a given mapping and to the two induced mappings is to find all interrelations between the following three statements:

(0.1) $f \in \mathfrak{M}_1$;

(0.2) $C(f) \in \mathfrak{M}_2$;

(0.3) $2^f \in \mathfrak{M}_3$.

There are some papers in which particular results concerning this problem are shown for various classes $\mathfrak{M}_i$ of mappings. In the present paper we will discuss possibly implications between (0.1)–(0.3) for the class of MO-mappings. We start with recalling some related results.

The following results concerning induced mappings for the classes of monotone, of open, and of OM-mappings are known. For monotone mappings see [10, Lemma 2.1, p. 750]; compare [6, Theorem 1.1, p. 121], [3, Lemma 2.3, p. 2], [2, Theorem 3.3, p. 4], and [5, Theorem 3.2, p. 241]. For open mappings see [5, Theorem 4.3, p. 243]; compare also [4, Theorem 3.2]). For OM-mappings see [5, Theorem 5.2, p. 244].

1. THEOREM. *Let a surjective mapping $f : X \to Y$ between continua X and Y be given. Then the following conditions are equivalent*:

(1.1) $f : X \to Y$ *is monotone*;

(1.2) $C(f) : C(X) \to C(Y)$ *is monotone*;

(1.3) $2^f : 2^X \to 2^Y$ *is monotone.*

2. THEOREM. *Let a surjective mapping $f : X \to Y$ between continua X and Y be given. Consider the following conditions*:

(2.1) $f : X \to Y$ *is open*;

(2.2) $C(f) : C(X) \to C(Y)$ *is open*;

(2.3) $2^f : 2^X \to 2^Y$ *is open.*

Then (2.1) *and* (2.3) *are equivalent, and each of them is implied by* (2.2).

3. THEOREM. *Let a surjective mapping $f : X \to Y$ between continua X and Y be given. Then the following conditions are equivalent*:

(3.1) $f : X \to Y$ *is an OM-mapping*;

(3.2) $C(f) : C(X) \to C(Y)$ *is an OM-mapping*;

(3.3) $2^f : 2^X \to 2^Y$ *is an OM-mapping.*

An example is known [5, Section 4, Example, p. 244] of an open surjective mapping $f : X \to Y$ between locally connected continua X and Y such that the induced mapping $C(f) : C(X) \to C(Y)$ is not open. It is so because of the following result, [1, Theorem 1].

4. THEOREM. *If a continuum X is locally connected, and for a mapping $f : X \to Y$ the induced mapping $C(f) : C(X) \to C(Y)$ is open, then f is monotone.*

As a consequence of this theorem the following corollary has been shown in [1, Corollary 2].

5. COROLLARY. *Let a continuum X be hereditarily locally connected, and a mapping $f : X \to Y$ be such that the induced mapping $C(f) : C(X) \to C(Y)$ is open. Then f is a homeomorphism.*

The following result is a consequence of Theorems 1 and 2, see [5, Theorem 5.3, p. 245].

6. COROLLARY. *If a mapping $f : X \to Y$ between continua X and Y is an MO-mapping, then 2^f is also an MO-mapping.*

Investigating the class $\mathfrak{M}$ of MO-mappings, H. Hosokawa asked in [4, Remark 3.7] if the condition $f \in \mathfrak{M}$ implies that $C(f) \in \mathfrak{M}$. Later, in [5, Section 8, Problem 2, p. 249] he asked if the implication holds under an additional assumption that the mapping f is open. Our next result presents a negative answer to both these questions. To formulate it we recall a countable family of open mappings of the closed unit interval onto itself. Let a positive integer k be given and let $m \in \{0, 1, \dots, k-1\}$. Define a surjection

$$g_k : [0,1] \to [0,1] \tag{7}$$

by the following conditions.

(7.1) If m is even, then $g_k(m/k) = 0$, and if m is odd, then $g_k(m/k) = 1$.

(7.2) For each m the restriction $g_k|[m/k, (m+1)/k] : [m/k, (m+1)/k] \to [0,1]$ is defined to be linear.

Thus this restriction, and hence the mapping g_k, is a surjection. Note that $g_k(0) = 0$ and $g_k(1)$ is either 1 or 0 according to k is either odd or even. Observe that g_1 is the identity, and g_2 is the *tent mapping* defined by

$$g_2(x) = \begin{cases} 2x, & \text{for } x \in [0, 1/2], \\ 2 - 2x, & \text{for } x \in [1/2, 1]. \end{cases} \tag{7.3}$$

Recall that two mappings $f_1 : X_1 \to Y_1$ and $f_2 : X_2 \to Y_2$ are said to be *topologically equivalent* provided that there exist homeomorphisms $h_X : X_1 \to X_2$ and $h_Y : Y_1 \to Y_2$ such that $f_2(h_X(x)) = h_Y(f_1(x))$ for each point $x \in X$. Observe that this relation is an equivalence in the class of mappings between topological spaces (see [12, p. 127]). It is known (see [12, (1.3), p. 184]) that a mapping of [0,1] into itself is open if and only if it is topologically equivalent to $g_k : [0,1] \to [0,1]$ for some positive integer k.

8. PROPOSITION. *If $g_2 : [0,1] \to [0,1]$ is the tent mapping, then the induced mapping $C(g_2)$ is an MO-mapping which is neither open nor monotone.*

PROOF. Since any nonempty subcontinuum of $[0,1]$ is a closed interval $[x, y]$ with $0 \le x \le y \le 1$, where $[x, x] = \{x\}$, hence one can assign to $[x, y] \subset [0,1]$ a point (x, y) of the triangle

$$T = \{(x, y) \in \mathbf{R}^2 : 0 \le x \le y \le 1\},$$

and, under this correspondence, the topology induced by the Hausdorff metric on $C([0,1])$ coincides with the Euclidean topology inherited from the plane $\mathbf{R}^2$ on T (see e.g. [11, p. 62]). To simplify notations we omit the homeomorphism between $C([0,1])$ and T. Thus the formula (7.3) for g_2 implies the following one for the induced mapping $C(g_2) : T \to T$:

$$C(g_2)((x, y)) = \begin{cases} (2x, 2y) & \text{if } 0 \le y \le 1/2, \\ (\min\{2x, 2 - 2y\}, 1) & \text{if } 0 \le x \le 1/2 \le y \le 1, \\ (2 - 2y, 2 - 2x) & \text{if } 1/2 \le x \le 1. \end{cases}$$

To see that $C(g_2)$ so defined is an MO-mapping let us consider two additional triangles: $T' = \{(x, y) \in T : x + y \le 1\}$ with vertices $(0,0)$, $(0,1)$, $(1/2, 1/2)$, and $T'' = \{(x, y) \in T' : 0 \le y \le 1/2\}$ with vertices $(0,0)$, $(0,1/2)$, $(1/2,1/2)$. Define a surjection $f_1 : T \to T'$ such that $f_1|T'$ is the identity and $f_1|(T\backslash T')$ is the symmetry with respect to the straight line $x + y = 1$. Thus f_1 is open, and we have

$$f_1((x, y)) = \begin{cases} (x, y) & \text{if } 0 \le x + y \le 1, \\ (1 - y, 1 - x) & \text{if } x + y \ge 1. \end{cases}$$

Next define a surjection $f_2 : T' \to T''$ such that $f_2|T''$ is the identity, and $f_2|(T'\backslash T'')$ projects points on the side of T'' that joins $(0, 1/2)$ with $(1/2, 1/2)$. Thus f_2 is monotone, and its formula is

$$f_2((x, y)) = \begin{cases} (x, y) & \text{if } 0 \le y \le 1/2, \\ (x, 1/2) & \text{if } y \ge 1/2. \end{cases}$$

Finally, let $h : T'' \to T$ be a homeomorphism defined by $h((x, y)) = (2x, 2y)$ for all $(x, y) \in T''$. It can be verified (details are left to the reader) that $C(g_2) = (h \circ f_2) \circ f_1$. Thus $C(g_2)$ is an MO-mapping. The proof is complete.

9. Proposition. *Let a mapping* $g_k : [0,1] \to [0,1]$ *be as in* (7). *Then for each integer* $k \ge 3$ *the induced mapping* $C(g_k)$ *is not an MO-mapping.*

Proof. Suppose on the contrary that for some $k \ge 3$ the induced mapping $C(g_k) : C([0,1]) \to C([0,1])$ can be represented as the composition of two mappings, $C(g_k) = f_2 \circ f_1$, where f_1 is open and f_2 is monotone. Let $Y = f_1(C([0,1]))$, and put

$$A = \left[0, \frac{1}{2k}\right], \quad B = \left[\frac{2}{k} - \frac{1}{2k}, \frac{2}{k}\right], \quad C = \left[\frac{2}{k}, \frac{2}{k} + \frac{1}{2k}\right].$$

Observe that $C(g_k)(A) = C(g_k)(B) = C(g_k)(C) = [0, 1/2]$. Let $\mathcal{U} = \{P \in C([0,1]) : H(P, A) < 1/4k\}$. Then

(9.1) the restriction $C(g_k)|\mathcal{U}$ is one-to-one,

whence $f_1|\mathcal{U}$ is one-to-one. We claim that

(9.2) $$f_1(A) = f_1(B).$$

Indeed, if not, we have $f_1(A) \neq f_1(B)$, but $f_2(f_1(A)) = f_2(f_1(B)) = [0, 1/2]$, and since f_2 is monotone, there is a continuum $M \subset Y$ with $f_1(A)$, $f_1(B) \in M$ and $f_2(M) = \{[0, 1/2]\}$. Let $\mathcal{C} \subset C([0,1])$ be the component of $f_1^{-1}(M)$ which contains A. Since f_1 is open, it is confluent, [12, Theorem 7.5, p. 148], so $f_1(\mathcal{C}) = M$, and thus $\mathcal{C}$ is a nondegenerate continuum containing A. Then $C(g_k)(\mathcal{C} \cap \mathcal{U})$ is a one-point set $\{[0, 1/2]\}$, contrary to (9.1). Thus (9.2) is established.

Let $\mathcal{V} = \{P \in C(B) : H(P, B) < 1/4k\}$. Then $C(g_k)|\mathcal{V}$ is one-to-one, whence $f_1|\mathcal{V}$ is one-to-one as well. Note that $\mathcal{V}$ is not a neighborhood of B.

Let $\{B_m\}$ be a sequence of continua in [0,1] satisfying $B_m \subset B$ and $2/k \notin B_m$ for each $m \in N$, and $B = \operatorname{Lim} B_m$. Observe that $(C(g_k))^{-1}(C(g_k)(B_m))$ has exactly k points. Therefore $f_1^{-1}(f_1(B_m))$ is a subset of the finite set $(C(g_k))^{-1}(C(g_k)(B_m))$, so it is finite. Openness of f_1 implies that f_1^{-1} is continuous, see [12, Theorem 4.32, p. 130], so

(9.3) $f_1^{-1}(f_1(B))$ is finite.

Let $\mathcal{A}$ be the (unique) order arc in $C([0,1])$ from B to $B \cup C$. By (9.3) the set $f_1(\mathcal{A})$ is a nondegenerate subcontinuum of Y. By (9.2) we see that $A \in f_1^{-1}(f_1(\mathcal{A}))$. Then the component $\mathcal{K}$ of $f_1^{-1}(f_1(\mathcal{A}))$ which contains A is a nondegenerate subcontinuum of $C([0,1])$ by confluence of f_1. Note that $C(g_k)(\mathcal{A}) = \{[0, 1/2]\}$, whence $C(g_k)(\mathcal{K}) = C(g_k)(\mathcal{A}) = \{[0, 1/2]\}$, contrary to (9.1). Thus the proof is finished.

As a consequence of Propositions 8 and 9 we have the following result.

10. THEOREM. *The identity g_1 and the tent mapping g_2 are the only two (up to equivalence) open mappings $f : [0,1] \to [0,1]$ for which the induced mapping $C(f)$ is an MO-mapping.*

11. REMARKS. (11.1) Taking as a mapping $f : X \to Y$ the mapping g_k for some integer $k \geq 3$ we see, by Proposition 9, that even in the case when f is open, the induced mapping $C(f)$ need not be an MO-mapping.

(11.2) Since openness of f is equivalent to that of 2^f (see Theorem 2), it follows from (11.1) that even if 2^f is an open mapping (an MO-mapping, in particular), then $C(f)$ need not be an MO-mapping.

The following three questions remain open. The first two of them were asked in [5, Section 8, Problem 2, p. 249].

12. QUESTIONS. (12.1) If 2^f is an MO-mapping, must f be an MO-mapping?

(12.2) If $C(f)$ is an MO-mapping, must f be an MO-mapping?

(12.3) If $C(f)$ is an MO-mapping, must 2^f be an MO-mapping?

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