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A CONFORMAL GAUGE INVARIANT FUNCTIONAL FOR WEYL STRUCTURES AND THE FIRST VARIATION FORMULA

By

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Abstract. We consider a new conformal gauge invariant functional which is a natural curvature functional on the space of Weyl structures. We derive the first variation formula of its functional and characterize its critical points.

1. Introduction

The problem of finding metrics and connections which are minima of some functional plays a central role in Riemannian geometry and conformal geometry. The Einstein equations are obtained as the Euler-Lagrange equations of the variational problem associated with the Hilbert-Einstein functional (cf. [2]). The Einstein-Weyl equations are a conformally invariant generalization of the Einstein equations (cf. [6], [7], [8], [9], [10], [12], [13], [14], [15], [16] and the references therein). Our first aim is to find a functional which has the Einstein-Weyl equations as the Euler-Lagrange equations of its functional. Motivated by this, we consider the variational problem on the space of Weyl structures. One of the most natural such problems is to find critical points of some functional and determine the Euler-Lagrange equations for its functional.

A C^{∞} manifold M is said to carry a Weyl structure if it has a torsion-free affine connection and a metric whose conformal class is preserved by this covariant derivative. Let $\mathfrak{M}$ be the space of all Riemannian metrics on M, $\mathfrak{C}$ the space of all affine connections on M and $\mathfrak{W}$ the space of all Weyl structures. In dimension four, Pedersen et al. [14] introduced the curvature functional $C : \mathfrak{W} \to \boldsymbol{R}$ by

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$$C(g, D) := \int_M |R^D|_g^2 v_g.$$

In their paper, they proved that if (g, D) is closed and satisfies the Einstein-Weyl equations, (g, D) is an absolute minimum of C using topological invariants and vice versa. However, they did not point out the Euler-Lagrange equations of the functional C. In this paper, we conformally generalize their functional to that of an arbitrary dimension and state clearly the Euler-Lagrange equations for its functional. We give the functional $C_n : \mathfrak{M} \times \mathfrak{C} \to \boldsymbol{R}$ by

$$C_n(g, D) := \int_M |R^D|_g^{n/2} v_g.$$

We denote by C_W, the restriction of C_n to the space $\mathfrak{W}$ of Weyl structures, and find critical points of C_W. Then, the first variation formula is given by Theorem 3.6 and the Euler-Lagrange equations as critical points of C_W is obtained in Theorem 3.7.

Next, we introduce the notion of conformal Yang-Mills fields which is a natural generalization of the Yang-Mills fields.

In dimension four, we have very simple interesting forms which are the mixtures of the Yang-Mills equation of D with respect to g and the Euler-Lagrange equation between $\check{R}^D$ and g which are described in Corollary 3.11.

In Appendix, we prove some properties of the curvature tensor R^D in order to calculate critical points of our functional.

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2. Preliminaries

In this paper, we always assume that M is an n-dimensional compact connected orientable C^∞ manifold. In this section, we introduce the conformal functional C_n and give all materials needed later. Let $D \in \mathfrak{C}$ be a torsion free affine connection and $g \in \mathfrak{M}$ a Riemannian metric on M. A couple $(g, D) \in \mathfrak{M} \times \mathfrak{C}$ is a *Weyl structure* on M if there exists a 1-form $\omega \in A^1(M)$ such that $Dg = \omega \otimes g$, that is, for $X, Y, Z \in \mathfrak{X}(M)$,

$$(D_X g)(Y, Z) = \omega(X) g(Y, Z). \tag{2.1}$$

Then, (M, g, D) is called a *Weyl manifold* and the equation (2.1) is called the *Weyl equation*. Let ∇ be the Levi-Civita connection of g. We define $\alpha := D - \nabla \in A^1(\mathrm{End}(TM))$ and review the following Lemma (cf. [15, p. 382, Lemma 2.1]).

LEMMA 2.1. *A couple $(g, D = \nabla + \alpha) \in \mathfrak{M} \times \mathfrak{C}$ satisfies the Weyl equation $Dg = \omega \otimes g$ if and only if*

$$\alpha_X Y = \frac{1}{2}\{g(X, Y)\omega^\sharp - \omega(X)Y - \omega(Y)X\} \quad \textit{for any } X, Y \in \mathfrak{X}(M),$$

where $\omega^\sharp \in \mathfrak{X}(M)$ is defined by $g(\omega^\sharp, Z) = \omega(Z)$ for $Z \in \mathfrak{X}(M)$.

Let $D_t \in \mathfrak{C}$ be a deformation of D. Then, we denote

$$\beta := \frac{d}{dt}\bigg|_{t=0} D_t \in T_D\mathfrak{C} \cong A^1(\mathrm{End}(TM)).$$

Here, the tangent space $T_D\mathfrak{C}$ of the totality $\mathfrak{C}$ of all affine connections on M at D, is identified with the space $A^1(\mathrm{End}(TM))$ of all $\mathrm{End}(TM)$-valued 1-forms.

Similarly, let $g_t \in \mathfrak{M}$ be a deformation of g. Then, we denote

$$h := \frac{d}{dt}\bigg|_{t=0} g_t \in T_g\mathfrak{M} \cong S^2(M).$$

Here, the tangent space $T_g\mathfrak{M}$ of the totality $\mathfrak{M}$ of all Riemannian metrics on M at g, is identified with the space $S^2(M)$ of all symmetric 2-tensor fields.

DEFINITION 2.2. We define a functional $C_n : \mathfrak{M} \times \mathfrak{C} \to \boldsymbol{R}$ by

$$C_n(g, D) := \int_M |R^D|_g^{n/2} v_g = \int_M \langle R^D, R^D \rangle_g^{n/4} v_g,$$

where

$$R^D(X, Y)Z := D_X D_Y Z - D_Y D_X Z - D_{[X, Y]}Z,$$

$|\cdot|_g$ is the norm induced by g and v_g is the volume form with respect to g.

PROPOSITION 2.3. *The functional $C_n : \mathfrak{M} \times \mathfrak{C} \to \boldsymbol{R}$ is conformal invariant, that is, for any $f \in C^\infty(M)$,*

$$C_n(e^{2f} g, D) = C_n(g, D).$$

We then call C_n the *conformal functional.*

Now, we show a one to one correspondence between $\mathfrak{W}$ and $\mathfrak{M} \times A^1(M)$.

LEMMA 2.4. *There is a natural isomorphism from* $\mathfrak{W}$ *to* $\mathfrak{M} \times A^1(M)$.

PROOF. For any $(g, D) \in \mathfrak{W}$, we can find uniquely $\omega \in A^1(M)$ such that $Dg = \omega \otimes g$. Then, we define a map $F : \mathfrak{W} \to \mathfrak{M} \times A^1(M)$ by

$$F(g, D) := (g, \omega).$$

For any $(g, \omega) \in \mathfrak{M} \times A^1(M)$, we define D by

$$D_X Y = \nabla_X Y + \frac{1}{2}\{g(X, Y)\omega^\sharp - \omega(X)Y - \omega(Y)X\}. \tag{2.2}$$

By Lemma 2.1, D is a torsion-free affine connection such that $Dg = \omega \otimes g$, which implies that F is bijective. □

PROPOSITION 2.5. *For any* $f \in C^\infty(M)$, *we consider a conformal transformation* $G_f : \mathfrak{M} \times \mathfrak{C} \to \mathfrak{M} \times \mathfrak{C}$ *given by* $(g, D) \mapsto (e^{2f} g, D)$. *Then, we have*

1. $G_f(\mathfrak{W}) = \mathfrak{W}$,
2. $F \circ G_f \circ F^{-1}(g, \omega) = (e^{2f} g, \omega + 2df)$,

where $F : \mathfrak{W} \to \mathfrak{M} \times A^1(M)$ *is the isomorphim as in Lemma* 2.4.

We denote also by $\tilde{D}\beta$, the exterior derivative $d^D \beta \in A^{k+1}(\mathrm{End}(TM))$ induced by a connection D, for $\beta \in A^k(\mathrm{End}(TM))$, that is,

$$(d^D \beta)(X_1, \ldots, X_{k+1}) := \sum_{i=1}^{k+1} (-1)^{i+1} (D_{X_i}\beta)(X_1, \ldots, \hat{X}_i, \ldots, X_{k+1}),$$

for $X_i \in \mathfrak{X}(M)$ $(i = 1, \ldots, k+1)$. Here $\hat{X}_i$ means to delete X_i. We also recall the symmetric 2-tensor $\check{R}^D$ which is defined in [2, p. 51, 1.131].

DEFINITION 2.6. For $(g, D) \in \mathfrak{M} \times \mathfrak{C}$, we define the symmetric 2-tensor $\check{R}^D$ as

$$\check{R}^D(X, Y) := \sum_{i,j,k=1}^{n} R^D(X, e_i, e_j, e_k) R^D(Y, e_i, e_j, e_k),$$

where R^D is the curvature tensor of D, $\{e_1, \ldots, e_n\}$ are orthonormal local frame fields with respect to g and $R^D(X, Y, Z, W) := g(R^D(X, Y)Z, W)$.

3. The First Variation Formula

In this section, we calculate the first variation of the functional $C_W : \mathfrak{W} \to \boldsymbol{R}$ defined by the restriction of C_n to the space $\mathfrak{W}$ of Weyl structures and characterize its critical points.

Fix $(g,\omega) \in \mathfrak{M} \times A^1(M) \cong \mathfrak{W}$ and consider a smooth deformation of Riemannian metrics $g_t \in \mathfrak{M}$ and 1-forms $\omega_t \in A^1(M)$ such that $g_0 = g$ and $\omega_0 = \omega$ $(-\varepsilon < t < \varepsilon)$. Then, we have

$$(3.1) \qquad \frac{d}{dt}\bigg|_{t=0} C_n(g_t,\omega_t) = \frac{d}{dt}\bigg|_{t=0} \int_M |R^{D^{g_t}}|_{g_t}^{n/2} v_{g_t} + \frac{d}{dt}\bigg|_{t=0} \int_M |R^{D_t}|_g^{n/2} v_g,$$

where D^{g_t}, D_t are the affine connections corresponding to $(g_t,\omega),(g,\omega_t)$, respectively.

We first calculate $\frac{d}{dt}\big|_{t=0} \int_M |R^{D_t}|_g^{n/2} v_g$ as follows.

Proposition 3.1. *We have the following formula*:

$$\frac{d}{dt}\bigg|_{t=0} \int_M |R^{D_t}|_g^{n/2} v_g = \frac{n}{4}\int_M \langle \eta^\sharp \otimes g - \mathrm{Id} \otimes \eta - \eta \otimes \mathrm{Id}, \tilde{D}^*(|R^D|_g^{(n-4)/2} R^D)\rangle_g v_g,$$

where $\eta = (d/dt)|_{t=0}\,\omega_t \in A^1(M)$, Id *is the identity map of* $\Gamma(TM)$ *and* $\tilde{D}^*$ *is the formal adjoint of* $\tilde{D}$ *with respect to* g.

Proof. We have

$$(3.2) \qquad \frac{d}{dt}\bigg|_{t=0} \int_M |R^{D_t}|_g^{n/2} v_g = \frac{n}{2}\int_M |R^D|_g^{(n-4)/2} \left\langle \frac{d}{dt}\bigg|_{t=0} R^{D_t}, R^D \right\rangle_g v_g.$$

When we set $A_t := D_t - D$, R^{D_t} is expressed as

$$(3.3) \qquad R^{D_t} = R^D + \tilde{D}A_t + A_t \wedge A_t,$$

where $(A_t \wedge A_t)(X,Y) := [A_t(X), A_t(Y)] \in \mathrm{End}(TM)$ for $X, Y \in \mathfrak{X}(M)$.

Differentiating the both hand sides of (3.3) at $t = 0$, we have

$$(3.4) \qquad \frac{d}{dt}\bigg|_{t=0} R^{D_t} = \tilde{D}\left(\frac{d}{dt}\bigg|_{t=0} D_t\right).$$

From (2.2), we have

$$(3.5) \qquad \frac{d}{dt}\bigg|_{t=0} D_t = \frac{1}{2}(\eta^\sharp \otimes g - \mathrm{Id} \otimes \eta - \eta \otimes \mathrm{Id}).$$

From (3.2), (3.4) and (3.5), we obtain Proposition 3.1. □

Next, we calculate $\frac{d}{dt}\big|_{t=0} \int_M |R^{D^{g_t}}|_{g_t}^{n/2} v_{g_t}$ as follows.

LEMMA 3.2. *We have the following formula*:

$$\frac{d}{dt}\bigg|_{t=0}\int_M |R^{D^{g_t}}|_{g_t}^{n/2} v_{g_t} = \frac{1}{2}\int_M \bigg\{\langle |R^D|_g^{(n-4)/2}(|R^D|_g^2 g - n\check{R}^D), h\rangle_g + n\Big\langle \gamma + \frac{1}{2}(\omega^\sharp \otimes h), \tilde{D}^*(|R^D|_g^{(n-4)/2} R^D)\Big\rangle_g\bigg\} v_g, \tag{3.6}$$

where $\gamma := \frac{d}{dt}\big|_{t=0} \nabla^{g_t} \in A^1(\mathrm{End}(TM))$ *and* $h := \frac{d}{dt}\big|_{t=0} g_t \in S^2(M)$.

PROOF. We have

$$\begin{aligned}\frac{d}{dt}\bigg|_{t=0}\int_M |R^{D^{g_t}}|_{g_t}^{n/2} v_{g_t} &= \int_M \frac{d}{dt}\bigg|_{t=0} |R^{D^{g_t}}|_{g_t}^{n/2} v_g + \int_M |R^D|_g^{n/2} \frac{d}{dt}\bigg|_{t=0} v_{g_t} \\ &= \frac{n}{4}\int_M |R^D|_g^{(n-4)/2} \frac{d}{dt}\bigg|_{t=0} \langle R^{D^{g_t}}, R^{D^{g_t}}\rangle_{g_t} v_g \\ &\quad + \frac{1}{2}\int_M |R^D|_g^{n/2}\langle g, h\rangle_g v_g.\end{aligned} \tag{3.7}$$

From Lemma A.4 in Appendix, we have the following equations:

$$\frac{d}{dt}\bigg|_{t=0} \langle R^{D^{g_t}}, R^{D^{g_t}}\rangle_{g_t} = -2\langle h, \check{R}^D\rangle_g + 2\Big\langle \frac{d}{dt}\bigg|_{t=0} R^{D^{g_t}}, R^D\Big\rangle_g. \tag{3.8}$$

When we set $B_t := D^{g_t} - D^g$, $R^{D^{g_t}}$ is expressed as

$$R^{D^{g_t}} = R^D + \tilde{D}B_t + B_t \wedge B_t. \tag{3.9}$$

Differentiating the both hand sides of (3.9) at $t = 0$, we have

$$\frac{d}{dt}\bigg|_{t=0} R^{D^{g_t}} = \tilde{D}\Big(\frac{d}{dt}\bigg|_{t=0} D^{g_t}\Big). \tag{3.10}$$

From $D^{g_t} = \nabla^{g_t} + (1/2)(\omega^\sharp \otimes g_t - \mathrm{Id} \otimes \omega - \omega \otimes \mathrm{Id})$, we have

$$\frac{d}{dt}\bigg|_{t=0} D^{g_t} = \gamma + \frac{1}{2}\omega^\sharp \otimes h. \tag{3.11}$$

From (3.8), (3.10) and (3.11), we have

$$\frac{d}{dt}\bigg|_{t=0} \langle R^{D^{g_t}}, R^{D^{g_t}}\rangle_{g_t} = -2\langle h, \check{R}^D\rangle_g + 2\Big\langle \tilde{D}\Big(\gamma + \frac{1}{2}(\omega^\sharp \otimes h)\Big), R^D\Big\rangle_g. \tag{3.12}$$

From (3.7) and (3.12), we have the equation (3.6). □

In order to formulate the right hand side of (3.6) in the form $\int_M \langle \bullet, h \rangle v_g$, we want to express $\gamma = (d/dt)|_{t=0} \nabla^{g_t}$ in terms of h. To do it, we need the Codazzi operator $\mathfrak{d}^\nabla$ with respect to the Levi-Civita connection ∇ (cf. [4, p. 20], [16, p. 103]).

Definition 3.3. For $h \in S^2(M)$, we define the *Codazzi operator* $\mathfrak{d}^\nabla$ as

$$(\mathfrak{d}^\nabla h)(X, Y, Z) := (\nabla_X h)(Y, Z) - (\nabla_Y h)(X, Z).$$

Then, we obtain

Lemma 3.4. *For any $h \in S^2(M) \cong T_g \mathfrak{M}$, we have*

$$g(\gamma_X Y, Z) = \frac{1}{2}((\nabla - \mathfrak{d}^\nabla) h)(Z, X, Y). \tag{3.13}$$

Proof. For ∇^{g_t}, we have

$$\begin{aligned} 2g_t(\nabla^{g_t}_X Y, Z) = {} & X(g_t(Y, Z)) + Y(g_t(Z, X)) - Z(g_t(X, Y)) \\ & + g_t(Z, [X, Y]) + g_t(Y, [Z, X]) - g_t(X, [Y, Z]). \end{aligned} \tag{3.14}$$

Differentiating the both hand sides of (3.14) at $t = 0$, we have

$$\begin{aligned} & 2\{h(\nabla_X Y, Z) + g(\gamma_X Y, Z)\} \\ & \quad = (\nabla_X h)(Y, Z) + (\nabla_Y h)(Z, X) - (\nabla_Z h)(X, Y) + 2h(\nabla_X Y, Z). \end{aligned} \tag{3.15}$$

Note that

$$\begin{cases} (\nabla_Y h)(X, Z) = (\nabla_Y h)(Z, X), \\ (\mathfrak{d}^\nabla h)(X, Y, Z) + (\mathfrak{d}^\nabla h)(Y, Z, X) + (\mathfrak{d}^\nabla h)(Z, X, Y) = 0. \end{cases} \tag{3.16}$$

From (3.15) and (3.16), we have (3.13), which completes the proof. □

From Lemma 3.4, we obtain

$$\begin{aligned} & \int_M \left\langle \gamma + \frac{1}{2}(\omega^\sharp \otimes h), \tilde{D}^*(|R^D|_g^{(n-4)/2} R^D) \right\rangle_g v_g \\ & \quad = \frac{1}{2} \int_M \langle h, ((\nabla - \mathfrak{d}^\nabla)^* + \omega)(\tilde{D}^*(|R^D|_g^{(n-4)/2} R^D)) \rangle_g v_g, \end{aligned}$$

where $(\nabla - \mathfrak{d}^\nabla)^*$ is the formal adjoint of a differential operator $\nabla - \mathfrak{d}^\nabla$ with respect to g.

From this equation, we immediately obtain

Proposition 3.5. *We have the following formula*:

$$\frac{d}{dt}\Big|_{t=0}\int_M |R^{D^{g_t}}|_{g_t}^{n/2} v_{g_t} = \frac{1}{2}\int_M \left\{ \left\langle |R^D|_g^{(n-4)/2}(|R^D|_g^2 g - n\check{R}^D) \right.\right.$$
$$\left.\left. + \frac{n}{2}\{((\nabla - \mathfrak{d}^\nabla)^* + \omega)(\tilde{D}^*(|R^D|_g^{(n-4)/2}R^D))\}, h \right\rangle_g \right\} v_g.$$

Combining (3.1) and Propositions 3.1 and 3.5, we have the first variation formula for C_W.

Theorem 3.6. *We have the following formula*:

$$\frac{d}{dt}\Big|_{t=0} C_n(g_t, \omega_t) = \frac{n}{4}\int_M \langle \eta^\sharp \otimes g - \mathrm{Id} \otimes \eta - \eta \otimes \mathrm{Id}, \tilde{D}^*(|R^D|_g^{(n-4)/2}R^D)\rangle_g v_g$$
$$+ \frac{1}{2}\int_M \left\{ \left\langle |R^D|_g^{(n-4)/2}(|R^D|_g^2 g - n\check{R}^D) \right.\right.$$
$$\left.\left. + \frac{n}{2}\{((\nabla - \mathfrak{d}^\nabla)^* + \omega)(\tilde{D}^*(|R^D|_g^{(n-4)/2}R^D))\}, h \right\rangle_g \right\} v_g.$$

From Theorem 3.6, one can determine the Euler-Lagrange equations of C_W.

Theorem 3.7. *A couple (g,ω) in $\mathfrak{M} \times A^1(M)$ is a critical point of C_W if and only if it satisfies*

$$\begin{cases} 2|R^D|_g^{(n-4)/2}\left(\check{R}^D - \dfrac{1}{n}|R^D|_g^2 g\right) - ((\nabla - \mathfrak{d}^\nabla)^* + \omega)(\tilde{D}^*(|R^D|_g^{(n-4)/2}R^D)) = 0, \\ \langle \eta^\sharp \otimes g - \mathrm{Id} \otimes \eta - \eta \otimes \mathrm{Id}, \tilde{D}^*(|R^D|_g^{(n-4)/2}R^D)\rangle_g = 0 \quad \textit{for any } \eta \in A^1(M), \end{cases}$$

where $\langle \bullet, \bullet \rangle_g$ denotes the pointwise inner product with respect to g.

Here, we recall the following property of Einstein metrics (cf. [2, p. 134, 4.72]).

Proposition 3.8. *Let ∇ be the Levi-Civita connection and $SR|_{\mathfrak{M}_1}$ a quadratic functional defined by $SR(g) := \int_M |R^\nabla|_g^2 v_g$ restricted to $\mathfrak{M}_1 := \{g \in \mathfrak{M}; \int_M v_g = 1\}$.*

An Einstein metric g (or more generally, a Riemannian metric with parallel Ricci tensor) is critical for a quadratic functional $SR|_{\mathfrak{M}_1}$ if and only if the curvature R^∇ of ∇ satisfies

$$\check{R}^\nabla = \frac{1}{n}|R^\nabla|_g^2 g. \tag{3.17}$$

From Theorem 3.7 and Proposition 3.8, we introduce the notion of conformal Yang-Mills connections. Let $[g]$ be a conformal class represented by $g \in \mathfrak{M}$.

DEFINITION 3.9. A torsion-free affine connection D is a *conformal Yang-Mills connection* with respect to $[g]$ if it satisfies

$$\tilde{D}^*(|R^D|_g^{(n-4)/2} R^D) = 0 \quad \text{for some (and then for any) } g \in [g]. \tag{3.18}$$

We then call R^D a *conformal Yang-Mills field* and the equation (3.18) a *conformal Yang-Mills equation.* Then, we have

COROLLARY 3.10. *Let D be a conformal Yang-Mills connection with respect to $[g]$. A couple $(D,[g])$ is critical for the conformal gauge fuctional C_W if and only if*

$$|R^D|_g^{(n-4)/2}\left(\check{R}^D - \frac{1}{n}|R^D|_g^2 g\right) = 0. \tag{3.19}$$

We consider that the equation (3.19) is a conformal generalization of the Einstein field equation characterized by the equation (3.17).

Especially, in dimenstion four, we have the interesting equations.

COROLLARY 3.11. *Let (M,g,D) be a 4-dimensional compact Weyl manifold and $(g,D) \in \mathfrak{W}$. Then, a couple (g,D) is a critical point of the functional $C_4 : \mathfrak{W} \times \mathfrak{C} \to \boldsymbol{R}$ if and only if it satisfies*

$$\begin{cases} \tilde{D}^* R^D = 0, \\ \check{R}^D = \frac{1}{4}|R^D|_g^2 g. \end{cases} \tag{3.20}$$

The above Proposition 3.8, Corollaries 3.10 and 3.11 suggests that a conformal Yang-Mills field is a natural generalization of the Einstein-Yang-Mills fields.

Finally, we note the Euler-Lagrange equations for the conformal functional $C_n : \mathfrak{M} \times \mathfrak{C} \to \boldsymbol{R}$.

REMARK 3.12. Let (M, g, D) be an n-dimensional compact Weyl manifold and $(g, D) \in \mathfrak{W}$. Then, a couple (g, D) is a critical point of C_n if and only if it satisfies

$$\begin{cases} |R^D|_g^{(n-4)/2}\left(\check{R}^D - \frac{1}{n}|R^D|_g^2 g\right) = 0, \\ \tilde{D}^*(|R^D|_g^{(n-4)/2} R^D) = 0. \end{cases}$$

4. Examples

In this section, we give some examples as critical points of our functional C_W.

EXAMPLE 4.1. *Let (M, g) be a 4-dimensional Einstein manifold and ∇ the Levi-Civita connection of g. Then, $(g, \nabla) \in \mathfrak{W}$ is a critical point of C_W.*

Let R^∇ be the curvature of $(g, \nabla) \in \mathfrak{W}$. We recall that any Einstein metric g has harmonic curvature (cf. [4, p. 20, 3.(i)]), accordingly,

$$\tilde{\nabla}^* R^\nabla = 0. \tag{4.1}$$

From Proposition 3.8, we have

$$\check{R}^\nabla = \frac{1}{4}|R^\nabla|_g^2 g. \tag{4.2}$$

From Corollary 3.11, (4.1) and (4.2), $(g, \nabla) \in \mathfrak{W}$ is a critical point of C_W.

EXAMPLE 4.2. *Let (M, g) be an n-dimensional isotropy irreducible homogeneous space with its canonical metric and the the Levi-Civita connection ∇. Then, (g, ∇) is a critical point of C_W.*

Since the norm $|R^D|_g$ is constant, we have

$$\tilde{D}^*(|R^D|_g^{(n-4)/2} R^D) = |R^D|_g^{(n-4)/2} \tilde{D}^* R^D = 0. \tag{4.3}$$

Thus, R^D is a conformal Yang-Mills field. We recall that isotropy irreducible homogeneous metrics are critical for the functional SR in Proposition 3.8 (cf. [2, p. 134, 4.73]) and the irreducibility of the isotropy representation and the homogeneity imply that g is an Einstein metric (cf. [2, p. 119, 4.13]). From this and Proposition 3.8, we have

$$\check{R}^\nabla = \frac{1}{n}|R^\nabla|_g^2 g. \tag{4.4}$$

From (4.3), (4.4) and Corollary 3.10, $(g, \nabla) \in \mathfrak{W}$ is a critical point of C_W.

REMARK 4.3. Every Weyl manifold (M, g, D) realized as an affine hypersurface of the Euclidean space is trivial, that is, there exists a metric $\hat{g}$ on M conformal to g such that $D\hat{g} = 0$ (see Matsuzoe [11]).

It would be an important problem to construct non-trivial Weyl structures which are critical for C_W.

Appendix A. Properties of the Curvature Tensor R^D

We prove some properties of the curvature tensor R^D in order to calculate critical points of our functional. We recall the Kulkarni-Nomizu product $\owedge$ to describe them.

DEFINITION A.1. For any 2-tensors h and k, the $(0,4)$-tensor $h \owedge k$ is defined by

$$\begin{aligned}(h \owedge k)(X, Y, Z, T) := h(X, Z)k(Y, T) + h(Y, T)k(X, Z)\\ - h(X, T)k(Y, Z) - h(Y, Z)k(X, T)\end{aligned}$$

for $X, Y, Z, T \in \mathfrak{X}(M)$.

The curvature tensor R^D of (g, D) has somewhat different features from the cuvature tensor R^∇ of the Levi-Civita connection ∇. And we recall the relation between R^D and R^∇ which is well known (cf. [8], [15]).

PROPOSITION A.2. *For $D = \nabla + \alpha$ in Lemma 2.1, we have*

$$R^D(X, Y)Z = R^\nabla(X, Y)Z + (d^\nabla \alpha)(X, Y)Z + \alpha_X(\alpha_Y Z) - \alpha_Y(\alpha_X Z),$$

where d^∇ is the exterior derivative associated with the Levi-Cività connection ∇.

We can also express the relation of R^D and R^∇ by g and ω (cf. [13, p. 104]).

PROPOSITION A.3. *The curvature R^D has the following relation with R^∇.*

$$
\begin{aligned}
R^D(X,Y)Z = R^\nabla(X,Y)Z &- \frac{1}{2}\left\{\left((\nabla_X\omega)Z + \frac{1}{2}\omega(X)\omega(Z)\right)Y\right. \\
&- \left((\nabla_Y\omega)Z + \frac{1}{2}\omega(Y)\omega(Z)\right)X + ((\nabla_X\omega)Y)Z - ((\nabla_Y\omega)X)Z \\
&\left.- g(Y,Z)\left(\nabla_X\omega^\sharp + \frac{1}{2}\omega(X)\omega^\sharp\right) + g(X,Z)\left(\nabla_Y\omega^\sharp + \frac{1}{2}\omega(Y)\omega^\sharp\right)\right\} \\
&- \frac{1}{4}|\omega|_g^2(g(Y,Z)X - g(X,Z)Y)
\end{aligned}
$$

for $X, Y, Z \in \mathfrak{X}(M)$.

Then, we have

Lemma A.4. *Let* (M,g,D) *be an n-dimensional Weyl manifold with the Weyl equation* $Dg = \omega\otimes g$ *and* R^D *the curvature tensor of* (g,D). *Then,* R^D *has the following properties*:

(1) $R^D(Y,X,Z,W) = -R^D(X,Y,Z,W)$,

(2) $R^D(X,Y,W,Z) = -R^D(X,Y,Z,W) - (d\omega\otimes g)(X,Y,Z,W)$,

(2) $$
\begin{aligned}
R^D(Z,X,Y,W) = R^D(Y,W,Z,X) &+ \frac{1}{2}\{(d\omega\otimes g)(X,Y,Z,W) \\
&+ (d\omega \owedge g)(X,Y,Z,W) + 2(d\omega\otimes g)(Y,Z,X,W) \\
&+ (d\omega\otimes g)(Z,W,X,Y)\}.
\end{aligned}
$$

Proof. The property (1) is given by a direct calculation.
From the Weyl equation $Dg = \omega\otimes g$, we have

$$
Y(g(W,Z)) = \omega(Y)g(W,Z) + g(D_YW,Z) + g(W,D_YZ). \tag{A.1}
$$

Differentiating the both hand sides of (A.1) about X, we have

$$
\begin{aligned}
X(Y(g(W,Z))) = X(\omega(Y))g(W,Z) &+ \omega(Y)\{\omega(X)g(W,Z) + g(D_XW,Z) \\
&+ g(W,D_XZ)\} + \omega(X)g(D_YW,Z) + g(D_XD_YW,Z) \\
&+ g(D_YW,D_XZ) + \omega(X)g(W,D_YZ) + g(D_XW,D_YZ) \\
&+ g(W,D_XD_YZ),
\end{aligned} \tag{A.2}
$$

and

$$Y(X(g(W,Z))) = Y(\omega(X))g(W,Z) + \omega(X)\{\omega(Y)g(W,Z) + g(D_Y W,Z) + g(W,D_Y Z)\} + \omega(Y)g(D_X W,Z) + g(D_Y D_X W,Z) + g(D_X W,D_Y Z) + \omega(Y)g(W,D_X Z) + g(D_Y W,D_X Z) + g(W,D_Y D_X Z). \tag{A.3}$$

From (A.2), (A.3) and

$$[X,Y](g(W,Z)) = \omega([X,Y])g(W,Z) + g(D_{[X,Y]}W,Z) + g(W,D_{[X,Y]}Z), \tag{A.4}$$

we have

$$d\omega(X,Y)g(W,Z) + R^D(X,Y,W,Z) + R^D(X,Y,Z,W) = 0.$$

Thus, we obtain (2).

From the first Bianchi identity (of torsion-free type), we have

$$R^D(X,Y)Z + R^D(Y,Z)X + R^D(Z,X)Y = 0, \tag{A.5}$$

and

$$S^D(X,Y,Z,W) := R^D(X,Y,Z,W) + R^D(Y,Z,X,W) + R^D(Z,X,Y,W) = g(R^D(X,Y)Z + R^D(Y,Z)X + R^D(Z,X)Y, W) = 0. \tag{A.6}$$

From the properties (1), (2) and the cyclic sum of $S^D(X,Y,Z,W)$, we have

$$-d\omega(X,Y)g(W,Z) - d\omega(Y,Z)g(X,W) - d\omega(Z,W)g(X,Y) - d\omega(W,X)g(Y,Z) + d\omega(Z,X)g(W,Y) + d\omega(W,Y)g(X,Z) + 2\{R^D(Z,X,Y,W) + R^D(W,Y,Z,X)\} = 0. \tag{A.7}$$

Using the Kulkarni-Nomizu product $\owedge$, from (A.7) we have

$$R^D(Z,X,Y,W) = R^D(Y,W,Z,X) + \frac{1}{2}\{(d\omega \otimes g)(X,Y,Z,W) + (d\omega \owedge g)(X,Y,Z,W) + 2(d\omega \otimes g)(Y,Z,X,W) + (d\omega \otimes g)(Z,W,X,Y)\}.$$

Thus, the property (3) is proved. □

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